DIFFERENTIAL EQUATIONS —MATH 131

LECTURE NOTES BY STEFAN WANER

Contents

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1. Introduction

What is a differential equation? In words, a **differential equation (DE)** is an equation involving some unknown function $y(x)$ (of possibly more than one variable) as well as derivatives of y.

Examples 1.1.

A. $\dfrac{dy}{dx} = e^x + x^2$ This is an equation involving the derivative of y, and is called a **first order** DE, since it involves only the first derivative of the unknown function y. A **solution** of a differential equation is a function $y(x)$ that satisfies the given equation. Thus the solution to the above equation is:

$$y = \int e^x + x^2 \; dx = e^x + \frac{x^3}{3} + C,$$

where C is any constant. Thus it seems that a DE can have many solutions.

Eg. here, $y = e^x + \dfrac{x^3}{3} + 9,\; e^x + \dfrac{x^3}{3} + 1,003,\;\; e^x + \dfrac{x^3}{3} + \pi$ are all particular solutions.

$e^x + \dfrac{x^3}{3} + C$ is called the **general solution,** since any solution is of this form. (You can check that it is, in fact, a solution by substituting y back in the original equation.)

B. Let $u(t)$ be the position of a particle at time t. Find $u(t)$ if

$$m\frac{d^2u}{dt^2} = ku + \frac{du}{dt}.$$

Here, m and k are constants. Our job is to find a function $u(t)$ which satisfies the equation. There is no obvious answer, but we will be able to solve it later in the course. This equation is called a **second order DE,** since it involves the second derivative of the unknown function u.

The DE's in Examples A and B are collectively referred to as **ordinary DE's,** since they deal with ordinary—as opposed to partial—derivatives. Thus, the DE in Example A is a first order ordinary DE, while the one in Example B is a second order ordinary DE.

C. Let $g(x, t)$ be a function of x and t. Then

$$\frac{\partial g}{\partial x} = \frac{\partial^2 g}{\partial t^2} + 4\sin(xt)$$

is a DE in the unknown function $g(x, t)$ of two variables x and t. This is called a **partial DE,** since it involves partial derivatives of the unknown function g. These are harder to solve, and you won't know how until Math 143 or thereabouts. (This course deals only with ordinary DE's.)

Finally, there are special types of ordinary DE's called **linear** DE's. These have the form

$$\textbf{Linear: } a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x) \qquad (*)$$

This is a linear equation of order n (since the highest derivative is the nth). The $a_n(x)$ are called the **coefficients** and may be constants or even horrible functions.

Examples 1.2. Linear DEs

(1) Equations A and B above are both linear DE's, since they can be rewritten with all the derivative terms on the left, making them look like (*).

(2) The equation

$$4\frac{d^2y}{dx^2}-3\frac{dy}{dx}-2y=e^x cosx$$

has coefficients $a_2(x)=4,\ a_1(x)=-3,\ a_0(x)=-2$. It is a second order linear DE *with constant coefficients.*

(3) The equation

$$\frac{d^2y}{dx^2}+3x^2\frac{dy}{dx}+4xy=e^x\sin x$$

has coefficients $a_2(x)=1,\ a_1(x)=3x^2,\ a_0(x)=4x$. It is a second order linear DE *with non-constant coefficients.*

(4) The equation

$$\frac{d^2y}{dx^2}+3y\frac{dy}{dx}+4xy=e^x\sin x$$

is not linear, since the coefficient of dy/dx is not a function of x.

(5) Here is a nice one:

$$y''+y=0$$

This in fact has solutions $y=\sin x$ and also $y=\cos x$. To check that they are in fact solutions, plug them in and see! Moreover, if A and B are any constants, then

$$y=A\sin x+B\cos x$$

is also a solution—again, check by plugging in. (In fact, this is the *general solution.*).

Methods of Solving Two broad methods:

I Analytical. This is the way we solved the first one above. It means giving an exact solution in terms of known functions by using various methods we shall study.

II Numerical. Many DE's are impervious to analytical methods, so there are ways of getting approximate solutions via the use, usually, of computers. These give solutions as accurately as you like, but in terms of numbers rather than known functions. For example, it might say that the solution to B above has the property that:

$$u(1) = 4.0985, u(1.3) = 5.69584, \text{etc.}$$

This does allow one to draw its graph, but doesn't tell one what $u(t)$ is, but it is useful in applications where you couldn't care whether it's a known function, and only need $u(1)$, $u(1.3)$ etc. We'll do mainly analytical methods, as they provide understanding. Currently available computer software does numerical solving.

Discussion Thread Suggestions Think about the difference between particular and general solutions, and about the constant of integration.

2. First Order Linear DEs

Recall that these involve only the first derivative, and are, in addition, linear. Here, y will always be an (unknown) function of x, and we are required to find y. Recalling that the general form of a 1st order linear DE is:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x),$$

we divide both sides by $a_1(x)$ to get an equation of the form

$y' + p(x)y = f(x)$	**Standard Form of First Order Linear DE**

We look at two specific types of these:

A. Easy: $(p(x) = 0)$

These have the form $y' = f(x)$, and so are like the very first one we looked at in Section 1. One solves them by doing a simple integration, and the solution is $y = \displaystyle\int f(x)\ dx$.

Examples of Easy Ones

(a) Solve $y' = \sin x$.

Solution: $y = \displaystyle\int f(x)\ dx = \int \sin x\ dx = -\cos x + C$. (general solution)

(b) Solve $y' = \dfrac{1}{1+x^2}$

Solution: $y = \displaystyle\int f(x)\ dx = \int \frac{1}{1+x^2}\ dx = \arctan x + C$. (general solution)

B. General First Order Linear:

As noted above, these have the form $y' + p(x)y = f(x)$. Here, $p(x)$ and $f(x)$ are given functions of x.

Examples

(a) $y' + xy = x^2 \quad (p(x) = x,\ q(x) = x^2)$
(b) $y' - 2xy = x \quad (p(x) = -2x,\ q(x) = x)$
(c) $xy' + 2y = \sin x$ (Not in the standard linear form, but dividing by x gives $y' + \dfrac{2}{x}y = \dfrac{\sin x}{x}$.)

Method of Solution of $y' + p(x)y = f(x)$.

Look at the left hand side. It looks a little like the product rule...Multiply both sides by a "certain" function $\mu(x)$. Get:

$$\mu(x)y' + p(x)\mu(x)y = \mu(x)f(x)$$

If only

$$p(x)\mu(x) \text{ was equal to } \mu'(x) \qquad (*)$$

Then we would have:

$$\mu(x)y' + \mu'(x)y = \mu(x)f(x)$$

ie.

