The Chain Rule for Functions of Several Variables

For functions of a single variable, we saw that, if y is a differentiable function of u and u is a differentiable function of x—say, $y = f(u(x))$ —then the derivative of y with respect to x is given by the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$ Differential form of the chain rule for $y = f(u(x))$

Here is one generalization of the chain rule to a function of two variables: We assume that y is a function of two variables u and v, where in turn u and v are functions of a single variable x. Then, by substitution, we can regard f as a function of x. For instance, let

$$y = u^2 - uv$$

where

$$u = x-2$$

and

$$v = x^2$$

Then substituting gives

$$\begin{aligned} y &= (x-2)^2 - (x-2)(x^2) \\ &= x^2 - 4x + 4 - x^3 + 2x^2 \\ &= -x^3 + 3x^2 - 4x + 4, \end{aligned}$$

a function of x only.

Chain Rule for a Function of Two Variables: y = f(u(x), v(x))

Suppose y is a function of two variables u and v with continuous partial derivatives $\frac{\partial y}{\partial u}$ and $\frac{\partial y}{\partial v}$, and that u and v are differentiable functions of x. Then y is a differentiable function of x, and

$$\frac{dy}{dx} = \frac{\partial y}{\partial u}\frac{du}{dx} + \frac{\partial y}{\partial v}\frac{dv}{dx}$$ Chain rule for $y = f(u(x), v(x))$

Quick Example

With $y = u^2 - uv$, $u = x-2$, and $v = x^2$ as above, we have

$$\frac{\partial y}{\partial u} = 2u - v, \quad \frac{\partial y}{\partial v} = -u, \quad \frac{du}{dx} = 1, \quad \frac{dv}{dx} = 2x$$

Therefore,

$$\frac{dy}{dx} = \frac{\partial y}{\partial u}\frac{du}{dx} + \frac{\partial y}{\partial v}\frac{dv}{dx}$$
$$= (2u-v)(1) + (-u)(2x) = 2u - v - 2ux$$

Substituting for u and v now gives $\frac{dy}{dx}$ as a function of the independent variable x:

$$\frac{dy}{dx} = 2(x-2) - x^2 - 2(x-2)x = -3x^2 + 6x - 4$$

This is the same answer we would get by differentiating $y = -x^3 + 3x^2 - 4x + 4$ directly.

Question What is the point of using the chain rule when we can find the same answer by direct substitution?

Answer When calculating derivatives we can usually use substitution rather than the chain rule, but using the chain rule is often algebraically less complicated, Also, we will give theoretical applications of the chain rule below.

Graphical Representation of Chain Rule

So far, we have looked at only one specific instance of the chain rule: y a function of u and v, which in turn are functions of x. What if y was a function of three or more variables, and what if each of those was a function of two variables? To clarify things, let us represent the situation we were given pictorially as in Figure 1.

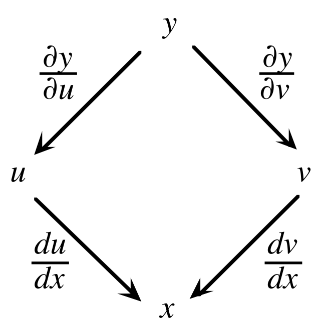

Figure 1

On top is y, from which arrows point to u and v, indicating that y is a function of u and v. On the two arrows we find the associated (partial) derivatives. Similarly, u is a function of x, and so there is an arrow pointing from u to x, along with the associated derivative. This diagram helps us write down the chain rule as follows: To get $\frac{dy}{dx}$:

1. Find all possible paths from y to x and multiply the (partial) derivatives along each path (Figure 2).

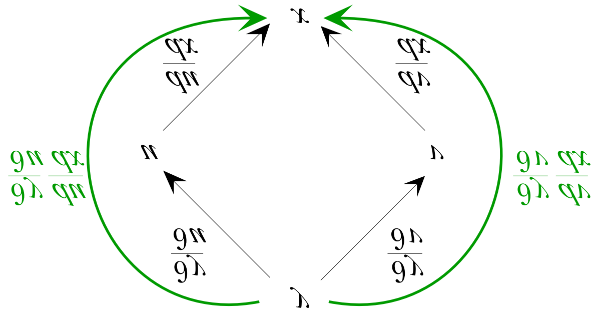

Figure 2

2. Now add all the products obtained in Step 1:

$$\frac{dy}{dx} = \frac{\partial y}{\partial u}\frac{du}{dx} + \frac{\partial y}{\partial v}\frac{dv}{dx}$$

Let us use this method to find the chain rule for a function of three variables.

