Supplementary Chapters to Accompany

Applied Calculus (*2nd. Ed.*)

by

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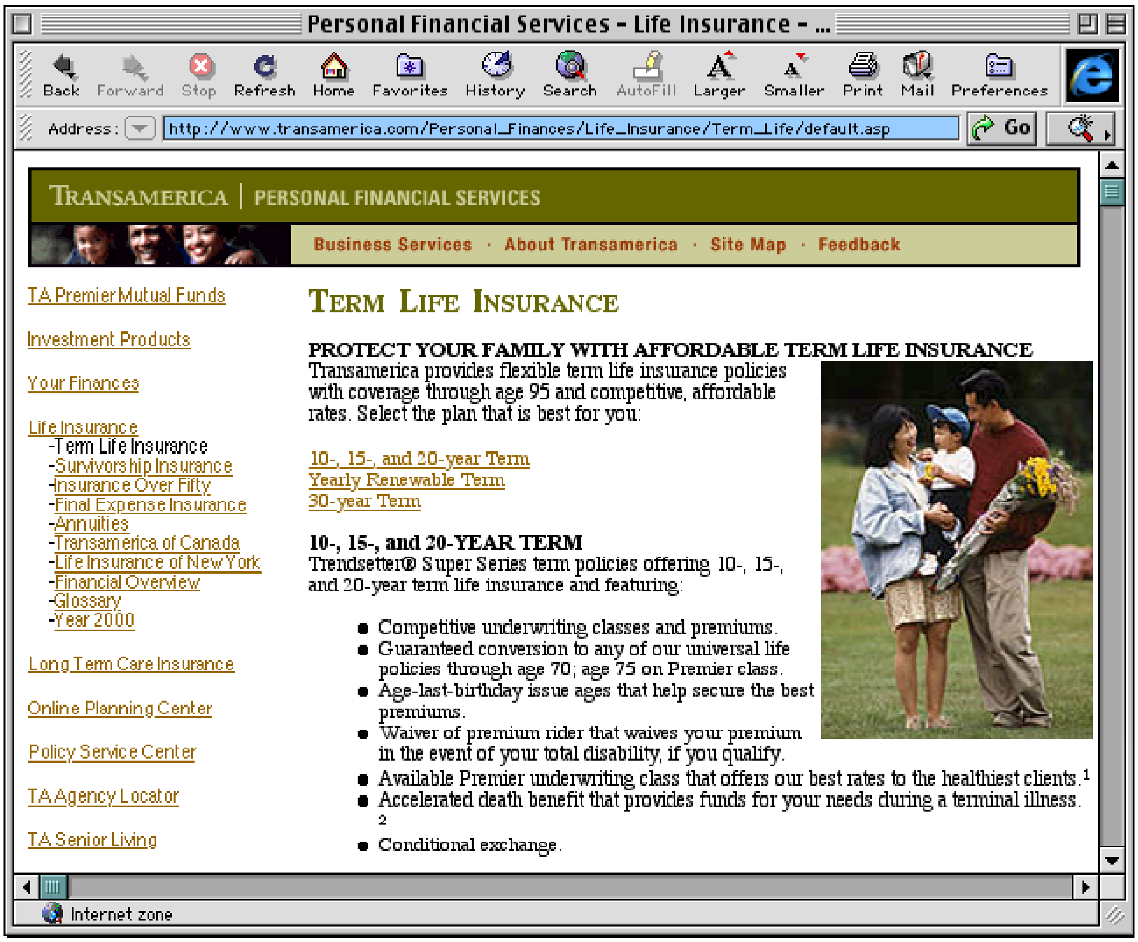
✣ Chapter P—Calculus Applied to Probability and Statistics

P.1 Continuous Random Variables and Histograms
P.2 Probability Density Functions: Uniform, Exponential, Normal, and Beta
P.3 Mean, Median, Variance, and Standard Deviation

You're the Expert —Creating a Family Trust



You are a financial planning consultant at a neighborhood bank. A 22-year-old client asks you the following question: "I would like to set up my own insurance policy by opening a trust account into which I can make monthly payments starting now, so that upon my death or my ninety-fifth birthday—whichever comes sooner—the trust can be expected to be worth \$500,000. How much should I invest each month?" Assuming a 5% rate of return on investments, how should you respond?

Introduction

To answer the question on the previous page, we must know something about the probability of the client's dying at various ages. There are so many possible ages to consider (particularly since we should consider the possibilities month by month) that it would be easier to treat his age at death as a *continuous* variable, one that can take on any real value (between 22 and 95 in this case). The mathematics needed to do probability and statistics with continuous variables is calculus.

The material on statistics in this chapter is accessible to any reader with a “common-sense” knowledge of probability, but it also supplements any previous study you may have made of probability and statistics without using calculus.

P.1 Continuous Random Variables and Histograms

Suppose that you have purchased stock in Colossal Conglomerate, Inc., and each day you note the closing price of the stock. The result each day is a real number X (the closing price of the stock) in the unbounded interval $[0, +\infty)$. Or, suppose that you time several people running a 50-meter dash. The result for each runner is a real number X, the race time in seconds. In both cases, the value of X is somewhat random. Moreover, X can take on essentially any real value in some interval, rather than, say, just integer values. For this reason we refer to X as a **continuous random variable**. Here is the formal definition.

Continuous Random Variable

A **random variable** is a function X that assigns to each possible outcome in an experiment a real number. If X may assume any value in some given interval I (the interval may be bounded or unbounded), it is called a **continuous** random variable. If it can assume only a number of separated values, it is called a **discrete** random variable.

Quick Examples

1. Roll a die and take X to be the number on the uppermost face. Then X is a discrete random variable with possible values 1, 2, 3, 4, 5 and 6.
2. Locate a star in the cosmos and take X to be its distance from the solar system in light years. Then X is a continuous random variable whose values are real numbers in the interval $(0, +\infty)$.
3. Open the business section of your newspaper and take X to be the closing price of Colossal Conglomerate stock. Then X can take on essentially any positive real value, so we can think of X as a continuous random variable.

If X is a random variable, we are usually interested in the **probability** that X takes on a value in a certain range. For instance, if X is the closing price of Colossal Conglomerate stock and we find that 60% of the time the price is between \$10 and \$20, we would say

The probability that X is between \$10 and \$20 is 0.6.

We can write this statement mathematically as follows.

$P(10 \le X \le 20) = 0.6$ The probability that $10 \le X \le 20$ is 0.6

We can use a bar chart, called a **probability distribution histogram,** to display the probabilities that X lies in selected ranges. This is shown in the following example.

Example 1 ***College Population by Age***

The following table shows the distribution of US residents (16 years old and over) attending college in 1980 according to age.[1]

Age (years)	15-19	20-24	25-29	30-34	35-?
Number in 1980 (millions)	2.7	4.8	1.9	1.2	1.8

Draw the probability distribution histogram for X = the age of a randomly chosen college student.

Solution Summing the entries in the bottom row, we see that the total number of students in 1980 was 12.4 million. We can therefore convert all the data in the table to probabilities by dividing by this total.

***X* = Age (years)**	15-19	20-24	25-29	30-34	35-?
Probability	0.22	0.39	0.15	0.10	0.15

The probabilities in the above table have been rounded, with the consequence that they add to 1.01 instead of the expected 1. In the category 15–19, we have actually included anyone at least 15 years old and less than 20 years old. For example, someone $19\frac{1}{2}$ years old would be in this range. We would like to write 15–20 instead, but this would be ambiguous, since we would not know where to count someone who was exactly 20 years old. However, the probability that a college student is *exactly* 20 years old (and not, say, 20 years and 1 second) is essentially 0, so it doesn't matter.[2] We can therefore rewrite the table with the following ranges.

***X* = Age (years)**	15-20	20-25	25-30	30-35	≥ 35
Probability	0.22	0.39	0.15	0.10	0.15

The table tells us that, for instance,

$$P(15 \le X \le 20) = 0.22$$

and

$$P(X \ge 35) = 0.15.$$

The probability distribution histogram is the bar graph we get from these data (Figure 1).

[1] Source: 1980 Census of Population, US Department of Commerce/Bureau of the Census.
[2] Also see the discussion after Example 2 below.

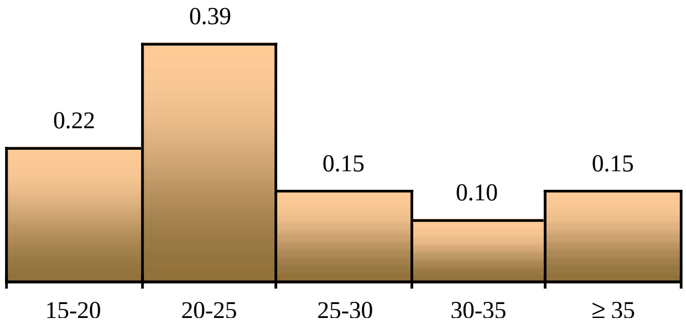


Figure 1

Before we go on... Had the grouping into ranges been finer—for instance into divisions of one year instead of five, then the histogram would appear smoother, and with lower bars (why?) (Figure 2).

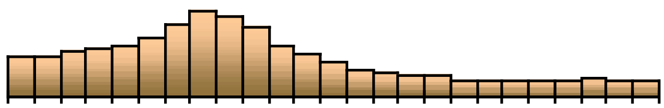


Figure 2

This smoother looking distribution suggests a smooth curve. It is this kind of curve that we shall be studying in the next section.

Example 2 ***Age of a Rented Car***

A survey finds the following probability distribution for the age of a rented car.[1]

Age (years)	0-1	1-2	2-3	3-4	4-5	5-6	6-7
Probability	0.20	0.28	0.20	0.15	0.10	0.05	0.02

Plot the associated probability distribution histogram, and use it to evaluate (or estimate) the following:

(a) $P(0 \le X \le 4)$ **(b)** $P(X \ge 4)$
(c) $P(2 \le X \le 3.5)$ **(d)** $P(X = 4)$

Solution The histogram is shown in Figure 3.

[1] As in the preceeding example we allow the brackets to intersect. However, since the probability that a car is *exactly* 1 or 2 or 3 or ... years old (to a fraction of a second) is essentially zero, we can ignore the apparent overlap. The discussion at the end of this example further clarifies this point.

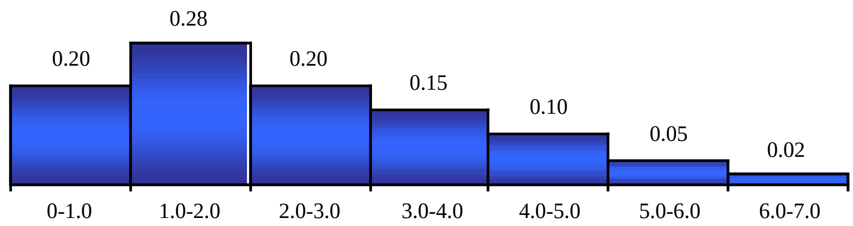


Figure 3

(a) We can calculate $P(0 \leq X \leq 4)$ from the table by adding the corresponding probabilities:

$$P(0 \leq X \leq 4) = 0.20 + 0.28 + 0.20 + 0.15 = 0.83$$

This corresponds to the shaded region of the histogram shown in Figure 4.

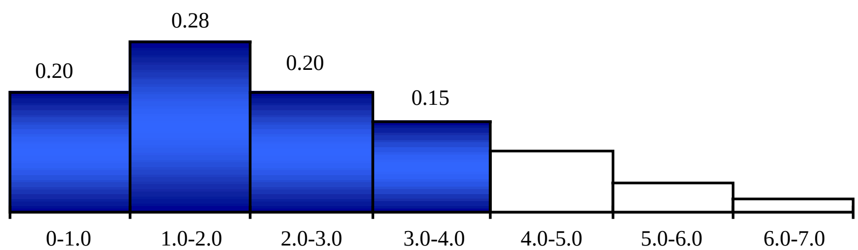


Figure 4

Notice that since each rectangle has width equal to 1 unit and height equal to the associated probability, its *area* is equal to the probability that X is in the associated range. Thus $P(0 \leq X \leq 4)$ is also equal to the area of the shaded region.

(b) Similarly, $P(X \geq 4)$ is given by the area of the *unshaded* portion of Figure 4, so

$$P(X \geq 4) = 0.10 + 0.05 + 0.02 = 0.17.$$

(Notice that $P(0 \leq X \leq 4) + P(X \geq 4) = 1$. Why?)

(c) To calculate $P(2 \leq X \leq 3.5)$, we need to make an educated guess, since neither the table nor the histogram has subdivisions of width 0.5. Referring to the graph, we can approximate the probability by the shaded area shown in Figure 5.