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x)$$

so

$$\mu(x)y = \int \mu(x)f(x)\ dx + C,$$

giving

$$\boxed{y = \frac{1}{\mu(x)}\left[\int \mu(x)f(x)\ dx + C\right] \text{ Gen Solution of First Order Linear DE}}$$

That's all well & good, but we still don't know what $\mu(x)$ is!
However, by (*), we must have $p(x)\mu(x) = \mu'(x)$. That is,

$$\frac{\mu'(x)}{\mu(x)} = p(x)$$

$$\text{ie. } \frac{d}{dx}[\ln \mu(x)] = p(x)$$

$$\text{or } \boxed{\mu(x) = e^{\int p(x)dx}}$$ **... Integrating Factor**

Thus our method of solution is given as follows:

Solving a First Order DE

(1) Make sure that the coefficient of y' is a 1 (by dividing if necessary).
(2) Set $\mu(x) = e^{\int p(x)dx}$.
(3) Then the general solution is $y = \dfrac{1}{\mu(x)}\left[\int \mu(x)f(x)\ dx + C\right]$.

Examples 2.1.
A. We solve $y' + 2y = e^{-x}$, subject to the initial condition $y(0) = 3$.
B. We solve $y' - 2xy = x$, subject to the initial condition $y(0) = 1$.
C. Solve $xy' + 2y = \sin x$.

Here is a bit of theory. We saw that, not only can we solve a first order linear DE by the above method, but also that the solution was ?forced? on us; that is, it is **unique.** A generalization of this is the following.

Theorem 2.2. ***Picard's Theorem*** *(Existence and uniqueness of solutions to 1st order DEs) Given any DE of the form*

$$\frac{dy}{dx} = f(x, y)$$

with initial conditions $y(x_0) = y_0$*, assume that both* f *and* $\partial f/\partial y$ *are continuous on some rectangle containing* (x_0, y_0)*. Then there is a unique solution* $y = \phi(x)$ *defined on some interval* $(x_0 - h, x_0 + h)$*.*

3. Separable DEs

Here we consider general DEs of the form

$$\frac{dy}{dx} = f(x, y)$$

but which can be rewritten in the form

$$M(x)dx = N(y)dy.$$

Thus we have "separated the xs from the ys," so to speak. We can now integrate both sides to get a relationship between x and y. If we can then solve for y in terms of x, then we have an explicit solution, else we just have an implicit solution.

Examples 3.1.

A. Solve $\dfrac{dy}{dx} = \dfrac{3x^2 + 4x + 2}{2(y-1)}$ subject to $y(0) = 1$.

B. Exponential growth: Solve $\dfrac{dy}{dx} = ky$ ($k > 0$ constant) subject to $y(0) = y_0$

4. Applications of First Order DEs

A. Exponential Decay

The rate of (radioactive) decay of a substance is proportional to the amount of substance left in the sample. Thus assume one begins at time $t = 0$ with a quantity Q_0 of the stuff. Our task is to find an equation for the quantity Q (of undecayed substance) left after time t.

Answer In words, the rate of decrease of Q is proportional to Q. That is,

$$-\frac{dQ}{dt} = kQ$$

where k is some constant. This is a separable DE with solution

$$Q = Ce^{-kt} \quad (C = \text{constant})$$

What does C represent? Well, at time $t = 0$, $Q = Q_0$; an initial condition.

Plugging in: $Q_0 = Ce^{-0} = C$. Thus $C = Q_0$. whence:

$$\boxed{Q = Q_0 e^{-kt}} \qquad \textbf{Exponential Decay}$$

Question How long does it take for half the sample to decay? (This time is called the **half-life.**)

Answer In "class," we obtain

$$\boxed{t_{1/2} = \frac{\ln 2}{k}} \qquad \textbf{Half-life}$$

Note $t_{1/2}$ is independent of the amount Q_0 you started with.

Example (From Calc 2) **Carbon dating** is based on the fact that radioactive Carbon-14 (which originates from the upper atmosphere when nitrogen is exposed to cosmic radiation) is absorbed by living plant and animal tissue along with non radioactive Carbon-12. When the plant or animal dies, the absorption stops, and the Carbon-14 decays into lead. When a fossil is analyzed, the amounts of both isotopes present are measured, and compared with the ratios assumed to be present in the environment at the time they lived.

Given that Carbon-14 has a half-life of 5750 years, and that tests on a certain fossil show that 95% of its Carbon-14 has decayed, how old is the fossil?

B. Exponential Growth

If you invest \$$P$ at an interest rate of k compounded every Δt years, then the amount you have at time $t + \Delta t$ is

$$A(t + \Delta t) = A(t) + kA(t)\Delta t.$$

For continuous compounding, we let $\Delta t \to 0$, and this leads to the DE

$$\frac{dA}{dt} = kA,$$

with solution

$$\boxed{A = A_0 e^{kt}} \qquad \textbf{Exponential Growth}$$

with

$$\boxed{t_D = \frac{\ln 2}{k}} \qquad \textbf{Doubling Time}$$

Example The population ten years ago was 1,000. It is now 3,250. Assuming exponential growth, what will it be in two years' time?

C. Epidemics Let a population of n susceptible people contain $p(t)$ infected people at time t. Then $q(t) = n - p(t)$ is the number of susceptible, as yet uninfected, people. We assume the disease is spread by contact, and that each person comes into contact with k other people from the population per unit time on average. (eg. k people per week, if this is our unit of time), and that the probability of getting infected through a single contact is α.

Now, $p(t)$ is the number of people infected at time t.

Question A little time, Δt later, how many people are now infected?

Answer First look at one unit of time later. Since each uninfected susceptible person has k contacts per unit time, $k \cdot p(t)/n$ of these contacts are with infected people. (The fraction of susceptible people infected is $p(t)/n$.) Thus, in a time period of Δt units of time, each uninfected person has $kp(t)\Delta t/n$ contacts. This is for a single uninfected person. Since there are $q(t)$ uninfected susceptible people, the total number of contacts between uninfected and infected people is $q(t)kp(t)\Delta t/n$. Finally, since the fraction α of these contacts results in new cases, the number of people infected in time Δt is

$$\Delta p = \frac{\alpha k p(t) q(t) \Delta t}{n}.$$

Thus,

$$\frac{\Delta p}{\Delta t} = \frac{\alpha k p(t) q(t)}{n}.$$

Taking limits as $\Delta t \to 0$, taking $\alpha k = \sigma$, and dropping the functional notation gives

$$\frac{dp}{dt} = \frac{\sigma p q}{n} = \frac{\sigma p(n-p)}{n}.$$

This is a separable equation, with

$$\frac{n\,dp}{p(n-p)} = \sigma dt$$

i.e.