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Example 1 Function of 3 Variables: y = f(u(t), v(t), w(t))

a. Obtain the chain rule and use it to find a formula for $\frac{dy}{dt}$ in the situation where y is a function of u, v, and w, each of which in turn is a function of t. (In symbols, $y = f(u(t), v(t), w(t))$.) Assume that all the (partial) derivatives you need exist and are continuous.

b. Apply the chain rule formula from part (a) to compute $\left.\frac{dy}{dt}\right|_{t=1}$ if $y = u^2 + v^2 + w^2$ where $u = t$, $v = t^2$, and $w = t^3$.

Solution

a. We are told that y is a function of u, v, and w, each of which in turn is a function of t. This leads to Figure 3.

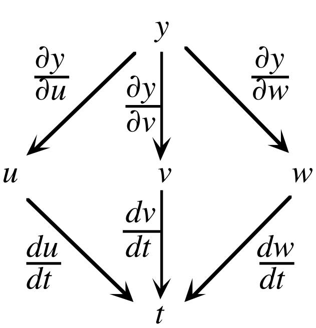


Figure 3

There are now three paths from y down to t, so we need to compute three products:

$$\frac{\partial y}{\partial u}\frac{du}{dt}, \quad \frac{\partial y}{\partial v}\frac{dv}{dt} \quad \text{and} \quad \frac{\partial y}{\partial w}\frac{dw}{dt}.$$

Adding these products gives us the desired chain rule:

$$\frac{dy}{dt} = \frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt} + \frac{\partial y}{\partial w}\frac{dw}{dt}$$

Chain rule for $y = f(u(t), v(t), w(t))$

b. We need to compute all the derivatives that occur in the figure:

$$y = u^2 + v^2 + w^2,$$

so

$$\frac{\partial y}{\partial u} = 2u, \quad \frac{\partial y}{\partial v} = 2v, \quad \frac{\partial y}{\partial w} = 2w.$$

Also,

$$u = t \Rightarrow \frac{du}{dt} = 1$$

$$v = t^2 \Rightarrow \frac{dv}{dt} = 2t$$

$$w = t^3 \Rightarrow \frac{dw}{dt} = 3t^2$$

Therefore,

$$\frac{dy}{dt} = \frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt} + \frac{\partial y}{\partial w}\frac{dw}{dt}$$

$$= (2u)(1) + (2v)(2t) + (2w)(3t^2) = 2u + 4vt + 6wt^2$$

To compute $\left.\frac{dy}{dt}\right|_{t=1}$ we must evaluate every term at $t = 1$, which we do by substitution:

$$t = 1 \Rightarrow u = t = 1, \quad v = t^2 = 1^2 = 1, \text{ and } w = t^3 = 1^3 = 1,$$

so

$$\left.\frac{dy}{dt}\right|_{t=1} = 2u + 4vt + 6wt^2$$

$$= 2(1) + 4(1)(1) + 6(1)(1^2) = 12.$$

Example 2 Computing Partial Derivatives: $z = f(x(u, v), y(u, v))$

Suppose that z is a function of x and y, and each of these is in turn a function of u and v. (In symbols, $z = f(x(u, v), y(u, v))$.) Assuming that all partial derivatives exist and are continuous, obtain formulas for $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

Solution The given information leads to Figure 4.