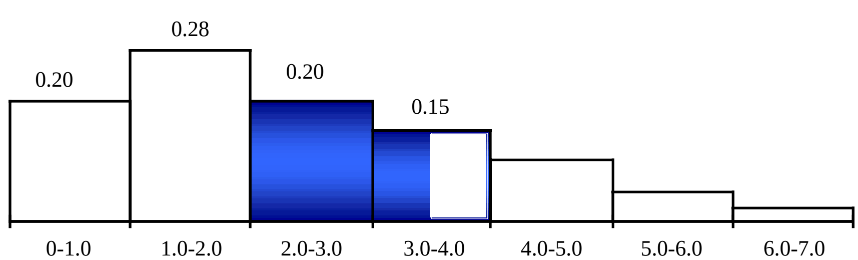


Figure 5

Thus,

$$P(2 \le X \le 3.5) \approx 0.20 + \frac{1}{2}(0.15) = 0.275.$$

(d) To calculate $P(X = 4)$, we would need to calculate $P(4 \le X \le 4)$. But this would correspond to a region of the histogram with zero area (Figure 6), so we conclude that $P(X = 4) = 0$.

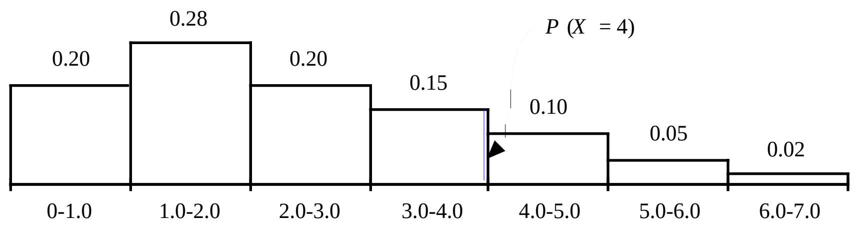


Figure 6

Question In the above example $P(X = 4)$ was zero. Is it true that $P(X = a)$ is zero for every number a in the interval associated with X?

Answer As a general rule, yes. If X is a *continuous* random variable, then X can assume infinitely many values, and so it is reasonable that the probability of its assuming any specific value we choose beforehand is zero.

Caution

If you wish to use a histogram to calculate probability as *area*, make sure that the subdivisions for X have width 1—for instance, $1 \le X \le 2$, $2 \le X \le 3$, and so on.

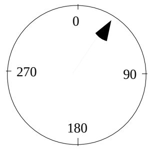
The histogram in Example 1 (Figure 1) had bars corresponding to larger ranges for X. The first bar has a width of 5 units, so its area is 5×0.22, which is 5 times the probability that $15 \le X \le 20$. If you wish to use a histogram to give probability as area, divide the area by the width of the intervals.

There is another way around this problem that we shall not use, but which is used by working statisticians: Draw your histograms so that the heights are not necessarily the probabilities but are chosen so that the *area* of each bar gives the corresponding probability. This is necessary if, for example, the bars do not all have the same width.

9.1 Exercises

In Exercises 1–10, identify the random variable (for example, "X is the price of rutabagas"), say whether it is continuous or discrete, and if continuous, give its interval of possible values.

1. A die is cast and the number that appears facing up is recorded.
2. A die is cast and the time it takes for the die to become still is recorded.
3. A dial is spun, and the angle the pointer makes with the vertical is noted. (See the figure.)



4. A dial is spun, and the quadrant in which the pointer comes to rest is noted.
5. The temperature is recorded at midday.

6. The US Balance of Payments is recorded (fractions of a dollar permitted).
7. The US Balance of Payments is recorded, rounded to the nearest billion dollars.
8. The time it takes a new company to become profitable is recorded.
9. In each batch of 100 computer chips manufactured, the number that fail to work is recorded.
10. The time it takes a TV set to break down after sale is recorded.

In Exercises 11–14, sketch the probability distribution histogram of the given continuous random variable.

11.

***X* = height of a jet fighter (ft.)**	0-20,000	20,000-30,000	30,000-40,000	40,000-50,000	50,000-60,000
Probability	0.1	0.2	0.3	0.3	0.1

12.

***X* = time to next eruption of a volcano (yrs.)**	0-2,000	2,000-3,000	3,000-4,000	4,000-5,000	5,000-6,000
Probability	0.1	0.3	0.3	0.2	0.1

13.

***X* = average temperature (˚F)**	0-50	50-60	60-70	70-80	80-90
Number of Cities	4	7	2	5	2

14.

***X* = Cost of a used car ($)**	0-2,000	2,000-4,000	4,000-6,000	6,000-8,000	8,000-10,000
Number of Cars	200	500	800	500	500

Applications

15. ***Farm Population, Female*** The following table shows the number of females residing on US farms in 1990, broken down by age.[1] Numbers are in thousands.

Age	0-15	15-25	25-35	35-45	45-55	55-65	65-75	75-95
Number	459	265	247	319	291	291	212	126

Construct the associated probability distribution (with probabilities rounded to four decimal places) and use the distribution to compute the following.

(a) $P(15 \le X \le 55)$ **(b)** $P(X \le 45)$ **(c)** $P(X \ge 45)$

[1] Source: Economic Research Service, US Department of Agriculture and Bureau of the Census, US Department of Commerce, 1990.

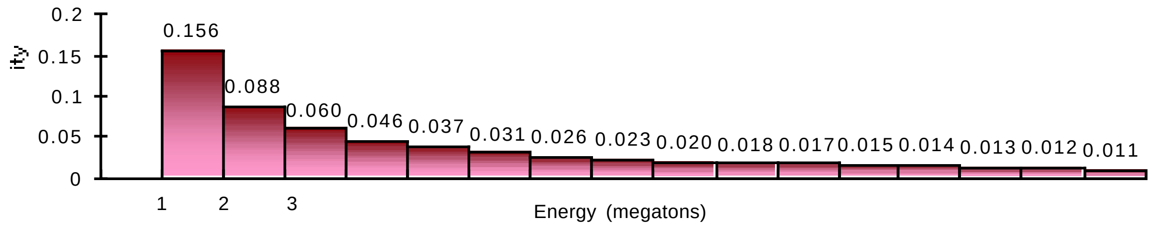
16. ***Farm Population, Male*** The following table shows the number of males residing on US farms in 1990, broken down by age.[1] Numbers are in thousands.

Age	0-15	15-25	25-35	35-45	45-55	55-65	65-75	75-95
Number	480	324	285	314	302	314	247	118

Construct the associated probability distribution (with probabilities rounded to four decimal places) and use the distribution to compute the following.

(a) $P(25 \le X \le 65)$ **(b)** $P(X \le 15)$ **(c)** $P(X \ge 15)$

17. ***Meteors*** The following histogram shows part of the probability distribution of the size (in megatons of released energy) of large meteors that hit the earth's atmosphere. (A large meteor is one that releases at least one megaton of energy, equivalent to the energy released by a small nuclear bomb.)[2]



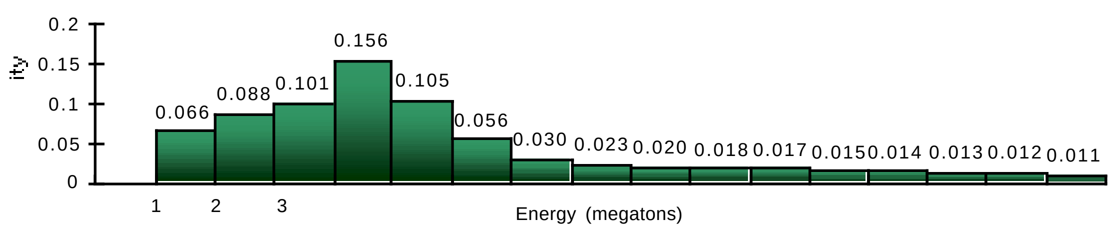
Calculate or estimate the following probabilities.

(a) That a large meteor hitting the earth's atmosphere will release between 1 and 4 megatons of energy.

(b) That a large meteor hitting the earth's atmosphere will release between 3 and 4.5 megatons of energy.

(c) That a large meteor will release at least 5 megatons of energy.

18. ***Meteors*** Repeat the preceding exercise using the following histogram for meteor impacts on the planet Zor in the Cygnus III system in Andromeda.



[1] *Ibid.*

[2] The authors' model, based on data released by NASA International Near-Earth-Object Detection Workshop/*The New York Times*, January 25, 1994, p. C1.

19. ***Quality Control*** An automobile parts manufacturer makes heavy-duty axles with a cross-section radius of 2.3 cm. In order for one of its axles to meet the accuracy standard demanded by the customer, the radius of the cross section cannot be off by more than 0.02 cm. Construct a histogram with X = the measured radius of an axle, using categories of width 0.01 cm, so that all of the following conditions are met.
(a) X lies in the interval $[2.26, 2.34]$.
(b) 80% of the axles have a cross-sectional radius between 2.29 and 2.31.
(c) 10% of the axles are rejected.

20. ***Damage Control*** As a campaign manager for a presidential candidate who always seems to be getting himself into embarrassing situations, you have decided to conduct a statistical analysis of the number of times per week he makes a blunder. Construct a histogram with X = the number of times he blunders in a week, using categories of width 1 unit, so that all of the following conditions are met.
(a) X lies in the interval $[0, 10]$.
(b) During a given week, there is an 80% chance that he will make 3 to 5 blunders.
(c) Never a week goes by when he doesn't make at least one blunder .
(d) On occasion, he has made 10 blunders in one week.

Communication and Reasoning Exercises

21. How is a random variable related to the outcomes in an experiment?

22. Give an example of an experiment and two associated continuous random variables.

23. You are given a probability distribution histogram with the bars having a width of 2 units. How is the probability $P(a \leq X \leq b)$ related to the area of the corresponding portion of the histogram?

24. You are given a probability distribution histogram with the bars having a width of 1 unit, and you wish to convert it into one with bars of width 2 units. How would you go about this?

P.2 Probability Density Functions: Uniform, Exponential, Normal, and Beta

We have seen that a histogram is a convenient way to picture the probability distribution associated with a continuous random variable X and that if we use subdivisions of 1 unit, the probability $P(c \le X \le d)$ is given by the area under the histogram between $X = c$ and $X = d$. But we have also seen that it is difficult to calculate probabilities for ranges of X that are not a whole number of subdivisions. The following example—based on an example in the previous section—introduces the solution to this problem.

Example 1 ***Car Rentals***

A survey finds the following probability distribution for the age of a rented car.

***X* = Age (years)**	0-1.0	1.0-2.0	2.0-3.0	3.0-4.0	4.0-5.0	5.0-6.0	6.0-7.0
Probability	0.20	0.28	0.20	0.15	0.10	0.05	0.02

The histogram of this distribution is shown in Figure 1a, and it suggests a curve something like the one given in Figure 1b.[1]

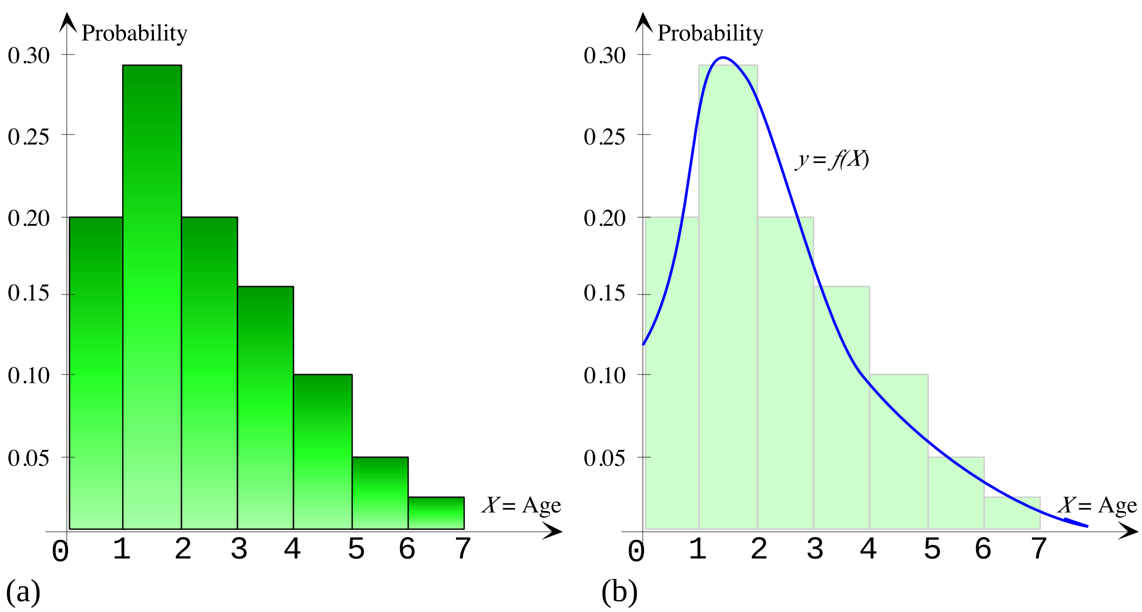


Figure 1

This curve is the graph of some function f, which we call a **probability density function**. We take the domain of f to be $[0,+\infty)$, since this is the possible range of values X can take (in

[1] There are many similarly shaped curves suggested by the bar graph. The question of finding the most appropriate curve is one we shall be considering below.

principle). In general, a probability distribution function will have some (possibly unbounded) interval as its domain.

Suppose now that as in Section P.1, we wanted to calculate the probability that a rented car is between 0 and 4 years old. Referring to the table, we find

$$P(0 \le X \le 4) = 0.20 + 0.28 + 0.20 + 0.15 = 0.83.$$

Referring to Figure 2, we can obtain the same result by adding the areas of the corresponding bars, since each bar has a width of 1 unit. Ideally, our probability density curve should have the property that the area under it for $0 \le X \le 4$ is the same, that is,

$$P(0 \le X \le 4) = \int_0^4 f(x)\ dx = 0.83.$$

(This area is shown in Figure 2 as well.)