$$\left(\frac{1}{p} + \frac{1}{n-p}\right) dp = \sigma dt,$$

whence

$$\ln\left(\frac{p}{n-p}\right) = \sigma t + C.$$

giving

$$\frac{p}{n-p} = Ae^{\sigma t}$$

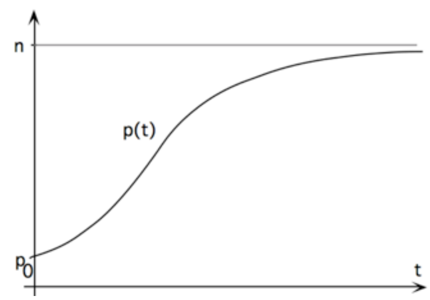
(by the usual arguments). To get the constant, we substitute the initial conditions $p = p_0$ at time $t = 0$, giving $A = \dfrac{p_0}{n-p_0}$, so that the solution is

$$\frac{p}{n-p} = \left(\frac{p_0}{n-p_0}\right)e^{\sigma t}.$$

We now solve for p to get the logistic equation:

$$\boxed{p = \frac{np_0}{p_0 + (n-p_0)e^{-\sigma t}}} \qquad \textbf{Logistic Equation}$$

Graph:



D. Mixing We do an example. Let a tank have Q_0 lbs. of salt in 100 gals. water. A solution of 1/4 lb/gal of salt-water is entering the tank at a rate of 3 gals/min, being stirred constantly. Water is also leaving the tank at the same rate. Find an expression for the amount of salt $Q(t)$ in the tank at time t.

Solution The total amount of salt is $Q = Q(t)$ at time t. Thus the concentration is $Q/100$ lbs. salt per gallon at time t. We write down the equation

Rate of increase in Q = Rate salt is entering – Rate salt is leaving

$$\frac{dQ}{dt} = \frac{1}{4}\cdot 3 - 3\cdot\frac{Q}{100}\text{lbs/min.}$$

This is a linear DE with solution

$$Q(t) = 25(1 - e^{-3t/100}) + Q_0e^{-3t/100}$$

$\uparrow$ Amount of new salt

$\uparrow$ a decay term: Amount of original salt present

5. Theory of linear differential equations

We saw in Section 1 that a linear DE or order n has the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad (*)$$

We are interested in finding solutions to (*), that is, a function or family of functions $y(x)$ satisfying this equation, possibly subject to some "initial condition(s)" and defined for all values of x in some interval. It is customary to assume that $a_n(x)$ is nowhere zero in the interval of interest, so, as in Section 2, we divide by it to get the "standard form:"

Standard Form of nth Order Linear DE

$$\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + p_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y = f(x)$$

(Instead of $p_1(x), p_2(x), p_3(x), \ldots$ we avoid subscripts by using $p(x), q(x), r(x), \ldots$ etc in equations of low order.

Examples 5.1. Linear DEs

A. First order linear DE in standard form: $y' + p(x)y = f(x)$
We already know how to solve these...

B. Second order linear DE in standard form: $y'' + p(x)y' + q(x)y = f(x)$

C. A mass m on a spring subjected to a varying force $F(t)$ has displacement $u(t)$ satisfying an equation of the form

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = F(t)$$

for given constants k and c. (We'll see why later.) This kind has **constant coefficients.**

D. Bessel's Equation

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$

for a constant α. This pops up in many applications, such as acoustics and electromagnetism.

Theorem 5.2. ***Existence and uniqueness theorem for linear DEs***
Suppose that $p_1(x), p_2(x), \ldots p_n(x)$ and $f(x)$ are continuous on an open interval (a, b) containing the point x_0. Then:

(1) The equation

$$\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y = f(x)$$

has an n-parameter family of solutions (called the ***general solution****) defined over the entire interval $(a, b.)$*

(2) There is exactly one solution $y(x)$ to this DE satisfying any specific set of "initial conditions"

$$y(x_0) = y_0,\ y'(x_0) = y_1, \quad \ldots, \quad y^{n-1}(x_0) = y_{n-1}.$$

Example 5.3.
The linear DE $y'' - 4y = 8$ has the general solution $y = Ae^{2x} + Be^{-2x} - 2$. Check that it is a solution mentally...
If we impose the initial conditions $y(0) = 2,\ y'(0) = 4$, we get the unique solution $y = 3e^{2x} + e^{-2x} - 2$. Check that it is a solution mentally...

General Strategy for Solving Linear DEs

To solve linear DE's we first consider some special kinds of linear DEs:

Definition 5.4. A linear DE is **homogeneous** if it has the form

$$\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + p_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y = 0,$$

so that here, $f(x) = 0$.
Given *any* linear DE

$$\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + p_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y = f(x),$$

then its **associated homogeneous equation** is obtained by replacing $f(x)$ by 0.

Thus, eg. the associated homogeneous equation of $y''+6y = 77x^2$ is $y''+6y = 0$.

What's so special about these? Well...

Theorem 5.5. ***Difference of solutions to a linear DE***
If $u(x)$ and $v(x)$ are any two particular solutions to the general linear DE, then $u(x) - v(x)$ is a solution to the associated homogeneous *DE.*

Example 5.6. Consider what this says about the linear DE $y' = \cos x$.

We prove the theorem in the video.

Corollary 5.7. *General form of Solution to a Linear DE*
The general solution to a linear DE has the form

$$y = CF + PI$$

*where CF is the **complementary function:** the general solution to the associated homogenous DE, and PI is a **particular integral:** any one particular solution to the (original) DE .*

We prove the corollary in the video.

Thus, all we need to do is learn two things:

(1) Finding complementary functions (meaning solving *homogeneous* linear DEs).
(2) Finding particular integrals (meaning finding a single solution to the original equation). One is enough.

Examples 5.8.

A. Solve the silly little first order linear DE $y' = \sin x$.

B. General First Order Linear DEs revisited
Let's go back to our solution of first order linear DEs:

$$y = \frac{1}{\mu(x)}\left[\int \mu(x)f(x)\ dx + C\right].$$

To get a PI, (particular solution of the original equation) just choose the constant C to be zero:

$$PI = \frac{1}{\mu(x)}\int \mu(x)f(x)\ dx.$$

To get the CF, put $f(x) = 0$, (for the associated homogenous equation) to get

$$CF = C \cdot \frac{1}{\mu(x)} \qquad \text{Solution of homogenous equation}$$

Adding them together:

$$\begin{aligned} y &= CF + PI \\ &= C \cdot \frac{1}{\mu(x)} + \frac{1}{\mu(x)}\int \mu(x)f(x)\ dx \\ &= \frac{1}{\mu(x)}\left[\int \mu(x)f(x)\ dx + C\right], \end{aligned}$$

which is the original formula for the general solution!

What do Complementary Functions Look Like?
We saw in the above example that the CF for a first order linear DE has the form $Cu(x)$, where C is some arbitrary constant.

So, you should not be surprised to know that, for a *second order* linear DE the CF has the form $Au(x) + Bv(x)$, where A and B are *two* arbitrary constants. For a *third order* linear DE, it would look like $Au(x) + Bv(x) + Cw(x)$, and so on.