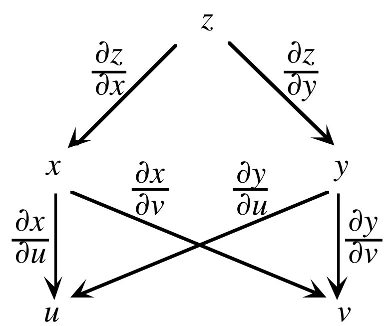


Figure 4

To compute $\frac{\partial z}{\partial u}$, notice that there are two paths in the figure from z to u; one via x, and the other via y. This gives

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}.$$

Similarly, following the two paths from z to v gives

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

Example 3 Computing Derivatives: z = f(x, y(x))

a. Suppose that z is a function of y and x, and that y is, in turn, a function of x. Assume that all (partial) derivatives exist and are continuous. Find a formula for $\frac{dz}{dx}$.

b. Suppose $z = x^2 + y^2$, where $y = e^{3x}$. Compute $\frac{dz}{dx}$.

Solution

a. The picture for the given situation is shown in Figure 5.

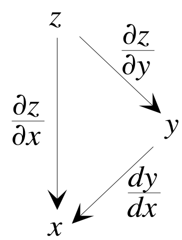


Figure 5

Notice that the figure shows that z is, indirectly, a function of x only. We write $z = f(x, y(x))$. We are asked to find the derivative of this function of x. From the diagram we have

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}.$$

b. $z = x^2 + y^2 \Rightarrow \frac{\partial z}{\partial x} = 2x$ and $\frac{\partial z}{\partial y} = 2y$

$$y = e^{3x} \Rightarrow \frac{dy}{dx} = 3e^{3x}.$$

Therefore,

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$
$$= 2x + 2y(3e^{3x}) = 2x + 6ye^{3x}.$$

Substituting for y allows us to express the answer in terms of x alone:

$$\frac{dz}{dx} = 2x + 6(e^{3x})e^{3x} = 2x + 6e^{6x},$$

Before we go on... If we take the information in part (b) of Example 3 and substitute directly for y as a function of x, we obtain

$$z = x^2 + y^2 = x^2 + (e^{3x})^2 = x^2 + e^{6x}$$

Therefore,

$$\frac{dz}{dx} = 2x + 6e^{6x},$$

the same answer we found using the chain rule.

In Example 3, note the difference between $\frac{\partial z}{\partial x}$ and $\frac{dz}{dx}$. $\frac{\partial z}{\partial x}$ is the partial derivative of $z = f(x, y)$ with respect to the first variable, x. On the other hand, $\frac{dz}{dx}$ is the derivative of the function $z = f(x, y(x))$ obtained by substituting for y as a function of x. In part (b) of the example, $\frac{\partial z}{\partial x} = 2x$ while $\frac{dz}{dx} = 2x + 6e^{6x}$. In this situation, we call $\frac{dz}{dx}$ the **total derivative** of z.

Elsewhere we saw how to compute $\frac{dy}{dx}$ when y is an implicit function of x. In the next example we consider an alternative (and quicker) method using the chain rule as formulated in Example 3.

Example 4 Using the Chain Rule for Implicit Differentiation

Suppose $3xy - e^{xy} = 5$. Find $\frac{dy}{dx}$.

Solution Write $z = 3xy - e^{xy}$. So, z is a function of x and y, where y in turn is a function of x: exactly the situation in Example 3.

Since z is constant (it equals 5) we know that

$$\frac{dz}{dx} = 0$$

However, the chain rule (Example 3) tells us that

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

$$0 = (3y - ye^{xy}) + (3x - xe^{xy})\frac{dy}{dx}$$

Solving for $\frac{dy}{dx}$ now gives

$$\frac{dy}{dx} = -\frac{3y - ye^{xy}}{3x - xe^{xy}}.$$

Chain Rule in Differential Form; Estimating Change

Let us go back for a moment to our first scenario: $y = f(u(x), v(x))$, so that

$$\frac{dy}{dx} = \frac{\partial y}{\partial u}\frac{du}{dx} + \frac{\partial y}{\partial v}\frac{dv}{dx}$$

Each of the derivatives in the above expression is actually the limit of a ratio. For instance,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x}, \qquad \frac{du}{dx} = \lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x}, \qquad \frac{dv}{dx} = \lim_{\Delta x \to 0}\frac{\Delta v}{\Delta x}$$