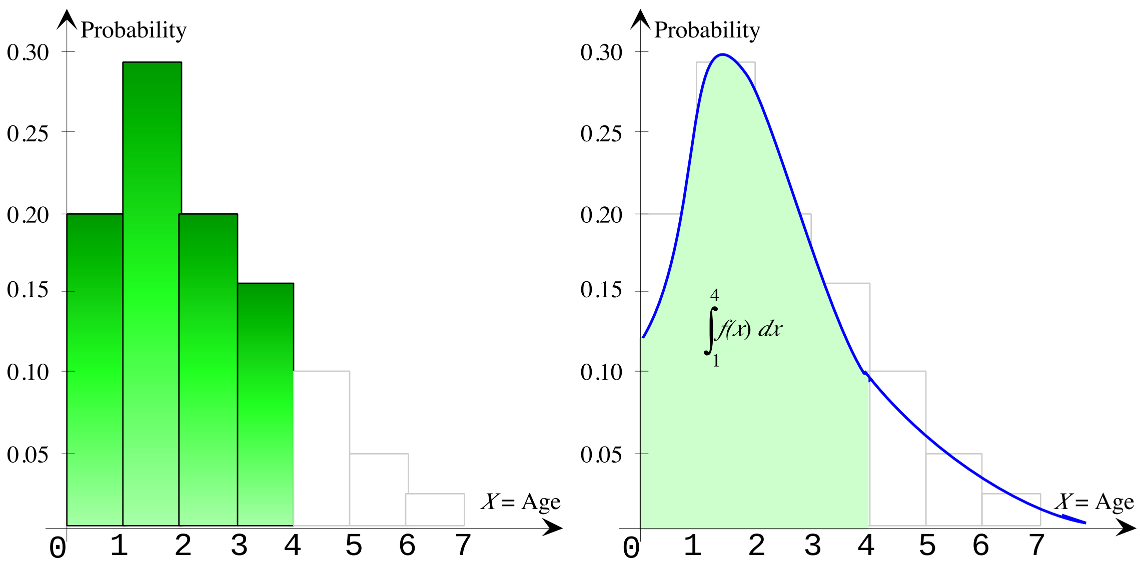


Figure 2

Now what happens if we want to find $P(2 \le X \le 3.5)$? In the previous section we estimated this by taking half of the rectangle between 3 and 4 (see Figure 3).

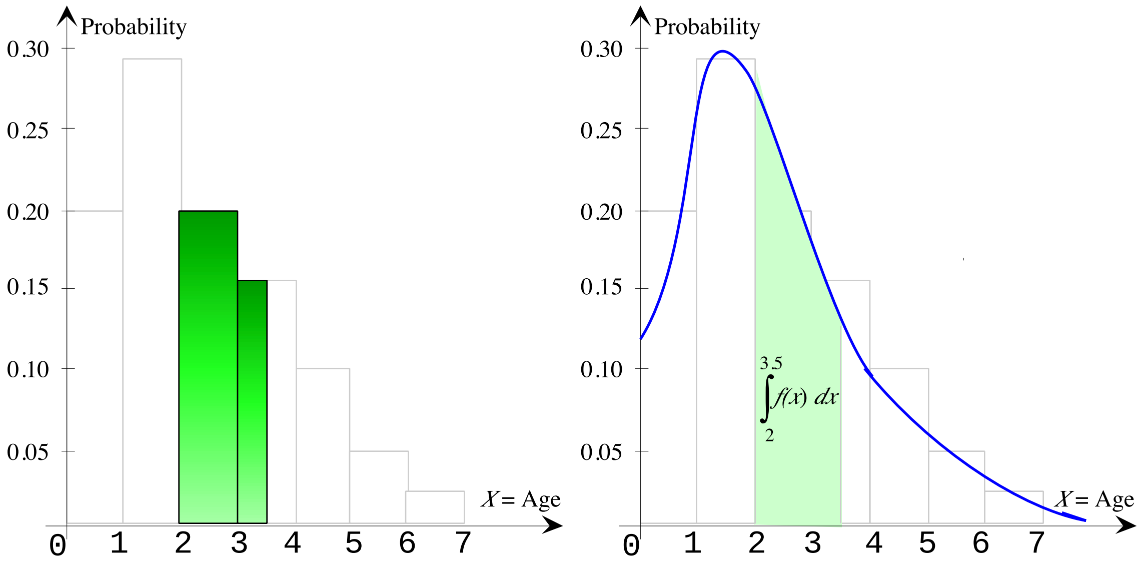


Figure 3

Instead, we could use the definite integral

$$P(2 \leq X \leq 3.5) = \int_2^{3.5} f(x)\, dx .$$

Before we go on... Although we haven't given you a formula for $f(x)$, we would *like* $f(x)$ to behave as described above. Here is something else we would like: Since a car has probability 1 of having an age between 0 and $+\infty$, we want

$$P(0 \leq X < +\infty) = \int_0^{+\infty} f(x)\, dx = 1.$$

The above example motivates the following.

Probability Density Function

A **probability density function** (or **probability distribution function**) is a function f defined on an interval (a, b) and having the following properties.

(a) $f(x) \geq 0$ for every x

(b) $\int_a^b f(x)\, dx = 1$

We allow a, b, or both to be infinite, as in the above example. This would make the integral in (b) an improper one.

Probability Associated with a Continuous Random Variable

A continuous random variable X is specified by a probability density function f. The probability $P(c \le X \le d)$ is specified by[1]

$$P(c \le X \le d) = \int_c^d f(x)\, dx\,.$$

Quick Example

Let $f(x) = \frac{2}{x^2}$ on the interval $[a, b] = [1, 2]$. Then property (a) holds, since $\frac{2}{x^2}$ is positive on the interval [1, 2]. For property (b),

$$\int_a^b f(x)\, dx = \int_1^2 \frac{2}{x^2}\, dx = \left[-\frac{2}{x}\right]_1^2 = -1 + 2 = 1.$$

If X is specified by this probability density function, then

$$P(1.5 \le x \le 2) = \int_{1.5}^2 \frac{2}{x^2}\, dx = \frac{1}{3}\,.$$

Note If X is specified by a probability density function f, then

$$P(X = c) = P(c \le X \le c) = \int_c^c f(x)\, dx = 0,$$

showing once again that there is a zero probability that X will assume any specified value.

Uniform Density Function

A **uniform density function** f is a density function that is constant, making it the simplest kind of density function. Since we require $f(x) = k$ for some constant k, requirement (b) in the definition of a probability density function tells us that

$$1 = \int_a^b f(x)\, dx = \int_a^b k\, dx = k(b-a).$$

Thus we must have

[1] This is not the most general situation. The most general definition of a random variable replaces the function f with an object known as a *probability measure*, but we shall not attempt to use this.

$$k = \frac{1}{b-a}.$$

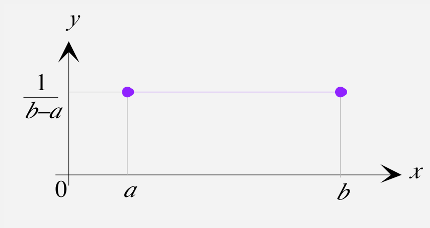
In other words, a uniform density function must have the following form.

Uniform Density Function

The **uniform density function on the interval [*a*, *b*]** is given by

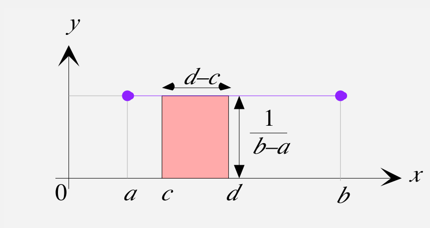
$$f(x) = \frac{1}{b-a}.$$

Its graph is a horizontal line.



Calculating Probability with a Uniform Density Function

Since probability is given by area, it is not hard to compute probabilities based on a uniform distribution.



$$P(c \le X \le d) = \text{Area of shaded rectangle} = \frac{d-c}{b-a}$$

Quick Example

Let X be a random real number between 0 and 5. Then X has a uniform distribution given by

$$f(x) = \frac{1}{5-0} = \frac{1}{5},$$

and $P(2 \le X \le 4.5) = \dfrac{4.5-2}{5-0} = 0.5$.

Example 2 ***Spinning a Dial***

Suppose that you spin the dial shown in Figure 4 so that it comes to rest at a random position. Model this with a suitable distribution, and use it to find the probability that the dial will land somewhere between 5˚ and 300˚.

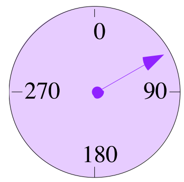


Figure 4

Solution We take X to be the angle at which the pointer comes to rest, so we use the interval $[0, 360]$. Since all angles are equally likely, the probability density function should not depend on x and therefore should be constant. That is, we take f to be uniform.

$$f(x) = \frac{1}{b-a} = \frac{1}{360-0} = \frac{1}{360}$$

Thus,

$$P(5 \le X \le 300) = \frac{300-5}{360-0} = \frac{295}{360} \approx 0.8194.$$

Before we go on... Notice that we could have used the integral formula and obtained the same answer.

$$\int_5^{300} \frac{1}{360}\,dx = \frac{1}{360}(300-5) = \frac{295}{360}$$

You can also check the following probabilities. Why are these the answers you expect?

$P(0 \le X \le 90) = 1/4$
$P(90 \le X \le 180) = 1/4$
$P(0 \le X \le 180) = 1/2$
$P(0 \le X \le 270) = 3/4$
$P(0 \le X \le 120) = 1/3$

Exponential Density Functions

Suppose that troubled saving and loan (S&L) institutions are failing continuously at a fractional rate of 5% per year. What is the probability that a troubled S&L will fail sometime within the next T years?

To answer the question, suppose that you started with 100 troubled S&Ls. Since they are failing continuously at a fractional rate of 5% per year, the number left after T years is given by the decay equation

$$\text{Number left} = 100e^{-0.05T},$$

so

$$\begin{aligned}\text{Number that fail} &= \text{Total number} - \text{Number left}\\ &= 100 - 100e^{-0.05T}\\ &= 100(1 - e^{-0.05T}).\end{aligned}$$

Thus, the percentage that will have failed by that time—and hence the probability that we are asking for—is given by

$$P = \frac{100(1 - e^{-0.05T})}{100} = 1 - e^{-0.05T}.$$

Now let X be the number of years a randomly chosen troubled S&L will take to fail. We have just calculated the probability that X is between 0 and T. In other words,

$$P(0 \le X \le T) = 1 - e^{-0.05T}.$$

Notice that this result can also be obtained by calculating a certain integral:

$$\int_0^T 0.05e^{-0.05x}\,dx = [e^{-0.05x}]_0^T = 1 - e^{-0.05T}.$$

Thus,

$$P(0 \le X \le T) = \int_0^T 0.05e^{-0.05x}\,dx,$$

and so we use $f(x) = 0.05e^{-0.05x}$ as a probability density function to model this situation.

Question Does this function satisfy the mathematical conditions necessary for it to be a probability density function?
Answer First, the domain of f is $[0, +\infty)$, since x refers to the number of years from now. Checking requirements (a) and (b) for a probability density function,

(a) $0.05e^{-0.05x} \ge 0$,

(b) $$\begin{aligned}\int_0^{+\infty} 0.05e^{-0.05x}\,dx &= \lim_{M\to+\infty} \int_0^M 0.05e^{-0.05x}\,dx\\ &= \lim_{M\to+\infty} [-e^{-0.05x}]_0^M\\ &= \lim_{M\to+\infty} (e^0 - e^{-0.05M}) = 1 - 0 = 1.\end{aligned}$$

There is nothing special about the number 0.05. Any function of the form

$$f(x) = ae^{-ax}$$

with a a positive constant is a probability density function. A density function of this form is referred to as an **exponential density function**.

Exponential Density Function

An **exponential density function** is a function of the form

$$f(x) = ae^{-ax} \quad (a \text{ a positive constant})$$

with domain $[0\ +\infty)$. Its graph is shown in Figure 5.

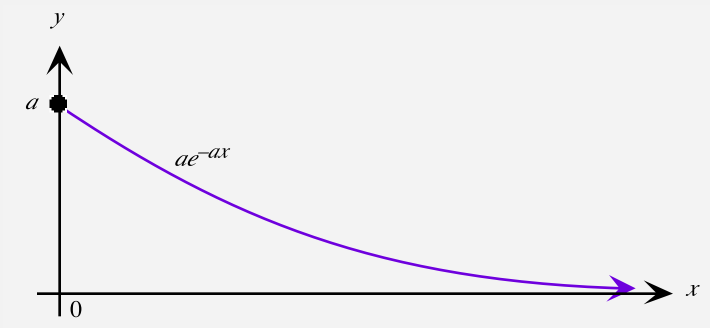


Figure 5

Quick Example

Continuing the example in the text, we find that the probability that a given troubled S&L will fail between 2 and 4 years from now is

$$P(2 \le X \le 4) = \int_2^4 0.05e^{-0.05x}\,dx = [-e^{-0.05x}]_2^4 = -e^{-0.2} + e^{-0.1} \approx 0.086.$$

The probability that it will last 5 or more years is

$$P(X \ge 5) = \int_5^{+\infty} 0.05e^{-0.05x}\,dx$$

$$= \lim_{M \to +\infty} \int_5^M 0.05e^{-0.05x}\,dx$$

$$= \lim_{M \to +\infty} [-e^{-0.05x}]_5^M = \lim_{M \to +\infty} (e^{-0.25} - e^{-0.05M}) = e^{-0.25} \approx 0.779.$$

So there is an 8.6% chance that a given S&L will fail between 2 and 4 years from now, and a 77.9% chance that it will last 5 or more years.