Further the functions $u(x), v(x), w(x), \ldots$ are individual solutions of the homogenous equation and also **linearly independent,** meaning that no one of them can be expressed as a linear combination of the others (that is, as a constant times one plus a constant times another plus a constant times yet another...) For instance,

$$\sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x},$$

expresses $\sinh x$ as a linear combination of e^x, and e^{-x}, so the three functions $\sinh x, e^x$, and e^{-x} are *not* linearly independent.

Q: So how do I tell at a glance whether a collection of, say, three solutions $u(x), v(x), w(x)$ of the homogenous equation is independent?
A: Take their **Wronskian:** [1]

$$W[u, v, w](x) = \begin{vmatrix} u(x) & v(x) & w(x) \\ u'(x) & v'(x) & w'(x) \\ u''(x) & v''(x) & w''(x) \end{vmatrix}$$

If you get any nonzero function, they are independent. If you get the zero function, they are dependent (i.e., not independent).

Examples 5.9.
A. Solve the less silly DE $y'' + y = 0$.
B. Solve the equally less silly DE $y'' - y = 0$.
C. Now solve $y^4 - y = 0$. (Part of that is left to the exercises...)

General form of the Wronskian:

$$W[u_1, \ldots, u_n](x) = \begin{vmatrix} u_1(x) & u_2(x) & \cdots & u_n(x) \\ u_1'(x) & u_2'(x) & \cdots & u_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(n-1)}(x) & u_2^{(n-1)}(x) & \cdots & u_n^{(n-1)}(x) \end{vmatrix}$$

Some underlying theory:

[1] Named after Josef Maria Hoene Wronski (1778–1853) whose only contribution to mathematics appears to have been the Wronskian, and who eventually went insane.

Theorem 5.10. ***Superposition Principle***
If $u_1(x), u_2(x), \ldots, u_k(x)$ are any solutions to a homogeneous linear DE, then so is any linear combination of them:

$$y(x) = A_1 u_1(x) + A_2 u_2(x) + \cdots + A_k u_k(x). \qquad (A_i \text{ arbitrary constants})$$

Proof in the videos.

Note This does not work for non-homogeneous ones—see the homework.

Finally, what we said above, which now seems more plausible in view of the superposition principle:

Theorem 5.11. ***Form of the CF***
The general solution of the homogenous equation

$$\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + p_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y = 0,$$

is

$$y(x) = A_1 u_1(x) + A_2 u_2(x) + \cdots + A_n u_n(x). \qquad (A_i \text{ arbitrary constants})$$

where $u_1(x), u_2(x), \ldots, u_n(x)$ are any n independent specific solutions.

Partial Proof That it *is* a solution follows from the superposition principle. that it is *the only* solution rests on the Existence and Uniqueness Theorem 5.2, and some linear algebra in combination with results about the Wronskian of solutions to homogeneous linear DE's (namely, that the Wronskian of such functions is either the zero function, or it is everywhere nonzero; one of the exercises deals with this for the second order DE case. The higher order cases are similar, but require a knowledge of minors in and the theory of determinants.)

6. Solving Homogeneous Second Order Linear DEs with Constant Coefficients

Here we see how to cook up the us and vs in the nice case of constant coefficients for second order homogeneous linear DEs. First, a linear 2nd order homogeneous DE with constant coefficients has the form

$$ay'' + by' + cy = 0 \quad \textbf{Homog. Linear DE Const. Coeffs. (*)}$$

where a, b, and c are constants ($a \neq 0$).

We wish to find the two independent solutions $u(x)$ and $v(x)$. We've seen that, no matter how we cook these up, (even by fortuitous guessing!), we will get the general solution this way as long as they are independent, namely:

$$y(x) = Au(x) + Bv(x) \qquad (A, B \text{ arbitrary constants})$$

We try as a good guess, a solution of the form:

$$y = e^{mx}$$

for some suitable constant m. In the video, we see how this leads to the **auxiliary equation**

$$am^2 + bm + c = 0.$$

From Baby Math, its roots depend on whether the discriminant $b^2 - 4ac$ is positive, zero, or negative.

Theorem 6.1. ***Distinct Real Roots***
If the discriminant $b^2 - 4ac > 0$, then the general solution of equation (*) is

$$y(x) = Ae^{m_1 x} + Be^{m_2 x},$$

where m_1 and m_2 are the roots of the auxiliary equation.

Examples 6.2.
A. Find the GS of $y'' + 5y' + 6y = 0$.
B. Find the GS of $y'' + y' - 2y = 0$ subject to $y(0) = y'(0) = 1$.

Theorem 6.3. ***Repeated Real Roots***
If the discriminant $b^2 - 4ac = 0$, then the general solution of equation (*) is

$$y(x) = Ae^{mx} + Bxe^{mx},$$

where m is the repeated root of the auxiliary equation.

Example 6.4.
Find the GS of $y'' + 2y' + y = 0$.

Theorem 6.5. ***Complex Roots***
If the discriminant $b^2 - 4ac < 0$, *then the general solution of equation (*) is*
$$y(x) = e^{\alpha x}[Acos\beta x + Bsin\beta x],$$
where the roots of the auxiliary equation are $m = \alpha \pm i\beta$.

Examples 6.6.
A. Find the GS of $y'' + y' + y = 0$.
B. Find the GS of $y'' + y = 0$.

7. Solving Nonhomogeneous Second Order Linear DEs with Constant Coefficients: Method of Undetermined Coefficients

Nonhomogeneous Second Order Linear DEs with constant coefficients have the form

$$ay'' + by' + cy = f(x) \quad \textbf{Linear DE Const. Coeffs. (*)}$$

with some nonzero $f(x)$. To solve these, we first recall Corollary 5.7:

Corollary 5.7. ***General form of Solution to a Linear DE***
The general solution to a linear DE has the form

$$y = CF + PI$$

where CF is the **complementary function:** the general solution to the associated homogenous DE, and PI is a **particular integral:** any one particular solution to the (original) DE .

In the preceding section we saw how to find CFs, so now we focus on finding PIs for some (but not all...) of these beasts, thereby completing the process of solving (some of) them completely. The **method of undetermined coefficients** is a "guessing method." We'll learn it via examples, which we go through in the video.

Examples 7.1.

A. Solve completely: $y'' - 3y' - 4y = 9$
B. Find a particular solution of $ay'' + by' + cy = K$ (Kconstant $\neq 0$)
C. Solve completely: $y'' + 2y' + y = 4x$
D. Solve completely: $y'' + y = 4x^2 - 1$
E. Solve completely: $y'' - 3y' - 4y = e^{2x}$
F. Solve completely: $y'' - 3y' - 4y = e^{-x}$ where we encounter a glitch...

Undetermined Coefficients

$f(x)$	What You Try for y
A polynomial	$y =$ A general polynomial of the same degree
$A\sin x, B\cos x,$ or their sum	$y = Q\sin x + R\cos x$
$Ae^{\alpha x}$	$y = Qe^{\alpha x}$

Glitches If any of the summands in the expression on the right is already a solution of the homogenous equation, then you multiply the entire expression on the right by x. If any one of the new terms you now get *still* a solution of the homogenous equation,, multiply again by x, and so on.