Therefore, for small Δx, we have

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}, \qquad \frac{du}{dx} \approx \frac{\Delta u}{\Delta x}, \qquad \frac{dv}{dx} \approx \frac{\Delta v}{\Delta x}$$

Substituting in the chain rule gives

$$\frac{\Delta y}{\Delta x} \approx \frac{\partial y}{\partial u}\frac{\Delta u}{\Delta x} + \frac{\partial y}{\partial v}\frac{\Delta v}{\Delta x}$$ For small Δx

or $$\Delta y \approx \frac{\partial y}{\partial u}\Delta u + \frac{\partial y}{\partial v}\Delta v$$ Multiply both sides by Δx

where Δy Δu, and Δv are the changes in y, u, and v produced by a small change Δx in x. Since u and v can be arbitrary functions of x, we can think of Δu and Δv as arbitrary changes in u and v respectively, and ignore the variable x.

Summarizing, we have the following.

Total Difference and Differential: y = f(u, v)

Suppose y is a function of two variables u and v with continuous partial derivatives $\frac{\partial y}{\partial u}$ and $\frac{\partial y}{\partial v}$. Then small changes Δu in u and Δv in v change y by the amount

$$\Delta y \approx \frac{\partial y}{\partial u}\Delta u + \frac{\partial y}{\partial v}\Delta v$$ Chain rule: Difference Form

We sometimes express the above approximation as

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv,$$ Chain rule: Differential Form

called the **total differential** of y.

Quick Example

If $y = u^2 + v^2$ and u and v are changed by small amounts Δu and Δv respectively, then the resulting change in y is given by

$$\Delta y \approx \frac{\partial y}{\partial u}\Delta u + \frac{\partial y}{\partial v}\Delta v$$
$$= 2u\Delta u + 2v\Delta v$$

We can use the total difference to estimate errors:

Example 5 Estimating the Error

The volume V of a right circular cone of height h and cross-sectional radius r at its base is given by $V = \frac{1}{3}\pi r^2 h$. Precision Cones Inc. manufactures little steel cones with a height of 3 mm and base of radius 0.5 mm. The error tolerance is ±0.001 mm for the radius of the base, and ±0.002 mm for the height. By how much can the volume vary?

Solution Since both the radius r and height h are changing, we think of V as a function of r and h: $V = f(r, h)$. The chain rule in difference form says

$$\begin{aligned}\Delta V &= \frac{\partial V}{\partial r}\Delta r + \frac{\partial V}{\partial h}\Delta h \\ &= \tfrac{2}{3}\pi rh\Delta r + \tfrac{1}{3}\pi r^2\Delta h = \tfrac{1}{3}\pi r[2h\Delta r + r\Delta h]\end{aligned}$$

Substituting $r = 0.5,\ h = 3,\ \Delta r = 0.001,\ \Delta h = 0.002$ gives

$$\Delta V = \tfrac{1}{3}\pi(0.5)[2(3)(0.001) + (0.5)(0.002)] = \frac{0.007\pi}{6} \approx 0.0037\text{mm}^3$$

Hence, the volume of a cone can vary by as much as 0.0037 mm^3.

Before we go on... The result in Example 5 gives the *largest possible* variation in the volume of the cones. The "correct" volume is

$$V = \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi(0.5)^2(3) = \frac{\pi}{4} \approx 0.7854\ \text{mm}^3.$$

Since the maximum possible variation is 0.0037 mm^3, the volume of the cones can vary from around $0.7854-0.0037 = 0.7817\ \text{mm}^3$ to $0.7854+0.0037 = 0.7891\ \text{mm}^3$.

Investigators are sometimes more interested in the *expected* variation—that is, the average variation in a large number of cones—than in the *maximum* variation we computed above. Formulas for the computation of expected variation come from formulas in statistics, and will not be discussed here.

Notes

1. Notice that we need not assume, in writing the expression

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$$

that u and v are differentiable—or even continuous—functions of x. We shall see that removing this assumption will prove useful in applications (see the *Before we go on* discussion following Example 6).

