Elementary Topology:
Math 167

Lecture Notes
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1. Set and Relations

A **set** is an undefined (primitive) notion. Roughly, it means a collection of things called **elements.** If a is an element of the set S, we write $a \in S$. If a is not an element of the set S, we write $a \notin S$. Some important sets are:

$\mathbb{Z}$, the set of all integers
$\mathbb{N}$, the set of all natural numbers (including 0)
$\mathbb{Z}^+$, the set of all positive integers
$\mathbb{Q}$, the set of rationals
$\mathbb{R}$, the set of reals
$\mathbb{C}$, the set of complex numbers.

We can describe a set in several ways:
(1) by listing its elements; e.g.. $S = \{6, 66, 666\}$
(2) in the form $\{x \mid P(x)\}$, where $P(x)$ is a predicate in x, for instance

$$S = \{x \mid x \text{ is a real number other than } 6\}$$

or

$$T = \{x \in \mathbb{Z} \mid x \text{ odd}\}.$$

Note: Two sets are equal if they have the same elements. That is,

$$A = B \text{ iff } (x \in A \Leftrightarrow x \in B)$$

Definitions 1.1 Let A and B be sets.
We say that A is **contained in** B , or $A \subset B$ if $x \in A \Rightarrow x \in B$.
$A \cap B$ is the **intersection** of A and B, and is the set $\{x \mid x \in A \text{ and } x \in B\}$.
$A \cup B$ is the **union** of A and B, and is the set $\{x \mid x \in A \text{ or } x \in B\}$.
$\emptyset$ is the **empty set,** $\emptyset = \{x \mid P(x)\}$, where $P(x)$ is any false predicate in x, such as ($x = 3$ and $x \neq 3$).
$A - B$ is the **complement** of B in A, and is the set $\{x \mid x \in A \text{ and } x \notin B\}$.
$A \times B$ is the **cartesian product** of A and B, and is the set of all ordered pairs,

$$\{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Interesting example: If A is any set, then $A \times I$ is the **cylinder on A.**
One can generalize this construction to obtain the **n-fold cartesian product,**

$$\prod_{i=1}^{n} A_i = A_1 \times \ldots \times A_n = \{(a_1, \ldots, a_n) \mid a_i \in A\}.$$

If $\{A_\alpha \mid \alpha \in \Omega\}$ is any collection of sets indexed by Ω, then:

$\bigcap_{\alpha \in \Omega} A_\alpha$ is the **intersection** of the A_α; $\bigcap_{\alpha \in \Omega} A_\alpha = \{x \mid x \in A_\alpha \text{ for all } \alpha \in \Omega\}$.

$\bigcup_{\alpha \in \Omega} A_\alpha$ is the **union** of the A_α; $\bigcup_{\alpha \in \Omega} A_\alpha = \{x \mid x \in A_\alpha \text{ for some } \alpha \in \Omega\}$.

Note: To prove that two sets A and B are equal, we need only prove that $x \in A \Leftrightarrow x \in B$. In other words, we must prove two things:

(a) $A \subset B$ (i.e., $x \in A \Rightarrow x \in B$)

(b) $B \subset A$ (i.e., $x \in B \Rightarrow x \in A$)

Proposition 1.2 *Properties of Sets*

The following hold for any three sets A, B and C and any indexed collection of sets B_α ($\alpha \in \Omega$):

(a) Associativity

$A \cap (B \cap C) = (A \cap B) \cap C \qquad A \cup (B \cup C) = (A \cup B) \cup C$

(b) Commutativity

$A \cap B = B \cap A \qquad A \cup B = B \cup A$

(c) Identity and Idempotent

$A \cup \emptyset = A \qquad A \cap \emptyset = \emptyset$

$A \cup A = A \qquad A \cap A = A$

(d) De Morgan's Laws

$A - (B \cup C) = (A - B) \cap (A - C) \qquad A - (B \cap C) = (A - B) \cup (A - C)$

Fancy Form:

$$A - \bigcup_{\alpha \in \Omega} B_\alpha = \bigcap_{\alpha \in \Omega} (A - B_\alpha) \qquad A - \bigcap_{\alpha \in \Omega} B_\alpha = \bigcup_{\alpha \in \Omega} (A - B_\alpha)$$

(e) Distributive Laws

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \qquad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Fancy Form:

$$A \cap \bigcup_{\alpha \in \Omega} B_\alpha = \bigcup_{\alpha \in \Omega} (A \cap B_\alpha) \qquad A \cup \bigcap_{\alpha \in \Omega} B_\alpha = \bigcap_{\alpha \in \Omega} (A \cup B_\alpha)$$

Proof We prove **(a), (b), (c)** and a bit of **(d)** in class. The rest in the exercise set.