Q&A
Q: S'pose you have, say, $f(x) = \sin x + e^{6x}$?
A: First find a PI for $ay''+by'+cy = \sin x$, then ditto for $ay''+by'+cy = e^{6x}$. Then your desired PI is the *sum of the two PIs* you just obtained. (You will see why in the exercises.)
Q: What if you have to solve, say, $ay'' + by' + cy = x \sin x$?
A: This is a product of two functions (x and $\sin x$) you know how to deal with. For x, you would try: $Px+Q$; for $\sin x$, you would try $R\cos x+Tsinx$. Then, for the product, you try

$$(Px + Q)(R\cos x + T\sin x) = Ux\cos x + Vx\sin x + W\cos x + Z\sin x$$

and solve for the constants U, V, W, Z by substituting. This works for any product.

Example 7.2. Solve $y'' + 4y = xe^x + x\sin(2x)$.

8. Homogeneous 2nd order DE's with Nonconstant Coefficients: Reduction of Order

Nonhomogeneous Second Order Linear DEs with nonconstant coefficients are harder to solve than those with constant coefficients. Recall that these have the (standard) form

$$\boxed{y''+p(x)y'+q(x)y=0 \quad \textbf{Linear Homog. DE NonConst. Coeffs. (*)}}$$

Here, $p(x)$ and $q(x)$ are given functions of x (not both constant in the case of interest here). In this case, we can't easily find a nice CF as we did with the constant coefficient ones. We'll eventually see how to find at least one solution using power series, although we need two independent solutions in general. In the meantime, we pretend that, from Mars, say, we have been given one of the solutions, $u(x)$, of equation (*). We now find out how to (sometimes) find the other one. To get the other one, we try a solution of the form

$$y(x) = h(x)u(x).$$

Substituting this into the DE gives, as we shall see in the lecture,

$$h\{u'' + pu' + q'u\} + h'\{2u' + pu\} + h''u = 0.$$

The first term in braces is zero, as u is a solution of (*). We recognize what is left as a linear equation for h'(x), and ultimately obtain

$$h(x) = \int e^{-\int\left(\frac{2u'}{u}+p\right)dx}dx,$$

which we can manipulate a little to get

$$\boxed{h(x) = \int \frac{e^{-\int p(x)dx}}{u^2(x)}dx}$$

This gives h(x), and hence the other solution,

$$v(x) = h(x)u(x).$$

Examples 8.1.

A. (a) Show that $y = x$ is a solution of Legéndre's equation

$$(1-x^2)y'' - 2xy' + 2y = 0 \quad (-1 < x < 1).$$

(b) Find a linearly independent second solution, and hence the GS.

B. Given that $y = \dfrac{\sin x}{\sqrt{x}}$ is a solution of $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$, obtain the general solution.

9. Variation of Parameters

Now that we (kind-of) know how to solve general homogeneous equations, it remains to find a way of solving the *non-homogeneousones* ones by finding PIs (particular integrals) *whether or not the coefficients are constant.*

Recall that a second order linear DE in general form looks like this:

$$y'' + p(x)y' + q(x)y = f(x) \qquad \ldots \qquad (*)$$

The Method

(1) First get the two linearly independent complementary functions (somehow or other). Call them $y_1(x)$ and $y_2(x)$.
(2) For the PI, we try a solution of the form

$$y = u(x)y_1(x) + v(x)y_2(x). \qquad \ldots \qquad (**)$$

Taking derivatives

$$y' = (u'y_1 + v'y_2) + (uy_1' + vy_2').$$

At this point, we *insist* that the first summand, $u'y_1 + v'y_2$, is zero. This gives us one equation:

$$u'y_1 + v'y_2 = 0 \qquad \ldots \qquad (1)$$

(We are so confident at this point, that, not only do we want to find a solution of the form (**), but we hope to find a "nice" u and v so that (1) also holds!)

So now,

$$\begin{aligned} y &= uy_1 + vy_2 \\ y' &= uy_1' + vy_2' \\ \text{and } y'' &= u'y_1' + v'y_2' + uy_1'' + vy_2''. \end{aligned}$$

Substituting and collecting terms gives

$$u\underbrace{[y_1'' + py_1' + qy_1]}_{0} + v\underbrace{[y_2'' + py_2' + qy_2]}_{0} + u'y_1' + v'y_2' = f(x)$$

giving

$$u'y_1' + v'y_2' = f(x)$$

as our second equation. So we must solve the system

$$\begin{aligned} u'y_1 + v'y_2 &= 0 \\ u'y_1' + v'y_2' &= f(x). \end{aligned}$$

This will have a unique solution if

$$\det \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \text{ is never zero.}$$

But this is just the Wronskian of two independent functions!!

Summary of Variation of Parameters Method *(In Four Very Easy Steps)*

(1) Get the CFs y_1 and y_2.
(2) Solve the system

$$\begin{aligned} u'y_1 + v'y_2 &= 0 \\ u'y_1' + v'y_2' &= f(x). \end{aligned}$$

for u' and v'.
(3) Integrate u' and v' to get u and v.
(4) The PI is then $yp = uy_1 + vy_2$.

Warning Make sure that the original DE is in the form (*) before you begin, so that the coefficient of y'' is 1. (Else you'll get the wrong $f(x)$.)

Example 9.1. Determine the GS of $y'' + y = \sec x; (0 < x < \pi/2)$.

10. Mechanical Vibrations

S'pose a mass m is suspended from a spring, stretching it a length ΔL. There are then two forces acting on the spring: gravity (magnitude mg, directed downward; $g = 9.8\text{m/sec}^2$ or 32ft/sec^2) and an elastic force (magnitude $k\Delta L$. This is **Hooke's Law.** k is the **elastic constant.**) When the spring is in equilibrium, these forces are equal:

$$mg = k\Delta L.$$

This gives a little formula for the calculation of the elastic constant of any spring:

$$\boxed{k = \frac{mg}{\Delta L} \qquad \textbf{Calculation of elastic constant } k}$$

Notes

(1) mg is the **weight** (measured in newtons (mks) or pounds (eng)).
(2) **Mass** m is measured in kilograms (mks) or slugs (eng) and equals weight$/g$.