2. The total differential form

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$$

can be obtained from the chain rule

$$\frac{dy}{dx} = \frac{\partial y}{\partial u}\frac{du}{dx} + \frac{\partial y}{\partial v}\frac{dv}{dx}$$

by formally "multiplying both sides by dx" and then canceling the terms dx, even though dx is not a real number:

$$\frac{dy}{\cancel{dx}}\cancel{dx} = \frac{\partial y}{\partial u}\frac{du}{\cancel{dx}}\cancel{dx} + \frac{\partial y}{\partial v}\frac{dv}{\cancel{dx}}\cancel{dx}$$

This formal technique allows us to obtain the total differential quickly in other situations, as the next example shows.

Example 6 Total Differential for z = f(x, y(x))

a. Find an expression for the total differential dz for $z = f(x, y(x))$.
b. The value of a stock in dollars is modeled by

$$v = 25 + 0.3t - 0.8ts$$

where t is time in years and s is an unspecified function of t. Currently, $t = 1$ and $s = 0.4$. What is the current value of the stock? If s increases by 0.05 and t increases by 0.02, estimate the change in the value of the stock.

Solution

a. In Example 3 we saw that the chain rule in the situation $z = f(x, y(x))$ takes the form

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

To obtain the differential form, multiply both sides by dx to obtain

$$\frac{dz}{dx}\cancel{dx} = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}\frac{dy}{\cancel{dx}}\cancel{dx}$$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

b. The current value of the stock is given by

$$\begin{aligned} v &= 25 + 0.3t - 0.8ts \\ &= 25 + 0.3(1) - 0.8(1)(0.4) = \$24.98. \end{aligned}$$

To obtain the differential form of the chain rule, notice that here, $v = f(t, s(t))$. So, we can use part (a) and rename the variables to obtain

$$dv = \frac{\partial v}{\partial t}\, dt + \frac{\partial v}{\partial s}\, ds$$

where

$$\frac{\partial v}{\partial t} = 0.3 - 0.8s, \quad \frac{\partial v}{\partial s} = -0.8t, \quad ds = 0.05, \quad dt = 0.02$$

giving

$$\begin{aligned} dv &= (0.3-0.8s)dt - 0.8t\, ds \\ &= (0.3-0.8(0.4))(0.02) - 0.8(1)(0.05) \\ &= -0.0404 \approx -\$0.04 \end{aligned}$$

Thus, the value of the stock will decline by approximately $0.04.

Before we go on... Notice that, in Example 6, we do not require s to be a differentiable function of t (see the notes just before the example). In particular, we could take s to be a random function of t satisfying certain prescribed conditions, such as specifying that s is normally distributed. In this way, we can model a process, such as the movement of a stock price, that includes random change.

Chain Rule Exercises

In Exercises 1–8, write down the appropriate chain rule formula for the stated derivative.

1. $z = f(x(t), y(t))$; $\dfrac{dz}{dt}$
2. $r = f(x(s), y(s))$; $\dfrac{dr}{ds}$
3. $y = f(x_1(t), x_2(t), x_3(t))$; $\dfrac{dy}{dt}$
4. $y = f(x(t_1, t_2), y(t_1, t_2))$; $\dfrac{\partial y}{\partial t_1}$
5. $z = f(x_1(s, t), x_2(s, t), x_3(s, t))$; $\dfrac{\partial z}{\partial t}$
6. $z = f(x_1(t), x_2(t), \ldots, x_n(t))$; $\dfrac{dz}{dt}$
7. $y = f(x, u(x), v(x))$; $\dfrac{dy}{dx}$
8. $x = f(s, t, r(s, t))$; $\dfrac{\partial x}{\partial t}$

In Exercises 9–14, use the chain rule to find the indicated derivative as a function of the independent variable(s).

9. $y = uv,\ u = x+1,\ v = x^2-2;\ \dfrac{dy}{dx}$

10. $y = uv - u,\ u = x,\ v = x^2-2;\ \dfrac{dy}{dx}$

11. $z = xy + x^2,\ x = e^t,\ y = e^{-t};\ \left.\dfrac{dz}{dt}\right|_{t=0}$

12. $z = \dfrac{x}{y},\ x = e^t,\ y = 1 + e^t;\ \left.\dfrac{dz}{dt}\right|_{t=0}$

13. $y = t^2 - 4tx,\ x = \dfrac{t+1}{t-1};\ \dfrac{dy}{dt}$

14. $y = t^2x^2 + x \ln t,\ x = t-1;\ \dfrac{dy}{dt}$

In Exercises 15–20, use the method of Example 4 to obtain $\dfrac{dy}{dx}$.