Example 3 ***Radioactive Decay***

Plutonium 239 decays continuously at a rate of 0.00284% per year. If X is the time a randomly chosen plutonium atom will decay, write down the associated probability density function, and use it to compute the probability that a plutonium atom will decay between 100 and 500 years from now.

Solution Using the discussion on failing S&Ls as our guide, we see that $a = 0.0000284$, so the probability density function is

$$f(x) = 0.0000284e^{-0.0000284x}.$$

For the second part of the question,

$$P(100 \leq X \leq 500) = \int_{100}^{500} (0.0000284e^{-0.0000284x})\, dx \approx 0.011.$$

Thus there is a 1.1% chance that a plutonium atom will decay sometime during the given 400 year period.

Normal Density Functions

Perhaps the most interesting class of probability density functions are the **normal density functions**, defined as follows.[1]

Normal Density Function

A **normal density function** is a function of the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

with domain $(-\infty, +\infty)$. The quantity μ is called the **mean** and can be any real number, while σ is called the **standard deviation** and can be any positive real number. The graph of a normal density function is shown in Figure 6.

[1] You may recall encountering exercises using the normal density function in the chapter on integration.

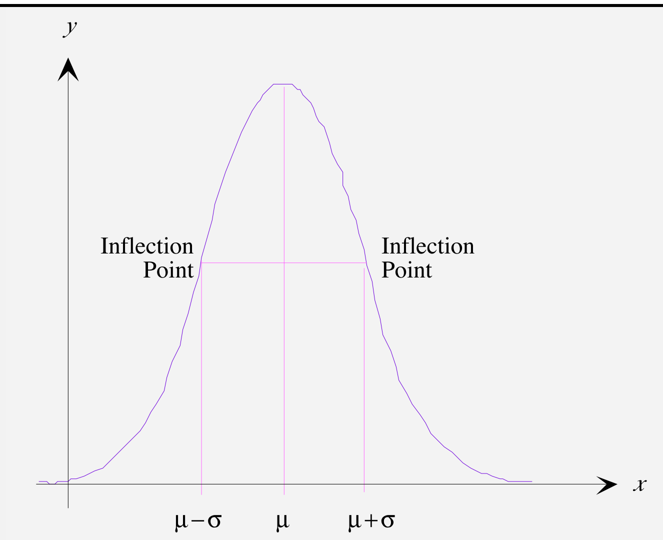


Figure 6

The following properties can be checked using calculus and a little algebra.

Properties of a Normal Density Curve

(a) It is "bell-shaped" with the peak occurring at $x = \mu$.
(b) It is symmetric about the vertical line $x = \mu$.
(c) It is concave down in the range $\mu - \sigma \leq x \leq \mu + \sigma$.
(d) It is concave up outside that range, with inflection points at $x = \mu-\sigma$ and $x = \mu+\sigma$.

The normal density function applies in many situations that involve measurement and testing. For instance, repeated imprecise measurements of the length of a single object, a measurement made on many items from an assembly line, and collections of SAT scores tend to be distributed normally. It is partly for this reason that the normal density curve is so important in quality control and in assessing the results of standardized tests.

In order to use the normal density function to compute probabilities, we need to calculate integrals of the form $\int_a^b f(x)\, dx$. However, the antiderivative of the normal density function cannot be expressed in terms of any commonly used functions. Traditionally, statisticians and others have used tables coupled with transformation techniques to evaluate such integrals. This approach is rapidly becoming obsolete as the technology of spreadsheets, hand-held computers and programmable calculators puts the ability to do numerical integration quickly and accurately in everybody's hands (literally). In keeping with this trend, we shall show how to use technology to do the necessary calculation in the next example.

Example 4 ***Quality Control***

Pressure gauges manufactured by Precision Corp. must be checked for accuracy before being placed on the market. To test a pressure gauge, a worker uses it to measure the pressure of a sample of compressed air known to be at a pressure of exactly 50 pounds per square inch. If the gauge reading is off by more than 1% (0.5 pounds), the gauge is rejected. Assuming that the reading of a pressure gauge under these circumstances is a normal random variable with mean 50 and standard deviation 0.5, find the percentage of gauges rejected.

Solution For a gauge to be accepted, its reading X must be 50 to within 1%, in other words, $49.5 \le X \le 50.5$. Thus, the probability that a gauge will be accepted is $P(49.5 \le X \le 50.5)$. X is a normal random variable with $\mu = 50$ and $\sigma = 0.5$. The formula tells us that

$$P(49.5 \le X \le 50.5) = \int_{49.5}^{50.5} f(x)\, dx ,$$

where
f is the normal distribution with mean $\mu = 50$ and standard deviation $\sigma = 0.5$;

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{0.5\sqrt{2\pi}}\, e^{-\frac{(x-50)^2}{0.5}} .$$

Graphing Calculator

There are two methods to calculate this integral on a graphing calculator.
Method 1: *Calculating the integral directly*
Using a TI-83, for example, you would enter

```
Y₁=(1/(0.5(2π)^0.5))e^(-(X-50)^2/0.5)
```

and then enter

```
fnInt(Y₁,X,49.5,50.5)
```

Method 2: *Using the built-in normal distribution function*
The TI-83 has a built-in normal distribution function, Press `[2nd]` [VARS] to obtain the selection of distribution functions. The second function, `normalcdf`, gives $P(a \le X \le b)$ directly. To compute $P(49.5 \le X \le 50.5)$, enter

```
normalcdf(49.5,50.5,50,.5)
```

Format: normalcdf(Lower bound, Upper bound, μ, σ)

Both methods yield an answer of approximately 0.6827. In other words, 68.27% of the gauges will be accepted. Thus, the remaining 31.73% of the gauges will be rejected.

Spreadsheet

Spreadsheet programs also come equipped with built-in statistical software that allows you to compute $P(a \le X \le b)$. To compute $P(49.5 \le X \le 50.5)$ in Excel, enter

```
=NORMDIST(50.5,50,0.5,1)-NORMDIST(49.5,50,0.5,1)
```

in any vacant cell.

Before we go on... As we mentioned above, the traditional and still common way of calculating normal probabilities is to use tables. The tables most commonly published are for the **standard normal distribution**, the one with mean 0 and standard deviation 1. If X is a normal variable with mean μ and standard deviation σ, the variable $Z = (X-\mu)/\sigma$ is a standard normal variable (see the exercises). Thus, to use a table we first write

$$P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right)$$

and then use the table to calculate the latter probability.

The following calculations, true for any normal random variable, are very useful to remember:

$$P(\mu-\sigma \le X \le \mu+\sigma) \approx 0.6827 \qquad \text{(See below.[1])}$$
$$P(\mu-2\sigma \le X \le \mu+2\sigma) \approx 0.9545$$
$$P(\mu-3\sigma \le X \le \mu+3\sigma) \approx 0.9973$$

Question Why can we assume that the reading of a pressure gauge is given by a normal distribution? Why is the normal distribution so common in this kind of situation?
Answer The reason for this is rather deep. There is a theorem in probability theory called the Central Limit Theorem, which says that a large class of probability density functions may be approximated by normal density functions. Repeated measurement of the same quantity gives rise to such a function.

Beta Density Functions

There are many random variables whose values are percentages or fractions. These variables have density functions defined on $[0,1]$. A large class of random variables, such as the percentage of new businesses that turn a profit in their first year, the percentage of banks that default in a given year, and the percentage of time a plant's machinery is inactive, can be modeled by a **beta density function**.

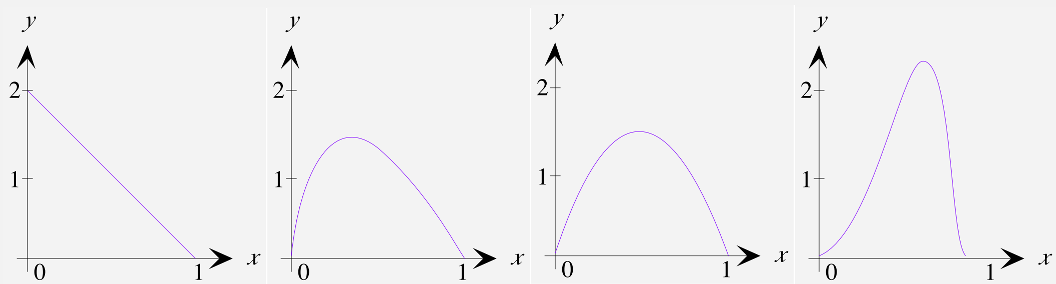
[1] If you use published four-figure tables and double the figure $P(0 \le Z \le 1) = 0.3413$ given there, you will get the incorrect answer (often quoted in texts) of 0.6826. The reason for the error is that doubling a rounded decimal also doubles the rounding error: to five figures, $P(0 \le Z \le 1) = 0.34134$, and thus doubling it gives $0.68268 \approx 0.6827$.

Beta Density Function

A **beta density function** is a function of the form

$$f(x) = (\beta+1)(\beta+2)x^{\beta}(1-x),$$

with domain [0, 1]. The number β can be any constant ≥ 0. Figure 7 shows the graph of $f(x)$ for several values of β.



$\beta = 0$, $f(x) = 2(1-x)$ | $\beta = 0.5$, $f(x) = 3.75x^{0.5}(1-x)$ | $\beta = 1$, $f(x) = 6x(1-x)$ | $\beta = 4$, $f(x) = 30x^4(1-x)$

Figure 7

Example 5 ***Downsizing in the Utilities Industry***

A utilities industry consultant predicts a cutback in the Canadian utilities industry during 2000–2005 by a percentage specified by a beta distribution with $\beta = 0.25$. Calculate the probability that Ontario Hydro will downsize by between 10% and 30% during the given five-year period.[1]

Solution The beta density function with $\beta = 0.25$ is

$$\begin{aligned} f(x) &= (\beta+1)(\beta+2)x^{\beta}(1-x) \\ &= 2.8125x^{0.25}(1-x) \\ &= 2.8125(x^{0.25} - x^{1.25}). \end{aligned}$$

Thus,

$$\begin{aligned} P(0.10 \le X \le 0.30) &= \int_{0.10}^{0.30} 2.8125(x^{0.25} - x^{1.25})\, dx \\ &= 2.8125 \int_{0.10}^{0.30} (x^{0.25} - x^{1.25})\, dx \end{aligned}$$

[1] This model is fictitious. Ontario Hydro did announce plans to downsize by 8.4% in 1995, however (*Report on Business* (Canada), Feb. 15, 1994, p. B1).

$$= 2.8125\left[\frac{x^{1.25}}{1.25} - \frac{x^{2.25}}{2.25}\right]_{0.10}^{0.30}$$
$$\approx 0.2968.$$

So there is approximately a 30% chance that Ontario Hydro will downsize by between 10% and 30%.

Before we go on... Figure 8 shows the density function. Notice that its shape is "in between" those for $\beta = 0$ and $\beta = 0.5$ in Figure 7.

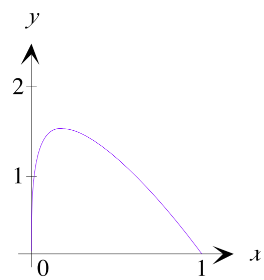


Figure 8

P.2 Exercises

In Exercises 1–12, check whether the given function is a probability density function. If a function fails to be a probability density function, say why.

1. $f(x) = 1$ on $[0, 1]$

2. $f(x) = x$ on $[0, 2]$

3. $f(x) = \frac{x}{2}$ on $[0, 1]$

4. $f(x) = 2$ on $[0, \frac{1}{2}]$

5. $f(x) = \frac{3}{2}(x^2-1)$ on $[0, 2]$

6. $f(x) = \frac{1}{3}(1-x^2)$ on $[0, 1]$

7. $f(x) = \frac{1}{x}$ on $[1, e]$

8. $f(x) = e^x$ on $[0, \ln 2]$

9. $f(x) = 2xe^{-x^2}$ on $[0, +\infty)$

10. $f(x) = -2xe^{-x^2}$ on $(-\infty, 0]$

11. $f(x) = xe^{-x^2}$ on $(-\infty, +\infty)$

12. $f(x) = |x|e^{-x^2}$ on $(-\infty, +\infty)$

In Exercises 13–16, find the values of k for which the given functions are probability density functions.