Now, what if an external varying downward force $F(t)$ is added, and the spring is displaced a further distance u from its equilibrium position? We denote its vertical position (measured downwards) by u. We take $u = 0$ at the equilibrium position. Then the mass experiences four forces:

(1) An elastic force $-k(u + \Delta L)$, directed upwards (up is negative)
(2) A damping force $-c\dot{u}$, proportional to the velocity, directed against the direction of velocity. ($c =$ **damping constant**)
(3) Gravity mg, directed downwards
(4) The external force $F(t)$

Adding these up gives the net (downward) force:

$$\begin{aligned} & mg - k(u + \Delta L) - c\dot{u} + F(t). \\ & = mg - ku - mg - c\dot{u} + F(t) \qquad \text{(using } k\Delta L = mg) \\ & = -ku - c\dot{u} + F(t) \end{aligned}$$

Newton's law now tells us that this net downward force equals mass $\times$ acceleration, $m\ddot{u}$. Thus,

$$m\ddot{u} = -ku - c\dot{u} + F(t)$$

giving

$$\boxed{m\ddot{u} + c\dot{u} + ku = F(t), \qquad \textbf{Motion of a stretched spring}}$$

which—surprise surprise—is a second order linear DE (with constant coefficients).

Notes

(1) This formula says that we don't have to take gravity into account. What happened to gravity? Put another way, springs on Mars behave just like springs on Earth (except for the equilibrium position..)
(2) m, c and k are all *nonnegative* constants.

We intend to solve the equation as generally as possible. This entails first finding the CF. In the videos, we solve for the CF in four cases:

A. Undamped motion $(c = 0)$
We obtain the GS

$$\boxed{u = R\cos(\omega_0 t + \delta), \ \textbf{where} \ \ \omega_o = \sqrt{\frac{k}{m}} \qquad \textbf{Simple harmonic motion}}$$

Examples 10.1. of Simple Harmonic Motion

A. Find the period of oscillation (that is, the time it takes for one complete cycle) and also the motion of a spring-mass system if $m = 10$ slugs (lb sec^2/ft), $k = 5$, given that it starts at rest, stretched 2 inches below the equilibrium position.
B. A mass of weight 10 lb stretches a steel spring 2 inches. Find the period of oscillation.
C. The end of the spring in Part (B) spring is stretched 2" below its equilibrium position and then released. Determine the subsequent motion.

B. Damped motion $(c \neq 0)$

Case I: Overdamped $(c^2 - 4mk > 0)$ We obtain the GS

$$\boxed{\begin{array}{c} u = Ae^{r_1 t} + Be^{r_2 t}, \ \textbf{where} \ \ r_i = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m} \\ \textbf{Overdamped harmonic motion} \end{array}}$$

Case II: Underdamped $(c^2 - 4mk < 0)$ We obtain the GS

$$\boxed{\begin{array}{c} u = Ee^{-\frac{c}{2m}t}\cos(\mu t + \delta), \ \textbf{where} \ \ \mu = \dfrac{1}{2m}\sqrt{4mk - c^2} \\ \textbf{Underdamped harmonic motion} \end{array}}$$

Case III: Critically damped $(c^2 - 4mk = 0)$ We obtain the GS

$$u = (A + Bt)e^{-\frac{c}{2m}t} \qquad \textbf{Critically damped harmonic motion}$$

Examples 10.2. of Damped Harmonic Motion

A. A 32 lb weight stretches a spring 2 feet. The weight is then pulled down an additional 6" and released. If the resistance of the medium is 4 lb/(ft/sec), find the subsequent motion, sketching its graph.

B. What value of c should a shock absorber provide to stop all vibrations of a car spring that can be compressed an inch by a 150 lb force?

Forced Vibrations

Here we consider a forcing function $F(t) = F_0 \cos \omega t$, and consider the undamped case ($c = 0$). Thus the equation is:

$$m\ddot{u} + ku = F_0 \cos \omega t.$$

Let $\omega_0 = \sqrt{\frac{k}{m}}$, the angular frequency in the unforced case.

Case 1. $\omega \neq \omega_0$ We obtain the solution

$$u = A \cos \omega_0 t + B \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t.$$

Case 2. $\omega = \omega_0$ We obtain the solution

$$u = A \cos \omega_0 t + B \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \cos \omega_0 t.$$

Note that here, $u \to +\infty$ as $t \to +\infty$ (resonance).

11. Laplace Transforms

Laplace transforms are a useful tool in solving DEs. In real life, it often happens that the "forcing" function (the right-hand side of a nonhomogeneous DE) is either discontinuous, non-differentiable, or impulsive. Such functions can easily be dealt with using Laplace Transforms. Furthermore, Laplace Transforms can be used to convert systems of differential equations into systems of linear algebraic equations (but more later?) First, we do some theory.

Definition 11.1. Let $f(t)$ be defined for $t \geq 0$. Then the **Laplace Transform (LT)** of f is the function $F = \mathcal{L}[f]$ of given by the formula

$$\boxed{F(s) = \int_0^{+\infty} e^{-st} f(t) dt,}$$

provided the integral exists.

Notes

(1) The natural domain of this function (of s) depends on the values of s for which the integral is defined.
(2) The domain of the original function must include $[0, +\infty)$, or else the integral is not defined. (We won't worry about cases when it is improper at 0).
(3) The defining integral is is an *improper integral* because of the upper limit, and it may well diverge. Whether it diverges or converges depends on the original function f.

Examples 11.2.

A. If $f(t) = 1$, then $F(s) = \dfrac{1}{s}$.
B. If $f(t) = e^{at}$, then $F(s) = \dfrac{1}{s-a}$ provided $s > a$.
C. If $f(t) = t$, then $F(s) = \dfrac{1}{s^2}$.
D. $f(t) = t^2$
E. $f(t) = t^n$
F. $f(t) = \cos bt$
G. $f(t) = \sin bt$
H. $f(t) = u_c(t) = \frac{1}{2}\left(1 + \frac{(t-c)}{|t-c|}\right)$
I. Linearity of LT (Proof in exercise set) The LT of a sum is the sum of the LTs, and the LT of a constant times a function is that constant times the LT of the function.

We now look to see when we can be guaranteed that the Laplace transform exists. First we need:

Theorem 11.3. ***Comparison Test*** *Let f and g be two functions that are integrable on the interval $[M, A]$ for every $A > M$ S'pose that, for $t \geq M$, $|f(t)| \leq g(t)$, and that $\int_M^{+\infty} g(t)dt$ converges. Then so does $\int_M^{+\infty} f(t)dt$.*

Proof. We prove the simpler case in which $f(t) \geq 0$ for all $t \geq M$. The integral $\int_M^{+\infty} f(t)dt$ is defined as $\lim_{A\to+\infty} \int_M^A f(t)dt$, so it suffices to prove that the limit exists (and is finite). Now , since $f(t) \geq 0$, the integral $\int_M^A f(t)dt$ is an increasing function of A (because its derivative with respect to A is $f(A) \geq 0$) and so one of two things can happen: either it is **unbounded,** in which case it approaches $+\infty$ as $A \to +\infty$, or it is **bounded,** in which case it approaches a finite limit L as $A \to +\infty$ (and hence converges). The same is true for $\int_M^A g(t)dt$. But since we are told that $\int_M^{+\infty} g(t)dt$ converges, $\int_M^A g(t)dt$ must be bounded. Also, since $f(t) \leq g(t)$ for all $t \geq M$, we have

$$\int_M^A f(t)dt \leq \int_M^A g(t)dt,$$

and so $\int_M^A f(t)dt$ too must be bounded, and hence convergent. □

Definition 11.4. The function f has a **jump discontinuity at** a if $\lim_{x\to a-} f(x)$ and $\lim_{x\to a+} f(x)$ both exist (and are finite). f is **piecewise continuous** if it has at most a finite number of jump discontinuities and is continuous everywhere else.