15. $x^2y - y^2 = 4$
16. $xy^2 - y = x$
17. $xe^y - ye^x = 1$
18. $x^2e^y - y^2 = e^x$
19. $\ln(y^2-y) + x = y$
20. $\ln(xy) - x \ln y = y$

21. Let $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_3$ each be a function of x_1, x_2, and x_3. Assume that x_1, x_2, and x_3 are, in turn, functions of t. Find a formula for each of $d\bar{x}_i/dt$ $(i = 1, 2, 3)$ in terms of the quantities $\partial\bar{x}_i/\partial x_j$ and dx_i/dt.

22. Let x_1, x_2, and x_3 each be a function of $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_3$. Assume that ϕ is a function of x_1, x_2, and x_3. Find a formula for each of $\partial\phi/\partial\bar{x}_i$ $(i = 1, 2, 3)$ in terms of the quantities $\partial x_i/\partial\bar{x}_j$ and $\partial\phi/\partial x_i$.

Δ Exercises 23–28 assume a knowledge of trigonometric functions.

23. Let $z = xy + x^2$, and $x = \cos t,\ y = \sin t$. Find $\dfrac{dz}{dt}$ when $t = 2\pi$

24. Let $z = \sin(xy)$, and $x = s + t,\ y = s^2 + t^2$. Find $\dfrac{\partial z}{\partial s}$ when $(s, t) = (0, \pi)$

25. Let $\phi = x^2 + y^2 + z^2$, where $x = \rho \sin u \cos v$, $y = \rho \sin u \sin v$, $z = \rho \cos u$. Use the chain rule to show that $\partial\phi/\partial u = 0$.

26. Let $z = 4y^2 + x$, where $y = \sinh x$. Compute $\partial z/\partial x$ and dz/dx.

27. *Laplacian in Polar Coordinates* Suppose that $F = F(x, y)$. Use the transformation equations $x = r\cos\theta$, $y = r\sin\theta$ and the chain rule to prove that

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}.$$

28. The coordinates of a point on a torus (a donut-shaped surface) are given by:

$x = (a + b\cos\phi)\cos\theta,\ y = (a + b\cos\phi)\sin\theta,$ and $z = b\sin\phi$

where

$0 \le \theta,\ \phi \le 2\pi$

Given that $\theta = \phi = t$, find $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$ when $t = \pi$.

Applications

29. *Precision* Precision Cylinders Inc. manufactures little steel cylinders with a height of 2 mm and base of radius 0.05 mm. The maximum variation in the radius of the base is ±0.002 mm and the height can vary by ±0.01 mm. By how much can the volume vary? (The volume of a cylinder is $V = \pi r^2 h$.)

30. *Precision* Precision Parallelepipeds Inc. (a subsidiary of Precision Cylinders, Inc.) manufactures little rectangular parallelepipeds with dimensions 3 mm × 4 mm × 1 mm. The three dimensions can vary, respectively, by up to ±0.001 mm, ±0.002 mm, and ±0.0001 mm. By how much can the volume vary?

31. *Resource Allocation* You manage an ice cream factory that makes two flavors: Creamy Vanilla and Continental Mocha. Your profit on x quarts of vanilla and y quarts of mocha is $P(x,y) = 3x + 2y - 0.01(x^2+y^2)$ dollars. Your current production level is 150 quarts of vanilla and 100 quarts of mocha per day. What is the effect on profit of reducing vanilla production by 2 quarts and increasing mocha production by 2 quarts?

32. *Resource Allocation* Repeat the preceding exercise using the profit function $P(x,y) = 3x + 2y - 0.005(x^2+y^2)$.