13. $f(x) = 2k$ on $[-1, 1]$

14. $f(x) = k$ on $[-2, 0]$

15. $f(x) = ke^{kx}$ on $[0, 1]$

16. $f(x) = kxe^{x^2}$ on $[0, 1]$

In Exercises 17–26, say which kind of probability density function is most appropriate for the given random variable: uniform, exponential, normal, beta, or none of these.

17. The time it takes a Carbon-14 atom to decay
18. The time it takes you to drive home
19. The SAT score of a randomly selected student
20. The value of a random number between 0 and 1
21. The time of day at a randomly chosen moment

22. The time it takes a careless driver to be involved in an accident
23. The fraction of fast-food restaurants that are profitable in their first year
24. The time it will take for the sun to die
25. The time it takes before a gambler loses on a bet
26. The length of a 2000 Ford Mustang® tailpipe

Applications

Unless otherwise stated, round answers to all applications to four decimal places.

27. ***Salaries*** Assuming that workers' salaries in your company are uniformly distributed between $10,000 and $40,000 per year, find the probability that a randomly chosen worker earns an annual salary between $14,000 and $20,000.

28. ***Grades*** The grade point averages of members of the Gourmet Society are uniformly distributed between 2.5 and 3.5. Find the probability that a randomly chosen member of the society has a grade point average between 3 and 3.2.

29. ***Boring Television Series*** Your company's new series "Avocado Comedy Hour" has been a complete flop, with viewership continuously declining at a rate of 30% per month. Use a suitable density function to calculate the probability that a randomly chosen viewer will be lost sometime in the next three months.

30. ***Bad Investments*** Investments in junk bonds are declining continuously at a rate of 5% per year. Use a suitable density function to calculate the probability that a dollar invested in junk bonds will be pulled out of the junk bond market within the next two years.

31. ***Radioactive Decay*** The half-life of Carbon-14 is 5,730 years. What is the probability that a randomly selected Carbon-14 atom will not yet have decayed in 4,000 years' time?

32. ***Radioactive Decay*** The half-life of Plutonium-239 is 24,400 years. What is the probability that a randomly selected Plutonium-239 atom will not yet have decayed in 40,000 years' time?

33. ***The Doomsday Meteor*** The probability that a "doomsday meteor" will hit the earth in any given year and release a billion megatons or more of energy is on the order of 0.000 000 01.[1]

(a) What is the probability that the earth will be hit by a doomsday meteor at least once during the next 100 years? (Use an exponential distribution with $a = 0.000\,000\,01$. Give the answer correct to 2 significant digits.)

(b) What is the probability that the earth has been hit by a doomsday meteor at least once since the appearance of life (about 4 billion years ago)?

---

[1] Source: NASA International Near-Earth-Object Detection Workshop/*The New York Times*, January 25, 1994, p. C1.)

34. ***Galactic Cataclysm*** The probability that the galaxy MX-47 will explode within the next million years is estimated to be 0.0003.

(a) What is the probability that MX-47 will explode within the next 5 million years? (Use an exponential distribution with $a = 0.0003$.)

(b) What is the probability that MX-47 will still be around 10 million years hence?

Exercises 35–42 use the normal probability density function and require the use of either technology or a table of values of the standard normal distribution.

35. ***Physical Measurements*** Repeated measurements of a metal rod yield a mean of 5.3 inches, with a standard deviation of 0.1. What is the probability that the rod is between 5.25 and 5.35 inches long?

36. ***IQ Testing*** Repeated measurements of a student's IQ yield a mean of 135, with a standard deviation of 5. What is the probability that the student has an IQ between 132 and 138?

37. ***Psychology Tests*** It is known that subjects score an average of 100 points on a new personality test. If the standard deviation is 10 points, what percentage of all subjects will score between 75 and 80?

38. ***Examination Scores*** Professor Easy's students earned an average grade of 3.5, with a standard deviation of 0.2. What percentage of his students earned between 3.5 and 3.9?

39. ***Operating Expenses*** The cash operating expenses of the regional *Bell* companies during the first half of 1994 were distributed about a mean of $29.87 per access line per month, with a standard deviation of $2.65. *Ameritech Corporation*'s operating expenses were $28.00 per access line per month.[1] Assuming a normal distribution of operating expenses, estimate the percentage of regional Bell companies whose operating expenses were closer to the mean than those of Ameritech.

40. ***Operating Expenses*** *Nynex Corporation*'s operating expenses were $35.80 per access line per month in the first half of 1994.[2] Referring to the distribution in the previous exercise, estimate the percentage of regional Bell companies whose operating expenses were higher than those of Nynex.

---

[1] Source: NatWest Securities/Company Reports/*The New York Times*, November 22, 1994, p. D1.

[2] *Ibid.*

41. ***Operating Expenses*** *SBC Corporation* (formerly Southwestern Bell) had operating expenses of \$27.70 per access line per month in the first half of 1994.[1] Could SBC justifiably claim that its operating expenses were among the lowest 25% of all the regional Bell companies? Explain. (Use the normal distribution of the above exercises.)

42. ***Operating Expenses*** *US West Corporation* had operating expenses of \$29.10 per access line per month in the first half of 1994.[2] Were US West's operating expenses closer to the mean than those of most other regional Bells? Explain. (Use the normal distribution of the above exercises.)

Cumulative Distribution If f is a probability density function defined on the interval (a, b), then the **cumulative distribution function** F is given by

$$F(x) = \int_a^x f(t)\, dt\ .$$

Exercises 43–52 deal with the cumulative distribution function.

43. Why is $F'(x) = f(x)$?

44. Use the result of the previous exercise to show that

$$P(c \le X \le d) = F(d) - F(c)$$

for $a \le c \le d \le b$.

45. Show that $F(a) = 0$ and $F(b) = 1$.

46. Can $F(x)$ can have any local extrema? (Give a reason for your answer.)

47. Find the cumulative distribution functions for the situation described in Exercise 27.
48. Find the cumulative distribution functions for the situation described in Exercise 28.
49. Find the cumulative distribution functions for the situation described in Exercise 29.
50. Find the cumulative distribution functions for the situation described in Exercise 30.
51. Find the cumulative distribution functions for the situation described in Exercise 31.
52. Find the cumulative distribution functions for the situation described in Exercise 32.

Communication and Reasoning Exercises

53. Why is a probability density function often more convenient than a histogram?

---

[1] *Ibid.*
[2] *Ibid.*

54. Give an example of a probability density function that is increasing everywhere on its domain.

55. Give an example of a probability density function that is concave up everywhere on its domain.

56. Suppose that X is a normal random variable with mean μ and standard deviation σ, and that Z is a standard normal variable. Using the substitution $z = (x - \mu)/\sigma$ in the integral, show that

$$P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right).$$

57. Your friend thinks that if f is a probability density function for the continuous random variable X, then $f(a)$ is the probability that $X = a$. Explain to your friend why this is wrong.

58. Not satisfied with your explanation in the previous exercise, your friend then challenges you by asking, "If $f(a)$ is not the probability that $X = a$, then just what does $f(a)$ signify?" How would you respond?

61. Your friend now thinks that if F is a *cumulative* probability density function for the continuous random variable X, then $F(a)$ is the probability that $X = a$. Explain why your friend is *still* wrong.

59. Once again not satisfied with your explanation in the previous exercise, your friend challenges you by asking, "If $F(a)$ is not the probability that $X = a$, then just what does $F(a)$ signify?" How would you respond?

P.3 Mean, Median, Variance, and Standard Deviation

Mean

In the last section we saw that if savings and loan institutions are continuously failing at a rate of 5% per year, then the associated probability density function is

$$f(x) = 0.05e^{-0.05x},$$

with domain $[0, +\infty)$. An interesting and important question to ask is: What is the average length of time such an institution will last before failing? To answer this question, we use the following.

Mean or Expected Value

If X is a continuous random variable with probability density function f defined on an interval with (possibly infinite) endpoints a and b, then the **mean** or **expected value** of X is

$$E(X) = \int_a^b xf(x)\ dx .$$

$E(X)$ is also called the **average value** of X. It is what we expect to get if we take the average of many values of X obtained in experiments.

Quick Example

Let X have probability density function given by $f(x) = 3x^2$, with domain $[0, 1]$. Then

$$E(X) = \int_a^b xf(x)\ dx = \int_0^1 (x\cdot 3x^2)\ dx = \int_0^1 3x^3\ dx = \left[\frac{3x^4}{4}\right]_0^1 = \frac{3}{4}.$$

We shall explain shortly why $E(X)$ is given by the integral formula.

Example 1 ***Failing S&Ls***

Given that troubled S&Ls are failing continuously at a rate of 5% per year, how long will the average troubled S&L last?

Solution If X is the number of years that a given S&L will last, we know that its probability density function is $f(x) = 0.05e^{-0.05x}$. To answer the question we compute $E(X)$.

$$E(X) = \int_a^b xf(x)\ dx$$

$$= \int_0^{+\infty} (0.05xe^{-0.05x})\ dx$$

$$= \lim_{M \longrightarrow +\infty} \int_0^{M} (0.05xe^{-0.05x})\ dx$$

Using integration by parts, we get

$$E(X) = \lim_{M \longrightarrow +\infty} -0.05\left[e^{-0.05x}(20x + 400)\right]_0^M = (0.05)(400) = 20.$$

Thus, the expected lifespan of a troubled S&L is 20 years.

Before we go on... Notice that the answer, 20, is the reciprocal of the failure rate 0.05. This is true in general: if $f(x) = ae^{-ax}$, then $E(X) = 1/a$.

Question Why is $E(X)$ given by that integral formula?
Answer Suppose for simplicity that the domain of f is a finite interval $[a, b]$. Break up the interval into n subintervals $[x_{k-1}, x_k]$, each of length Δx, as we did for Riemann sums. Now, the probability of seeing a value of X in $[x_{k-1}, x_k]$ is approximately $f(x_k)\Delta x$ (the approximate area under the graph of f over $[x_{k-1}, x_k]$). Think of this as the fraction of times we expect to see values of X in this range. These values, all close to x_k, then contribute approximately $x_k f(x_k)\Delta x$ to the average, if we average together many observations of X. Adding together all of these contributions, we get

$$E(X) \approx \sum_{k=1}^{n} x_k f(x_k)\Delta x\ .$$

Now these approximations get better as $n \longrightarrow \infty$, and we notice that the sum above is a Riemann sum converging to

$$E(X) = \int_a^b xf(x)\ dx\ ,$$

which is the formula we have been using.

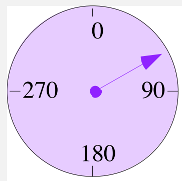
Question What are the expected values of the standard distributions we discussed in the preceding section?
Answer Let's compute them one by one.

Mean of a Uniform Distribution

If X is uniformly distributed on $[a, b]$, then

$$E(X) = \frac{a+b}{2}.$$

Quick Example Suppose that you spin the dial shown so that it comes to rest at a random position X.



Then $E(X) = \frac{0+360}{2} = 180°$.

This formula is not surprising, if you think about it for a minute. We'll leave the actual computation as one of the exercises.

Mean of an Exponential Distribution

If X has the exponential distribution function $f(x) = ae^{-ax}$, then

$$E(X) = \frac{1}{a}.$$

Quick Example

If Internet startup companies are failing at a rate of 10% per year, then the expected lifetime of an Internet startup company is

$$E(X) = \frac{1}{0.1} = 10 \text{ years.}$$

We saw why this formula works in Example 1.

Mean of a Normal Distribution

If X is normally distributed with parameters μ and σ, then

$$E(X) = \mu.$$

Quick Example

If the final exam scores in your class are normally distributed with mean 72.6 and standard deviation 8.3, then the expected value for a test score is

$$E(X) = \mu = 72.6.$$

That is why we called μ the mean, but we ought to do the calculation. Here we go.

$$E(X) = \int_{-\infty}^{+\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\, dx$$

$$= \int_{-\infty}^{+\infty} (\sigma w+\mu)\frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw \quad \text{after substituting } w = (x-\mu)/\sigma$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} we^{-w^2/2}\, dw \quad + \mu \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw \quad .$$

Now, the first integral can be done easily (substitute $v = -w^2/2$) and converges to 0. The second integral we recognize as the area under another normal distribution curve (the one with $\mu = 0$ and $\sigma = 1$), so it is equal to 1. Therefore, the whole thing simplifies to

$$E(X) = \mu$$

as claimed.

Mean of a Beta Distribution

If X has the beta distribution function $f(x) = (\beta+1)(\beta+2)x^\beta(1-x)$, then

$$E(X) = \frac{\beta + 1}{\beta + 3} \ .$$

Again, we shall leave this as an exercise.

Example 2 ***Downsizing in the Utilities Industry***

A utilities industry consultant predicts a cutback in the Canadian Utilities industry during 2000–2005 by a percentage specified by a beta distribution with $\beta = 0.25$. What is the expected size of the cutback by Ontario Hydro?[1]

Solution Since $\beta = 0.25$,

$$E(X) = \frac{\beta + 1}{\beta + 3} = \frac{1.25}{3.25} \approx 0.38.$$

Therefore, we can expect about a 38% cutback by Ontario Hydro.

[1] This model is fictitious. Ontario Hydro did announce plans to downsize by 8.4% in 1995, however (*Report on Business* (Canada), Feb. 15, 1994, p. B1).