Fact: Piecewise continuous functions are Riemann integrable.

Definition 11.5. The function $f(t)$ has **exponential order** e^{at} if there exist positive numbers M and K such that $|f(t)| \leq Ke^{at}$ for $t \geq M$.

Theorem 11.6. ***Existence of the Laplace Transform*** *S'pose that:*

(i) f is piecewise continuous on every interval $[0, A]$ (ie: it has only finitely many discontinuities in each finite interval, but may have infinitely many on $[0, +\infty]$)

(ii) $f(t)$ has exponential order e^{at}.

Then $F(s)$ exists for $s > a$.

Proof. Because f is piecewise continuous, so is $e^{-st}f(t)$ and hence the integral

$$\int_0^M e^{-st}f(t)dt$$

exists for every $M > 0$. As f has exponential order,

$$|f(t)e^{-st}| \leq e^{at}e^{-st} = e^{(a-s)t}$$

from some value of t onwards. also, $\int_0^{+\infty} e^{(a-s)t}\,dt$ converges for $s > a$, whence, by the Comparison test, so does the integral $\int_0^{+\infty} e^{-st} f(t)dt$ that defines $F(s)$. $\square$

Further properties of the LT:

Proposition 11.7. *If $F(s)$ exists for $s > k$, then:*
(a) $\mathcal{L}[e^{at} f(t)](s)$ exists for $s > a + k$ and

$$\mathcal{L}[e^{at} f(t)](s) = F(s - a) \qquad \textbf{\textit{Translation Rule}}$$

(b) $\mathcal{L}[t^n f(t)](s)$ exists for $s > k$ and

$$\mathcal{L}[t^n f(t)](s) = (-1)^n \frac{d^n}{ds^n} F(s) \qquad \textbf{\textit{Derivative of LT}}$$

(c) $\mathcal{L}[f'](s)$ exists for $s > k$ and

$$\mathcal{L}[f'](s) = sF(s) - f(0) \qquad \textbf{\textit{LT of Derivative}}$$

(d) $\mathcal{L}[f''](s)$ exists for $s > k$ and

$$\mathcal{L}[f''](s) = s^2 F(s) - sf(0) - f'(0) \qquad \textbf{\textit{LT of 2nd Derivative}}$$

(e) $\mathcal{L}[f^{(n)}](s)$ exists for $s > k$ and

$$\mathcal{L}[f^{(n)}](s) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{(n-1)}(0)$$

$f(t)$	$F(s)$	$f(t)$	$F(s)$
1	$\dfrac{1}{s}$	$e^{at}f(t)$	$F(s-a)$
t	$\dfrac{1}{s^2}$	$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}$
t^n	$\dfrac{n!}{s^{n+1}}$	$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}$
e^{at}	$\dfrac{1}{s-a}$	$u_c(t)$	$\dfrac{e^{-sc}}{s}$
$\cos bt$	$\dfrac{s}{s^2+b^2}$	$u_c(t)f(t-c)$	$e^{-sc}F(s)$
$\sin bt$	$\dfrac{b}{s^2+b^2}$	$\delta(t-c)$	e^{-sc}
$\cosh bt$	$\dfrac{s}{s^2-b^2}$	y'	$sY-y(0)$
$\sinh bt$	$\dfrac{b}{s^2-b^2}$	y''	$s^2Y-sy(0)-y'(0)$
$t^n f(t)$	$(-1)^n\dfrac{d^n}{dx^n}F(s)$	$y^{(n)}$	$s^nY-s^{n-1}y(0)$ $-\cdots-y^{(n-1)}(0)$

Table 1. Laplace Transforms

12. Table of Laplace Transforms

13. Solving DEs with Laplace Transforms

We essentially learn to do this by practice:

Examples 13.1.
A. Solve $y'' - y' - 2y = 0$ subject to $y(0) = 1, y'(0) = 0$.
B. Solve $y'' + 3y' + 2y = e^{-3t}$ subject to $y(0) = 0, y'(0) = 1$.
C. Arbitrary boundary conditions:
Solve $y'' - y = 0$ subject to $y(0) = A, y'(0) = B$.

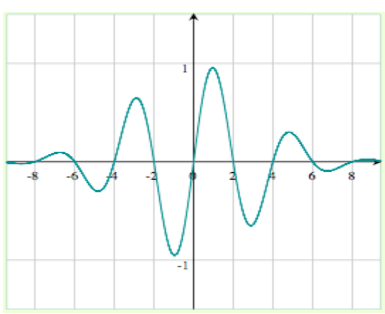
We next obtain some following additional formulas to use with more interesting types of DE:

Proposition 13.2.
(a) $\mathcal{L}[u_c(t)](s)$ *exists for* $s > 0$ *and*

$$\mathcal{L}[u_c(t)](s) = \frac{e^{-sc}}{s}.$$

More generally,
(b) If $F(s)$ *exists, then so does* $\mathcal{L}[u_c(t)f(t-c)](s)$ *and*

$$\mathcal{L}[u_c(t)f(t-c)](s) = e^{-sc}F(s). \qquad \textbf{\textit{Shifted function}}$$



$f(x) = \sin(\pi x/2)e^{-x^2/20}$

$u_2(x)f(x-2)$

The usefulness of these is seen in the following examples.

Examples 13.3. of multi-step functions in class, culminating in LT's of the following.
A. Square wave: $q(t) = u(t) - u_1(t) + u_2(t) - \cdots$
B. Sawtooth wave: $s(t) = t - u_1(t) - u_2(t) - \cdots$
C. Rectified sine wave: $r(t) = \sin t + 2u_\pi(t)\sin(t-\pi) + 2u_{2\pi}(t)\sin(t - 2\pi) + \cdots$
D. Triangular wave: $g(t) = t - 2(t-1)u_1(t) + 2(t-2)u_2(t) - \cdots$

Solving DE's with Discontinuous and Periodic Forcing Functions

Examples 13.4. A. On-Off forcing in a circuit: $y''+y = 1-u_\pi(t)$; $y(0) = y'(0) = 0$.

B. On-Off forcing with damping: $y''+2y'+2y = 1-u_\pi(t)$; $y(0) = y'(0) = 0$.
C. Square Wave Forcing: $y'' + y = q(t)$; $y(0) = y'(0) = 0$.