33. *Investing* The value of the Amex Gold BUGS[1] Index in 2003–2004 can be approximated by

$$b(t) \approx 150 - 14.5t + 1.8t^2 + s\sqrt{t} \qquad (0 \le t \le 8)$$

where t is the time in months since January 2003, and s is a function of t that varies randomly.

[1] BUGS stands for "basket of unhedged gold stocks."

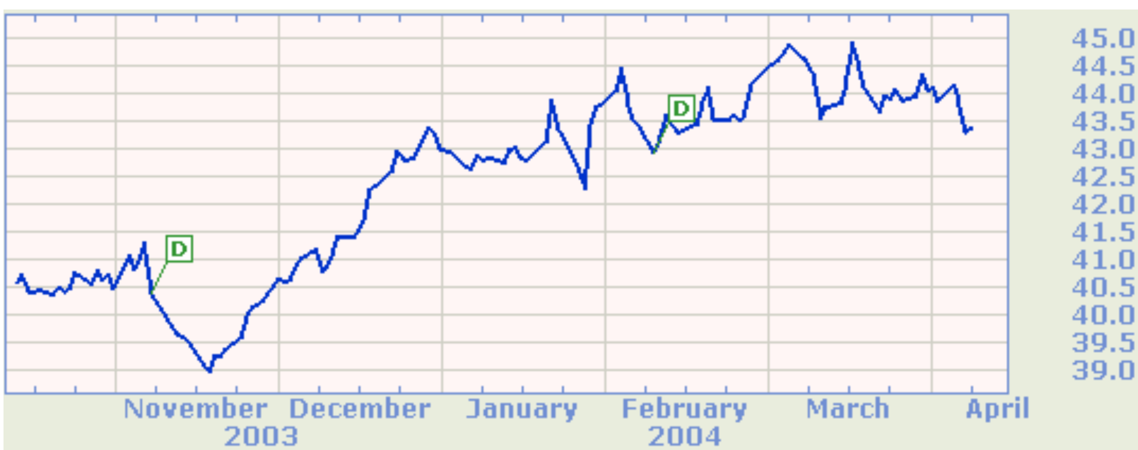
Source: http://www.amex.com
<EXCR.33>

Currently, $t = 4$ and $s = 1.5$. Predict the approximate value of b at $t = 4.1$ assuming that s will decrease by 0.5.

34. ***Investing*** The price of Consolidated Edison common stock (ED) from December 1, 2003 to April 1, 2004 can be approximated by

$$p(t) \approx 40.5 + 3t - 0.55t^2s^2 \quad (0 \le t \le 4)$$

where t as time in months since December 1 2003, and s is a function of t that varies randomly.



Source: http://money.excite.com
<EXCR.34>

It is now February 1, 2003 and $s = -2$. Predict the approximate price of ED stock 0.01 month later assuming that the value of s will increase by 1.5.

35. ***Production*** The automobile assembly plant you manage has a Cobb-Douglas production function given by

$$P = 10x^{0.3}y^{0.7}$$

where P is the number of automobiles it produces per year, x is the number of employees, and y is the daily operating budget (in dollars). You maintain a production level of 1000 automobiles per year. If you currently employ 150 workers and are hiring new workers at a rate of 10 per year, how fast is your daily operating budget changing?

36. ***Production*** Refer back to the Cobb-Douglas production formula in the preceding exercise. Assume that you maintain a constant work force of 200 workers and wish to increase production in order to meet a demand that is increasing by 100 automobiles per year. The current demand is 1000 automobiles per year. How fast should your daily operating budget be increasing?

37. ***Employment*** An employment research company estimates that the value of a recent MBA graduate to an accounting company is

$$V = 3e^2 + 5g^3$$

where V is the value of the graduate, e is the number of years of prior business experience, and g is the graduate school grade point average. A company that currently employs graduates with a 3.0 average wishes to maintain a constant employee value of $V = 200$ but finds that the grade point average of its new employees is dropping at a rate of 0.2 per year. How fast must the experience of its new employees be growing in order to compensate for the decline in grade point average?