Before we go on... What $E(X)$ really tells us is that the *average* downsizing of many utilities will be 38%. Some will cut back more, and some will cut back less.

There is a generalization of the mean that we shall use below. If X is a random variable on the interval (a, b) with probability density function f, and if g is any function defined on that interval, then we can define the **expected value of *g*** to be

$$E(g(X)) = \int_a^b g(x)f(x)\,dx\,.$$

Thus, in particular, the mean is just the expected value of the function $g(x) = x$. We can interpret this as the average we expect if we compute $g(X)$ for many experimental values of X.

Variance and Standard Deviation

Statisticians use the variance and standard deviation of a continuous random variable X as a way of measuring its dispersion, or the degree to which is it "scattered." The definitions are as follows.

Variance and Standard Deviation

Let X be a continuous random variable with density function f defined on the interval (a, b), and let $\mu = E(X)$ be the mean of X. Then the **variance** of X is given by

$$Var(X) = E((X-\mu)^2) = \int_a^b (x-\mu)^2 f(x)\,dx\,.$$

The **standard deviation** of X is the square root of the variance,

$$\sigma(X) = \sqrt{Var(X)}\,.$$

Notes

1. In order to calculate the variance and standard deviation, we need first to calculate the mean.

2. $Var(X)$ is the expected value of the function $(x-\mu)^2$, which measures the square of the distance of X from its mean. It is for this reason that $Var(X)$ is sometimes called the *mean square deviation*, and $\sigma(X)$ is called the *root mean square deviation*. $Var(X)$ will be larger if X tends to wander far away from its mean, and smaller if the values of X tend to cluster near its mean.

3. The reason we take the square root in the definition of $\sigma(X)$ is that $Var(X)$ is the expected value of the *square* of the deviation from the mean, and thus is measured in square units. Its square root $\sigma(X)$ therefore gives us a measure in ordinary units.

Question What are the variances and standard deviations of the standard distributions we discussed in the previous section?
Answer Let's compute them one by one. We'll leave the actual computations (or special cases) for the exercises.

Variance and Standard Deviation of Some Distributions

Uniform Distribution
If X is uniformly distributed on $[a,b]$, then

$$Var(X) = \frac{(b-a)^2}{12}$$

and

$$\sigma(X) = \frac{b-a}{\sqrt{12}} .$$

Exponential Distribution
If X has the exponential distribution function $f(x) = ae^{-ax}$, then

$$Var(X) = \frac{1}{a^2}$$

and

$$\sigma(X) = \frac{1}{a} .$$

Normal Distribution
If X is normally distributed with parameters μ and σ, then

$$Var(X) = \sigma^2$$

and

$$\sigma(X) = \sigma.$$

This is what you might have expected

Beta Distribution
If X has the beta distribution function $f(x) = (\beta+1)(\beta+2)x^\beta(1-x)$, then

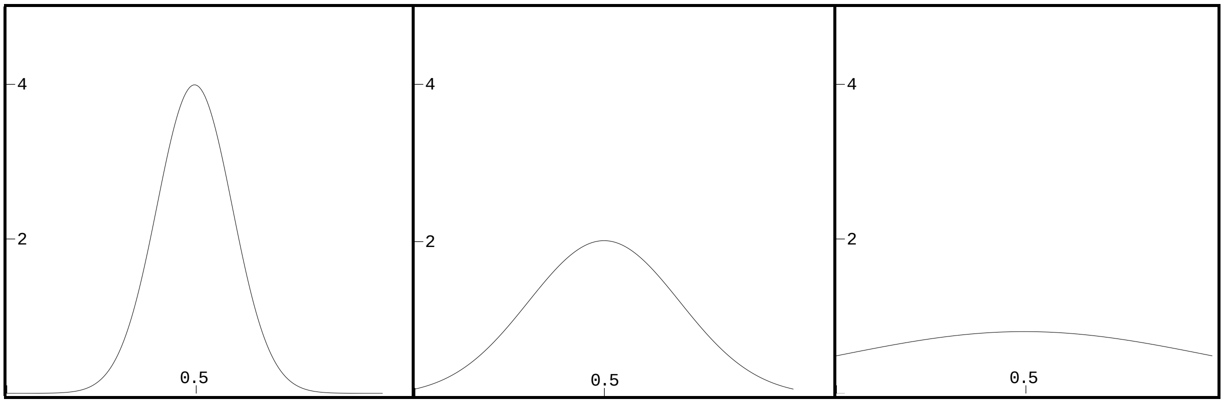
$$Var(X) = \frac{2(\beta+1)}{(\beta+4)(\beta+3)^2}$$

and

$$\sigma(X) = \sqrt{\frac{2(\beta+1)}{(\beta+4)(\beta+3)^2}} .$$

You can see the significance of the standard deviation quite clearly in the normal distribution. As we mentioned in the previous section, σ is the distance from the maximum at

μ to the points of inflection at $\mu-\sigma$ and $\mu+\sigma$. The larger σ is, the wider the bell. Figure 1 shows three normal distributions with three different standard deviations (all with $\mu = 0.5$).



$\sigma = 0.1$ $\sigma = 0.2$ $\sigma = 0.5$

Figure 1

Again, a small standard deviation means that the values of X will be close to the mean with high probability, while a large standard deviation means that the values may wander far away with high probability.

Median

The *median income* in the US is the income M such that half the population earn incomes $\le M$ (so the other half earn incomes $\ge M$). In terms of probability, we can think of income as a random variable X. Then the probability that $X \le M$ is $1/2$, and the probability that $X \ge M$ is also $1/2$.

> **Median**
> Let X be a continuous random variable. The **median** of X is the number M such that
>
> $$P(X \le M) = \frac{1}{2}.$$
>
> Then, $P(M \le X) = \frac{1}{2}$ also.

If f is the probability density function for X and f is defined on (a, b), then we can calculate M by solving the equation

$$P(a \le X \le M) = \int_a^M f(x)\, dx = \frac{1}{2}$$

for M. Graphically, the vertical line $x = M$ divides the total area under the graph of f into two equal parts. (Figure 2).

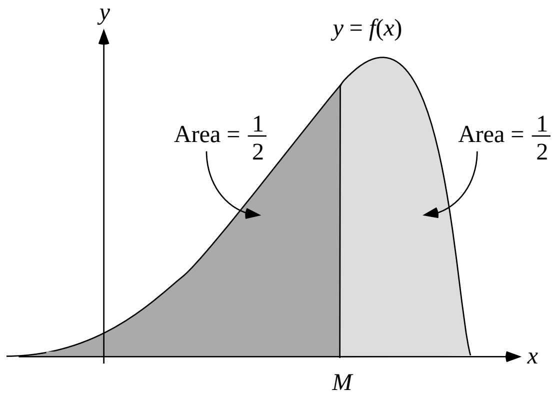


Figure 2

Question What is the difference between the median and the mean?
Answer Roughly speaking, the median divides the area under the distribution curve into two equal parts, while the mean is the value of X at which the graph would *balance*. If a probability curve has as much area to the left of the mean as to the right, then the mean is equal to the median. This is true of uniform and normal distributions, which are *symmetric* about their means. On the other hand, the medians and means are different for the exponential distributions and most of the beta distributions, because their areas are not distributed symmetrically.

Example 3 *Lines at the Post Office*

The time in minutes between individuals joining the line at an Ottawa Post Office is a random variable with the exponential distribution

$$f(x) = 2e^{-2x},\ (x \geq 0).$$

Find the mean and median time between individuals joining the line and interpret the answers.

Solution The expected value for an exponential distribution $f(x) = ae^{-ax}$ is $1/a$. Here, $a = 2$, so $E(X) = 1/2$. We interpret this to mean that, on average, a new person will join the line every half a minute, or 30 seconds. For the median, we must solve

$$\int_a^M f(x)\,dx = \frac{1}{2}.$$

That is,

$$\int_0^M (2e^{-2x})\,dx = \frac{1}{2}.$$

Evaluating the integral gives

$$-\left[e^{-2x}\right]_0^M = \frac{1}{2},$$

or

$$1 - e^{-2M} = \frac{1}{2}$$

so

$$e^{-2M} = \frac{1}{2},$$

or

$$-2M = \ln\left(\frac{1}{2}\right) = -\ln 2.$$

Thus,

$$M = \frac{\ln 2}{2} \approx 0.3466 \text{ minutes.}$$

This means that half the people get in line less than 0.3466 minutes (about 21 seconds) after the previous person, while half arrive more than 0.3466 minutes later. The mean time for a new person to arrive in line is larger than this because there are some occasional long waits between people, and these pull the average up.

Sometimes we cannot solve the equation $\int_a^M f(x)\,dx = 1/2$ for M analytically, as the next example shows.

Example 4 ***Median***

Find the median of the random variable with beta density function for $\beta = 4$.

Solution Here,

$$\begin{aligned} f(x) &= (\beta+1)(\beta+2)x^\beta(1-x) \\ &= 30x^4(1-x). \end{aligned}$$

Thus we must solve

$$\int_0^M (30x^4(1-x))\,dx = \frac{1}{2}.$$

That is,

$$30\int_0^M (x^4 - x^5)\,dx = \frac{1}{2}.$$

So

$$30\left[\frac{M^5}{5} - \frac{M^6}{6}\right] = \frac{1}{2},$$

or, multiplying through and clearing denominators,

$$12M^5 - 10M^6 - 1 = 0.$$

This is a degree six polynomial equation that has no easy factorization. Since there is no general analytical method for obtaining the solution, the only method we can use is numerical. Figure 3 shows three successive views of a graphing calculator plot of `Y = 12X^5 - 10X^6 - 1`, obtained by zooming in towards one of the zeros.

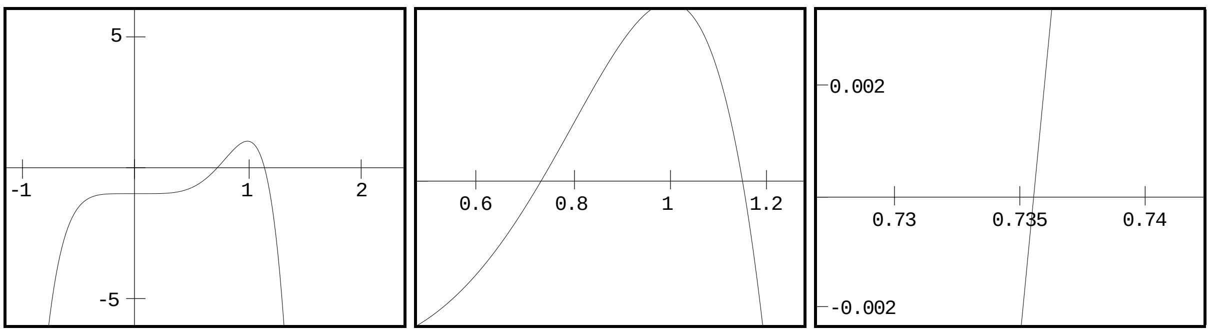


Figure 3

We are interested only in the zero that occurs between 0 and 1 (why?), and find that $M \approx 0.735$ to within ± 0.001.

Before we go on...

Question This method required us first to calculate $\int_a^M f(x)\,dx$ analytically. What if even this is impossible to do?

Answer We can solve the equation $\int_a^M f(x)\,dx = 1/2$ graphically by having the calculator compute and graph this function of M by numerical integration. For example, to redo the above example on the TI-82 or compatible models, enter

```
Y1 = fnInt(30T^4(1-T),T,0,X)-0.5
```

which corresponds to

$$y = \int_0^x 30t^4(1-t)\,dt - \frac{1}{2},$$

a function of x. Since the median of M is the solution obtained by setting $y = 0$, we can obtain the answer by plotting Y_1 and finding its x-intercept. The plot should be identical to the one we obtained above (why?).

9.3 Exercises

Find the expected value $E(X)$, the variance $Var(X)$ and the standard deviation $\sigma(X)$ for each of the density functions in Exercises 1–20.

1. $f(x) = \frac{1}{3}$ on $[0, 3]$

2. $f(x) = 3$ on $[0, \frac{1}{3}]$

3. $f(x) = \frac{x}{50}$ on $[0, 10]$

4. $f(x) = 5x$ on $[0, \sqrt{2/5}\,]$

5. $f(x) = \frac{3}{2}(1 - x^2)$ on $[0, 1]$

6. $f(x) = \frac{3}{4}(1 - x^2)$ on $[-1, 1]$

7. $f(x) = e^x$ on $[0, \ln 2]$

8. $f(x) = \frac{1}{x}$ on $[1, e]$

9. $f(x) = 0.1e^{-0.1x}$ on $[0, +\infty)$

10. $f(x) = 4e^{4x}$ on $(-\infty, 0]$

11. $f(x) = 0.03e^{0.03x}$ on $(-\infty, 0]$

12. $f(x) = 0.02e^{-0.02x}$ on $[0, +\infty)$

13. $f(x) = \frac{2}{x^3}$ on $[1, +\infty)$

14. $f(x) = \frac{1}{2\sqrt{x}}$ on $(0, 1]$

15. Normal density function with $\mu = 1$ and $\sigma = 1$ on $(-\infty, +\infty)$

16. Normal density function with $\mu = -1$ and $\sigma = 1$ on $(-\infty, +\infty)$

17. Beta density function with $\beta = 0.5$

18. Beta density function with $\beta = 1.5$

19. Beta density function with $\beta = 3.2$

20. Beta density function with $\beta = 4.6$

Use a graphing calculator or computer to find $E(X)$, $Var(X)$ and $\sigma(X)$ for each of the density functions in Exercises 21–24. (Round all answers to four significant digits.)