14. Review of Power Series

Definition 14.1. A **power series about** 0 is a series of the form

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k + \dots .$$

The c_k are called the **coefficients** of the series.

Examples 14.2.

A. If all the coefficients $c_k = 1$, then we have the geometric series $1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} x^k$ This series converges to $\frac{1}{1-x}$ for $|x| < 1$and diverges otherwise.

B. $e^x = 1 + 1 \cdot x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \quad = \quad \sum_{k=0}^{\infty} \frac{1}{k!}x^k$ has $c_k = \frac{1}{k!}$

C. $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots \quad = \quad \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$

D. $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots \quad = \quad \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$

Note that

$$\sum_{k=0}^{\infty} c_k x^k = \sum_{n=0}^{\infty} c_n x^n = \sum_{m=0}^{\infty} c_m x^m,$$

so, in $\sum_{k=0}^{\infty} c_k x^k$, the index k is sometimes called a "dummy index."

Definition 14.3. A **power series about the point** a is a series of the form

$$\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_k(x-a)^k + \dots .$$

Example 14.4. If $0 < x < 2$, then

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots \quad = \quad \sum_{k=1}^{\infty} \frac{(-1)^k (x-1)^k}{k}$$

Convergence of power series

Examples 14.5. For which x do the following power series converge?

A. $\sum_{k=0}^{\infty} k^2 x^k$

B. $\sum_{k=0}^{\infty} \frac{(x-3)^k}{k}$

Theorem 14.6 (Theorem on Power Series)**.**
Given any power series $\sum_{k=0}^{\infty} b_k(x-a)^k$ *, there is an associated* **interval of convergence** *centered at* $x = a$ *such that the series converges for values of* x *in that interval and diverges outside it. The interval can be any kind of interval whatsoever, but is always centered at* a*. The half-width of this interval is called the* **radius of convergence.**

The proof of the theorem hinges on the following lemma:

Lemma 14.7. *If* $\sum_{k=0}^{\infty} b_k(x-a)^k$ *converges for any* x *with* $|x-a| = r$*, then it converges absolutely for* every x *with* $|x-a| < r$*.*

To prove the lemma, use the fact that the terms in a convergent series approach zero and therefore are eventually smaller than 1:

$$|b_n||x-a|^n < 1,$$
$$\text{so that } |b_n| < \frac{1}{|x-a|^n}.$$

This inequality allows you to prove quickly that the series converges for all x with $|x-a| < r$.

Examples 14.8. Find the intervals of convergence and radius of convergence of the following power series:

A. $\displaystyle\sum_{k=0}^{\infty} k^2 x^k$

B. $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$

C. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k(x-3)^k}{k}$

D. $\displaystyle\sum_{k=0}^{\infty} \frac{(x+3)^k}{k^2 2^k}$

15. Series Solution Near an Ordinary Point

We are going to be solving a general second order linear DE of the form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0 \tag{1}$$

We shall assume that $P(x)$, $Q(x)$, and $R(x)$ are analytic functions (functions equal to their Taylor series near each point).

Definition 15.1. A point x_0 with $P(x_0) \neq 0$ is called an **ordinary point** of equation (1). Otherwise, x_0 is a **singular point.**

Note that P must remain non-zero in a small interval of an ordinary point, and so we can divide by $P(x)$ to get a DE of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \tag{2}$$

Our job will be to seek a solution of (1) of the form

$$y(x) = \sum_{n=0}^{\infty} a_m(x-x_0)^n = a_0 + a_1(x-x_0) + \cdots + a_n(x-x_0)^n + \cdots$$

Examples 15.2.

(A) Airy's equation $y'' - xy = 0$. Every point is an ordinary point. We expand this one about $x_0 = 0$.
(B) Airy's equation about $x_0 = 1$. First four terms only.

The following theorem guarantees the existence of nice series solutions near an ordinary point:

Theorem 15.3. *The DE (1) is guaranteed to have a series solution*

$$y = \sum_{n=0}^{\infty} a_m(x-x_0)^n = Ay_1(x) + By_2(x)$$

where $y_1(x)$ and $y_2(x)$ are linearly independent series solutions with radius of convergence at least as large as the minimum radius of convergence of $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$. □

16. Series Solution Near Regular Singular Points

Definition 16.1. A **regular singular point** for the DE

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

is a singular point x_0 such that:

$$\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)} \quad \text{and}$$
$$\lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)}$$

are both finite. In other words, when we divide by $P(x)$, the singularity in the coefficient of y' is "no worse than" $1/(x-x_0)$, while that of y is no worse than $1/(x-x_0)^2$. A singular point that is not regular is **irregular.**

Examples 16.2.

(A) $2x(x-2)^2y'' + 3xy' + (x-2)y = 0$ has a regular singular point at $x = 0$ and an irregular singular point at $x = 2$.

(B) $(x-\pi/2)^2y'' + (\cos x)y' + (\sin x)y = 0$ has a regular singular point at $x = \pi/2$. To see why, apply l'Hospital to the relevant limit.

(C) **The Euler equation** $x^2y'' + \alpha xy' + \beta y = 0$ has $x = 0$ as a regular singular point. You found its general solution in the homework some time back.

We now assume that our singular point happens to be $x = 0$. (If we have a singular point at x_0, then the substitution $t = x - x_0$ will transform the DE into one for with a singular point at 0. So, from now on, $x_0 = 0$.

Trying to find a power series solution for a DE with singular points will always lead to problems in determining the coefficients. Instead, we use a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{r+n}$$

Example 16.3. Solve $2x^2y'' - xy' + (1+x)y = 0$. When we equate the coefficient of x^r to zero we will find:

$$2r^2 - 3r + 1 = 0$$

which is called the **indicial equation.** When there are two roots that do not differ by an integer, we get two nice independent solutions.

When the roots of the indicial equation are equal of differ by an integer, we have the following theorem:

Theorem 16.4. *If a second order linear DE has $x = 0$ as a regular singular point, and if r_1 and r_2 are the roots of the indicial equation, then:*

(1) If $r_1 = r_2$*, a second solution is given by*

$$y_2(x) = y_1(x) \ln |x| + |x|^{r_1} \sum_{n=1}^{\infty} a_n x^n$$

in each of the intervals $(-\rho, 0)$ *and* $(0, \rho)$ *for some* ρ*.*

(2) If $r_1 - r_2 = N$*, a positive integer, then*

$$y_2(x) = a_1 y_1(x) \ln |x| + |x|^{r_2} \left[1 + \sum_{n=1}^{\infty} a_n x^n \right]$$

(3) If r_i *are complex of the form* $a \pm ib$*, then*

$$y_1(x) = |x|^a \cos(b \ln |x|) \left[1 + \sum_{n=1}^{\infty} a_n x^n \right]$$

$$y_2(x) = |x|^a \sin(b \ln |x|) \left[1 + \sum_{n=1}^{\infty} a_n x^n \right]$$

□