38. ***Grades***[2] A production formula for a student's performance on a difficult English examination is given by

$$g = 4hx - 0.2h^2 - 10x^2$$

where g is the grade the student can expect to obtain, h is the number of hours of study for the examination, and x is the student's grade point average. The instructor finds that students' grade point averages have remained constant at 3.0 over the years, and that students currently spend an average of 15 hours studying for the examination. However, scores on the examination are dropping at a rate of 10 points per year. At what rate is the average study time decreasing?

Communication & Reasoning Exercises

39. Your friend Etrenam is annoyed at having to do homework on the chain rule, and says "The discussion at the start of this section shows that the chain rule is not

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[2] Based on an Exercise in *Introduction to Mathematical Economics* by A.L. Ostrosky Jr. and J.V. Koch (Waveland Press, Illinois, 1979.)

necessary: Just substitute and then take the derivatives directly." Comment on his reasoning.

40. Your other friend Imogen is telling everyone that you can compute the approximate change in any function of several variables, even if you don't know the exact form of *any* of the variables—all you need to know are their values and their changes. Comment on her claim.

41. Obtain the formula

$$f(x, y) = f(a, b) + \left.\frac{\partial f}{\partial x}\right|_{(a,b)}(x-a) + \left.\frac{\partial f}{\partial y}\right|_{(a,b)}(y-b)$$

for the approximate value of a function $f(x, y)$ near the point (a, b), assuming that we know its value at (a, b), and also the value of its partial derivatives at (a, b).

42. The formula in the preceding exercise is called the *linear approximation of f near (a, b)*. If f is a function of one variable x only, what is the resulting formula, and how is it related to Taylor's Theorem?

Answers to Odd-Numbered Exercises

1. $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

3. $\frac{dy}{dt} = \frac{\partial y}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial y}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial y}{\partial x_3}\frac{dx_3}{dt}$

5. $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial z}{\partial x_3}\frac{\partial x_3}{\partial t}$

7. $\frac{dy}{dx} = \frac{\partial y}{\partial x} + \frac{\partial y}{\partial u}\frac{du}{dx} + \frac{\partial y}{\partial v}\frac{dv}{dx}$

9. $3x^2+2x-2$

11. 2

13. $2t - 4\frac{t+1}{t-1} + \frac{8t}{(t-1)^2}$

15. $-2xy/(x^2-2y)$

17. $(ye^x - e^y)/(xe^y-e^x)$

19. $(y-y^2)/(-1+3y-y^2)$

21. $\frac{d\bar{x}_1}{dt} = \frac{\partial \bar{x}_1}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \bar{x}_1}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial \bar{x}_1}{\partial x_3}\frac{dx_3}{dt}$, $\frac{d\bar{x}_2}{dt} = \frac{\partial \bar{x}_2}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \bar{x}_2}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial \bar{x}_2}{\partial x_3}\frac{dx_3}{dt}$,
$\frac{d\bar{x}_3}{dt} = \frac{\partial \bar{x}_3}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \bar{x}_3}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial \bar{x}_3}{\partial x_3}\frac{dx_3}{dt}$

23. 1

24. $\pi^2\cos(\pi^3)$

29. $0.00425\pi \approx 0.01335$ mm^3

31. Your profit will be unchanged.

33. 122.8

35. The daily operating budget is dropping at a rate of $2.40 per year.

37. The daily operating budget should be increasing at a rate of $10.61 per year.
Their prior experience must increase at a rate of approximately 0.97 years every year.

39. He is correct as far as the computation of derivatives using the chain rule is concerned, provided we know formulas for all the functions involved. However, the chain rule is also used to estimate change, even when the exact form of the various functions is not known, where direct substitution would be impossible.

41. $\Delta f \approx \left.\frac{\partial f}{\partial x}\right|_{(a,b)}\Delta x + \left.\frac{\partial f}{\partial y}\right|_{(a,b)}\Delta y = \left.\frac{\partial f}{\partial x}\right|_{(a,b)}(x-a) + \left.\frac{\partial f}{\partial y}\right|_{(a,b)}(y-b)$

so $f(x, y) \approx f(a, b) + \left.\frac{\partial f}{\partial x}\right|_{(a,b)}(x-a) + \left.\frac{\partial f}{\partial y}\right|_{(a,b)}(y-b)$