21. $f(x) = \frac{4}{\pi(1+x^2)}$ on $[0, 1]$

22. $f(x) = \frac{3}{\pi\sqrt{1-x^2}}$ on $[\frac{1}{2}, 1]$

23. $f(x) = 2xe^{-x^2}$ on $[0, +\infty)$

24. $f(x) = -2xe^{-x^2}$ on $(-\infty, 0]$

Find the medians of the random variables with the probability density functions given in Exercises 25–34.

25. $f(x) = 0.25$ on $[0, 4]$

26. $f(x) = 4$ on $[0, 0.25]$

27. $f(x) = 3e^{-3x}$ on $[0, +\infty)$

28. $f(x) = 0.5e^{-0.5x}$ on $[0, +\infty)$

29. $f(x) = 0.03e^{0.03x}$ on $(-\infty, 0]$

30. $f(x) = 0.02e^{-0.02x}$ on $[0, +\infty)$

31. $f(x) = 2(1 - x)$ on $[0, 1]$

32. $f(x) = \frac{1}{x^2}$ on $[1, +\infty)$

33. $f(x) = \frac{1}{2\sqrt{x}}$ on $(0, 1]$

34. $f(x) = \frac{1}{x}$ on $[1, e]$

35. ***Mean of a Uniform Distribution*** Verify the formula for the mean of a uniform distribution by computing the integral.

36. ***Mean of a Beta Distribution*** Verify the formula for the mean of a beta distribution by computing the integral.

37. ***Variance of a Uniform Distribution*** Verify the formula for the variance of a uniform distribution by computing the integral.

38. ***Variance of an Exponential Distribution*** Verify the formula for the variance of an exponential distribution by computing the integral.

39. ***Median of an Exponential Random Variable*** Show that if X is a random variable with density function $f(x) = ae^{-ax}$ on $[0, +\infty)$, then X has median $\ln 2/a$.

40. ***Median of a Uniform Random Variable*** Show that if X is uniform random variable taking values in the interval $[a, b]$, then X has median $(a+b)/2$.

Use technology to find the medians of the random variables with the probability density functions given in Exercises 41–50. (Round all answers to two decimal places.)

41. $f(x) = \frac{3}{2}(1 - x^2)$ on $[0, 1]$

42. $f(x) = \frac{3}{4}(1 - x^2)$ on $[-1, 1]$

43. beta density function with $\beta = 2$

44. beta density function with $\beta = 3$

45. beta density function with $\beta = 2.5$

46. beta density function with $\beta = 0.5$

47. $f(x) = \frac{4}{\pi(1+x^2)}$ on $[0, 1]$

48. $f(x) = \frac{3}{\pi\sqrt{1-x^2}}$ on $[\frac{1}{2}, 1]$

49. $f(x) = 2xe^{-x^2}$ on $[0, +\infty)$

50. $f(x) = -2xe^{-x^2}$ on $(-\infty, 0]$

The **mean square** of a random variable X with density function f is given by the formula

$$E(X^2) = \int_a^b x^2 f(x)\, dx .$$

51–60. In Exercises 1–10, compute $E(X^2)$. In each case, compute also $E(X^2) - E(X)^2$.

61. Compare the answers in 51–60 to those in 1–10, and hence suggest a formula expressing $E(X^2)$ in terms of $E(X)$ and $Var(X)$.

62. Calculate $E(e^{tX}) = \int_a^b e^{tx} \cdot f(x)\, dx$ with $f(x) = 0.1e^{-0.1x}$ on $[0, +\infty)$. Then evaluate $\frac{d}{dt}E(e^{tX})\Big|_{t=0}$ and $\frac{d^2}{dt^2}E(e^{tX})\Big|_{t=0}$, comparing these answers with the answer to Exercise 59. What do you notice?

Applications

63. ***Salaries*** Assuming that workers' salaries in your company are uniformly distributed between \$10,000 and \$40,000 per year, calculate the average salary in your company.

64. ***Grades*** The grade point averages (gpa's) of members of the Gourmet Society are uniformly distributed between 2.5 and 3.5. Find the average gpa in the Gourmet Society.

65. ***Boring Television Series*** Your company's new series “Avocado Comedy Hour” has been a complete flop, with viewership continuously declining at a rate of 30% per month. How long will the average viewer continue to watch the show?

66. ***Bad Investments*** Investments in junk bonds are declining continuously at a rate of 5% per year. How long will an average dollar remain invested in junk bonds?

67. ***Radioactive Decay*** The half-life of carbon-14 is 5,730 years. How long, to the nearest year, do you expect it to take for a randomly selected carbon-14 atom to decay?

68. ***Radioactive Decay*** The half-life of plutonium-239 is 24,400 years. How long, to the nearest year, do you expect it to take for a randomly selected plutonium-239 atom to decay?

69. ***The Doomsday Meteor*** The probability that a “doomsday meteor” will hit the earth in any given year and release a billion megatons or more of energy is on the order of 0.000 000 01.[1] When do you expect the earth to be hit by a doomsday meteor? (Use an exponential distribution with $a = 0.000\,000\,01$.)

70. ***Galactic Cataclysm*** The probability that the galaxy MX-47 will explode within the next million years is estimated to be 0.0003. When do you expect MX-47 to explode? (Use an exponential distribution with $a = 0.0003$.)

Exercises 71–74 use the normal probability density function and require the use of technology for numerical integration. (Alternatively, see Exercise 61.) Find the root mean square value for X (i.e., $\sqrt{E(X^2)}$) in each exercise.

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[1] Source: NASA International Near-Earth-Object Detection Workshop (*New York Times*, January 25, 1994, p. C1.)

71. ***Physical Measurements*** Repeated measurements of a metal rod yield a mean of 5.3 inches, with a standard deviation of 0.1.

72. ***IQ Testing*** Repeated measurements of a student's IQ yield a mean of 135, with a standard deviation of 5.

73. ***Psychology Tests*** It is known that subjects score an average of 100 points on a new personality test, with a standard deviation of 10 points.

74. ***Examination Scores*** Professor May's students earned an average grade of 3.5 with a standard deviation of 0.2.

75. ***Learning*** A graduate psychology student finds that 64% of all first semester calculus students in Prof. Mean's class have a working knowledge of the derivative by the end of the semester.

(a) Take X = percentage of students who have a working knowledge of calculus after 1 semester, and find a beta density function that models X, assuming that the performance of students in Prof. Mean's is average.

(b) Find the median of X (rounded to two decimal places) and comment on any difference between the median and the mean.

76. *Plant Shutdowns* An automobile plant is open an average of 78% of the year.

(a) Take X = fraction of the year for which the plant is open, and find a beta density function that models X.

(b) Find the median of X (rounded to two decimal places) and comment on any difference between the median and the mean.

Communication and Reasoning Exercises

77. Sketch the graph of a probability distribution function with the property that its median is larger than its mean.

78. Sketch the graph of a probability distribution function with a large standard deviation and a small mean.

79. Complete the following sentence. The ___ measures the degree to which the values of X are distributed, while the ___ is the value of X such that half the measurements of X are below and half are above (for a large number of measurements).

80. Complete the following sentence. (See Exercise 61.) Given two of the quantities ___, ___ and ___, we can calculate the third using the formula ___ .

81. A value of *X* for which the probability distribution function *f* has a local maximum is called a **mode** of the distribution. (If there is more than one mode, the distribution is called bimodal (2 modes), trimodal (3 modes), etc. as the case may be.) If a distribution has a single mode, what does it tell you?

82. Referring to Exercise 81, sketch a bimodal distribution whose mean coincides with neither of the modes.

You're the Expert—Creating a Family Trust

Your position as financial consultant to the clients of Family Bank, Inc., often entails your having to give financial advice to clients with complex questions about savings. One of your newer clients, Malcolm Adams, recently graduated from college and 22 years old, presents you with a perplexing question. "I would like to set up my own insurance policy by opening a trust account into which I can make monthly payments starting now, so that upon my death or my ninety-fifth birthday—whichever comes sooner—the trust can be expected to be worth $500,000. How much should I invest each month?"

This is not one of those questions that you can answer by consulting a table, so you promise Malcolm an answer by the next day and begin to work on the problem. After a little thought, you realize that the question is one about *expected value*—the expected future value of an annuity into which monthly payments are made. Since the annuity would terminate upon his death (or his ninety-fifth birthday), you decide that you need a model for the probability distribution of the lifespan of a male in the United States. To obtain this information, you consult mortality tables and come up with the histogram in Figure 1 (you work with the actual numbers, but they are not important for the discussion to follow).[1]

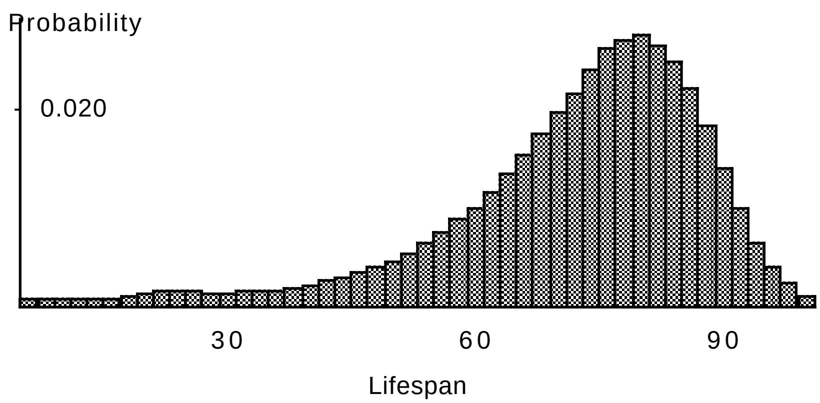


Figure 1

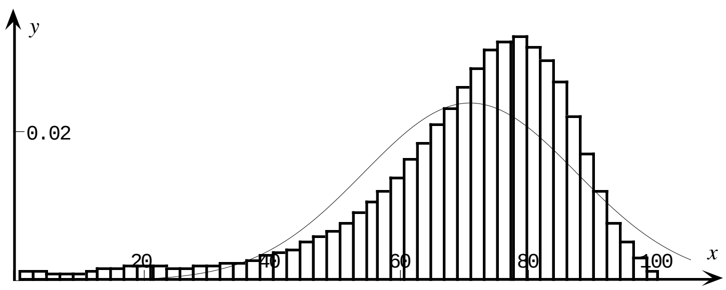
From the data you calculate that the mean is $\mu = 70.778$ and the standard deviation is $\sigma = 16.5119$.

Next, you decide to model these data with a suitable probability density function. You rule out the uniform and exponential density functions, since they have the wrong shape, and you first try the normal distribution. The normal distribution is

[1] Probabilities are normalized (scaled) so that the total area of the histogram is one square unit. Thus the area (not the height) of each bar is the probability of mortality in a two-year period. The data on which the histogram is based were obtained from the 1980 Standard Ordinary Mortality Table, Male Lives (Source: Black/Skipper, *Life Insurance*, Eleventh Edition (Englewood Cliffs, NJ: Prentice-Hall, Inc., 1987), p. 314).

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$
$$= \frac{1}{16.5119\sqrt{2\pi}}\ e^{-\frac{(x-70.778)^2}{2(16.5119)^2}}\ .$$

Figure 2 shows the graph of the normal density function superimposed on the actual data.



Normal Density Function
($\mu = 70.778$, $\sigma = 16.5119$)
Figure 2

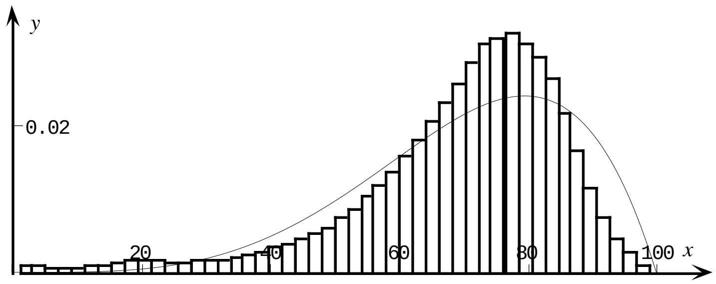
This does not seem like a very good fit at all! Since the actual histogram looks as though it is "pushed over" to the right, you think of the beta distribution, which has that general shape. The beta distribution is given by

$$f(x) = (\beta+1)(\beta+2)x^{\beta}(1-x),$$

where β can be obtained from the mean μ using the equation

$$\mu = \frac{\beta+1}{\beta+3}\ .$$

There is one catch: the beta distribution assumes that X is between 0 and 1, whereas your distribution is between 0 and 100. This doesn't deter you: all you need to do is scale the X-values to $^1/_{100}$ of their original value. Thus you substitute $\mu = {}^1/_{100}(70.778) = 0.70778$ in the above equation and solve for β, you obtain $\beta = 3.8442$. You then plot the associated beta function (after scaling it to fit the range $0 \le X \le 100$) and again discover that, although better, the fit still leaves something to be desired (Figure 3).



Beta Density Function
(β = 3.8442)
Figure 3

Now you just want some function that fits the data. You turn to your statistical software and ask it to find the cubic equation that best fits the data using the least squares method. It promptly tells you that the cubic function that best fits the data is

$$f(x) = ax^3 + bx^2 + cx + d,$$

where

$a = -3.815484 \times 10^{-7}$
$b = 5.7399145 \times 10^{-5}$
$c = -0.0020856085$
$d = 0.0190315095.$

Its graph is shown in Figure 4. Note that the curve, although erratic for small values of X, fits the large peak on the right more closely than the others.

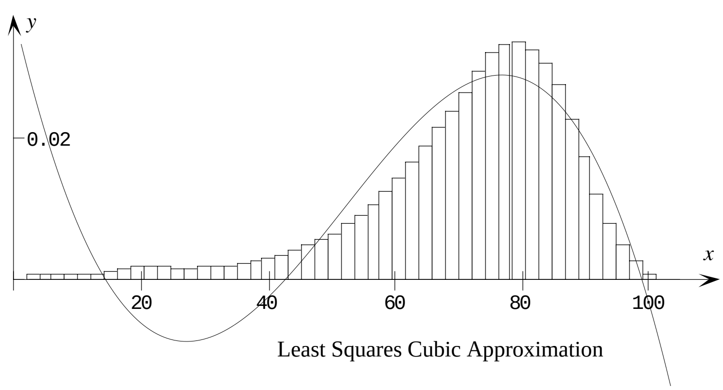


Figure 4

Encouraged, you use the same software to obtain a quartic (degree 4) approximation, and you find:

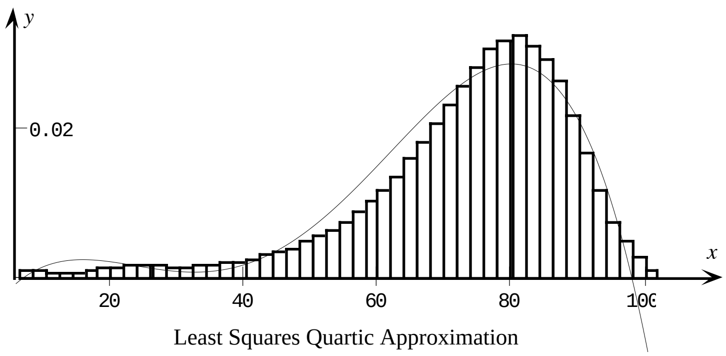
$$f(x) = ax^4 + bx^3 + cx^2 + dx + e,$$

where

$a = -\ 9.583507 \times 10^{-9}$
$b = 1.650155 \times 10^{-6}$
$c = -8.523081 \times 10^{-5}$

$$d = 0.0016190575$$
$$e = -0.007865381.$$

Figure 5 shows the result.



Least Squares Quartic Approximation

Figure 5

This seems like the best fit of them all—especially for the range of X you are interested in: $22 \le X \le 95$. (Malcolm is 22 years old and the trust will mature at 95.)

Now that you have the density function you wish to use, you use it to find the expected future value of an annuity into which monthly payments are made. The simplest formula for the future value V of an annuity is

$$V = 12P\left(\frac{\left(1 + \frac{i}{12}\right)^{12n} - 1}{i}\right).$$

(This is a standard formula from finance. This formula assumes that interest is paid at the end of each month.) Here, P is the monthly payment—the quantity that Malcolm wants to know—i is the interest rate, and n is the number of years for which payments are made. Since Malcolm will be making investments starting at age 22, this means that $n = x - 22$, so the future value of his annuity at age x is

$$V(x) = 12P\left(\frac{\left(1 + \frac{i}{12}\right)^{12(x-22)} - 1}{i}\right).$$

As for the interest rate i, you decide to use a conservative estimate of 5%.

Now the expected value of $V(X)$ is given by

$$E(V) = \int_{22}^{95} V(x)f(x)\,dx \quad ,$$

where $f(x)$ is the quartic approximation to the distribution function. Since Malcolm wants this to be \$500,000, you set

$$500{,}000 = \int_{22}^{95} V(x)f(x)\,dx$$

$$= \int_{22}^{95} 12P\left(\frac{\left(1+\frac{i}{12}\right)^{12(x-22)}-1}{i}\right)f(x)\,dx$$

$$= P\int_{22}^{95} 12\left(\frac{\left(1+\frac{0.05}{12}\right)^{12(x-22)}-1}{0.05}\right)f(x)\,dx \quad .$$

Solving for P,

$$P = \frac{500{,}000}{\int_{22}^{95} 12\left(\frac{\left(1+\frac{0.05}{12}\right)^{12(x-22)}-1}{0.05}\right)f(x)\,dx} \quad .$$

You now calculate the integral numerically (using the quartic approximation to $f(x)$), obtaining

$$P \approx \frac{500{,}000}{3{,}409.8019} \approx \$146.64 \text{ per month.}$$

The next day, you can tell Malcolm that at a 5% interest rate, his family can expect the trust to be worth $500,000 upon maturity if he deposits $146.64 each month.

Exercises

1. How much smaller will the payments be if the interest rate is 6%?
2. How much larger would the payments be if Malcolm began payments at the age of 30?
3. Repeat the original calculation using the normal distribution described above. Give reasons for the discrepancy between the answers, explaining why your answer is smaller or larger than the one calculated above.
4. Repeat Exercise 3 using the cubic distribution.
5. If Malcolm wanted to terminate the trust at age 65, which model would you use for the probability density? Give reasons for your choice.
6. Suppose you were told by your superior that, since the expected male lifespan is 71, you could have saved yourself a lot of trouble by using the formula for the future value of an annuity maturing at age 71. Based on that, the payment comes out to about $198 per month. Why is it higher? Why is it the wrong amount?

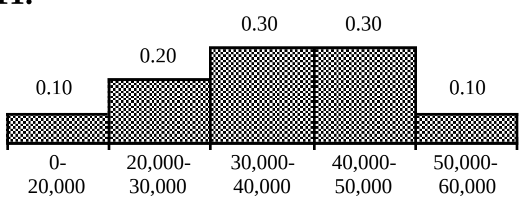
7. Explain how an insurance company might use the above calculations to compute life insurance premiums.

Answers to Odd-Numbered Exercises

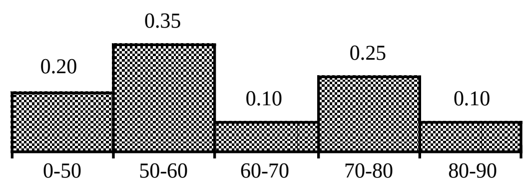
P.1

1. X is the number of the uppermost face; discrete **3.** X is the angle the pointer makes with the vertical; continuous with interval of values [0, 360). **5.** X is the temperature at midday; continuous. There are many possible intervals of values, such as $(-\infty, +\infty)$, $[-1{,}000, 1000]$, or $[-150, 150]$ (degrees Fahrenheit) which would be reasonable on earth. **7.** X is the US Balance of Payments, rounded to the nearest billion dollars; discrete. **9.** X is the number of computer chips that fail to work in a batch of 100; discrete.

11.



13.



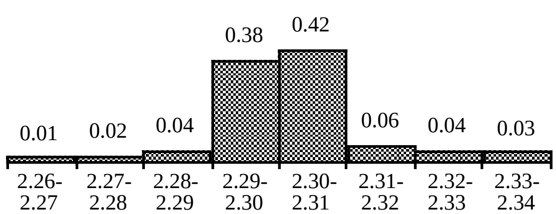
15.

Age	0-15	15-25	25-35	35-45	45-55	55-65	65-75	75-95
Probability	0.2077	0.1199	0.1118	0.1443	0.1317	0.1317	0.0959	0.0570

(a) 0.5077 **(b)** 0.5837 **(c)** 0.4163

17. (a) 0.304 **(b)** 0.083 **(c)** 0.237

19.



21. A random variable assigns a number to each outcome in an experiment. **23.** It is half the corresponding area.

P.2

1. Yes **3.** No; the integral $\neq 1$ **5.** No; the function is not ≥ 0 **7.** Yes **9.** Yes **11.** No; both conditions fail **13.** 1/4 **15.** ln 2 **17.** Exponential **19.** Normal **21.** Uniform **23.** Beta **25.** Exponential **27.** 0.2 **29.** 0.5934 **31.** 0.6164 **33. (a)** 0.000 0010 **(b)** 1 **35.** 0.3829

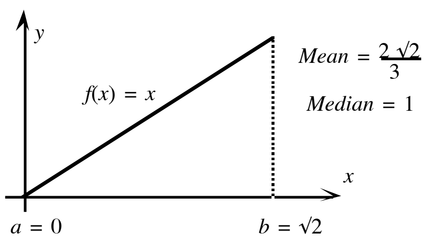
37. 0.01654 **39.** 51.96% **41.** Yes. The probability that a regional Bell had lower operating expenses than SBC was 0.2064. In other words, approximately 21% of the companies should have had lower operating costs than SBC (according to the normal distribution). **43.** By the Fundamental Theorem of Calculus, $F(x)$ as given is an antiderivative of $f(x)$. In other words, $F'(x) = f(x)$, as required. **45.** By definition of $F(x)$, $F(a) = \int_a^a f(t)\,dt$, which is zero because the lower and upper limits agree, and $F(b) = \int_a^b f(t)\,dt$

= 1. **47.** $F(x) = \dfrac{x-10{,}000}{30{,}000}$ **49.** $1-e^{-0.3x}$ **51.** $1-e^{-0.000121x}$ **53.** A probability density function allows us to compute probabilities algebraically using a single function (often specified by a formula) rather than numerically by adding the values of the bars in a histogram. **55.** An example is $f(x) = 3x^2$ on [0, 1]. **57.** The probability associated with a continuos random variable is given by the area under the probability density function curve; $P(a \le X \le b) = \int_a^b f(x)\,dx$. Thus the probability that $X = a$ is $\int_a^a f(x)\,dx = 0$. **61.** If F is the cumulative probability density, then $F(a)$ is the probability that $X \le a$, and not the probability that $X = a$.

§9.3

1. $E(X) = 3/2$, $Var(X) = 3/4$, $\sigma(X) = \sqrt{3}/2$ **3.** $E(X) = 20/3$, $Var(X) = 50/9$, $\sigma(X) = \sqrt{50}/3$ **5.** $E(X) = 3/8$, $Var(X) = 0.05975$, $\sigma(X) = 0.2437$ **7.** $E(X) = 0.3863$, $Var(X) = 0.03909$, $\sigma(X) = 0.1977$ **9.** $E(X) = 10$, $Var(X) = 100$, $\sigma(X) = 10$ **11.** $E(X) = -33.3333$, $Var(X) = 1111.1111$, $\sigma(X) = 33.3333$ **13.** $E(X) = 3/2$, $Var(X) = 3/4$, $\sigma(X) = \sqrt{3}/2$ **15.** $E(X) = 1$, $Var(X) = 1$, $\sigma(X) = 1$ **17.** $E(X) = 0.4286$, $Var(X) = 0.05442$, $\sigma(X) = 0.2333$ **19.** $E(X) = 0.6774$, $Var(X) = 0.0304$, $\sigma(X) = 0.1742$ **21.** $E(X) = 0.4413$, $Var(X) = 0.07852$, $\sigma(X) = 0.2802$ **23.** $E(X) = 0.8862$, $Var(X) = 0.2146$, $\sigma(X) = 0.4633$ **25.** 2 **27.** 0.2310 **29.** −23.1049 **31.** 0.2929 **33.** 0.25 **35−40.** Proofs **41.** 0.35 **43.** 0.61 **45.** 0.65 **47.** 0.41 **49.** 0.83 **51.** $E(X^2) = 3$, $E(X^2)-E(X)^2 = 3/4$ **52.** $E(X^2) = 1/27$, $E(X^2)-E(X)^2 = 1/108$ **55.** $E(X^2) = 1/5$, $E(X^2)-E(X)^2 = 0.059375$ **57.** $E(X^2) = 0.1883$, $E(X^2)-E(X)^2 = 0.0391$ **59.** $E(X^2) = 200$, $E(X^2) - E(X)^2 = 100$ **61.** Comparing answers suggests that $E(X^2) - E(X)^2 = Var(X)$. Thus, $E(X^2) = E(X)^2 + Var(X)$. **63.** \$25,000 **65.** $3\frac{1}{3}$ months **67.** 8,267 years **69.** 100,000,000 years **71.** 5.3009 **73.** 100.4988 **75.** **(a)** $\beta = 2.5556$, $f(x) = 16.1975x^{2.5556}(1-x)$ **(b)** $M(X) = 0.66$, a little smaller than the mean. This indicates that more students scored below the mean than above it.

77.



79. Missing words: variance, median. **81.** Values of X are more likely to be close to the mode than anywhere else. Thus an interval about the mode determines the most popular values of X.