Definition 1.3 A **partitioning** of a set S is a representation of S as a disjoint union of subsets, or **partitions**:

$$S = \bigcup_{\alpha \in \Omega} S_\alpha,$$

where $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$.

Examples 1.4

A. The set $\mathbb{Z}$ can be partitioned into the odd and even integers.

B. Partition $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ into the sets L_x where, for $x \in \mathbb{R}$, $L_x = \{(a, b) \mid a = x\}$ is the vertical line with first coordinate x.

C. The set of $n \times n$ matrices can be partitioned into subsets each of which contains matrices with the same determinant.

Definition 1.5 A **relation** on a set S is a subset R of $S \times S$. If $(a, b) \in R$, we write aRb, and say that ***a* stands in the relation *R* to *b*.**

Examples 1.6

A. Equality on any set A **B.** $\neq$ on any set A

C. $<$ on $\mathbb{Z}$ **D.** $m \approx n$ if $m-n \in 2\mathbb{Z}$, on $\mathbb{Z}$

E. Row equivalence on the set of $m \times n$ matrices

F. Any partitioning of a set S gives one: Define xRy if x and y are in the same partition S_α.

Definition 1.7 An **equivalence relation** on a set S is a relation $\approx$ on S such that, for all a, b and $c \in S$:

(a) $a \approx a$	**Reflexivity**
(b) $a \approx b \Rightarrow b \approx a$	**Symmetry**
(c) $(a \approx b$ and $b \approx c) \Rightarrow a \approx c$	**Transitivity**

Examples 1.8

A. Equality on any set

B. Define a relation on $\mathbb{R}^2$ by $(a, b) \approx (c, d)$ iff $a = c$.

C. Row equivalence on the set of $m \times n$ matrices

D. The cone relation Take the cylinder $A \times I$ and declare

$$(a, t) \approx (b, s) \text{ iff either } t = s = 1 \text{ or } (a, t) = (b, s).$$

E. Any partitioning of S yields an equivalence relation

Definition 1.9 If $\approx$ is an equivalence relation on S, then the **equivalence class** of the element $s \in S$ is the subset

$$[s] = \{t \in S \mid t \approx s\}$$

Examples 1.10

A. The equivalence classes associated with the relation $(a, b) \approx (c, d)$ iff $a = c$ on $\mathbb{R}^2$.

B. The equivalence classes associated with the cone relation above.

C. The equivalence classes in $\mathbb{Z}$ of equivalence mod 2.

Lemma 1.11 ***Equivalence Classes***

Let $\approx$ be any equivalence relation on S. Then

(a) If $s, t \in S$, then $[s] = [t]$ iff $s \approx t$.

(b) Any two equivalence classes are either disjoint or equal

(c) The equivalence classes form a **partition** of the set S: that is, a decomposition of S into a disjoint union of subsets.

Theorem 1.12 ***Equivalence Relations "are" Partitions***

There is a one-to-one correspondence between equivalence relations on a set S and partitions of S. Under this correspondence, an equivalence class corresponds to a set in the partition.

Definition 1.13 If S is a set and $\approx$ is an equivalence relation on it, the **quotient** or **identification** set, $S/\approx$, is defined as the set of equivalence classes.

Examples 1.14

A. Look at $\mathbb{R}^2/\approx$ where $(a, b) \approx (c, d)$ iff $a = c$ on $\mathbb{R}^2$.

B. Finally, the **cone on A**, $\quad CA = A\times I/\approx$

C. A **based set** is just a pair (A, a_0) where A set and $a_0 \in A$ is a "distinguished" **basepoint**. If (A, a_0) and (B, b_0) are based sets, then their **smash product $(A\wedge B,\ *)$** is the set $(A\times B)/\approx$, where we declare $(a_1, b_1) \approx (a_0, b_0)$ if either $a_1 = a_0$ or $b_1 = b_0$. The associated partitioning consists of single points (if neither point is the basepoint) and a single partition consisting of all the rest. We take the basepoint $*$ of $A\wedge B$ to be the equivalence class of (a_0, b_0).

Exercise Set 1

1. Prove Proposition 1.2 (d) and (e)

2. Prove that

$$A\times \bigcup_{\alpha\in\Omega} S_\alpha = \bigcup_{\alpha\in\Omega}(A \times S_\alpha) \quad \text{and} \quad A\times \bigcap_{\alpha\in\Omega} S_\alpha = \bigcap_{\alpha\in\Omega}(A \times S_\alpha)$$

3. Show that, if $X \subset A$ and $Y \subset B$, then $(A\times B)-(X\times Y) = A\times(B-Y) \cup (A-X)\times B$.

4. Let S^1 be the unit circle in the plane; $S^1 = (x, y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$

(a) What does $S^1\times S^1$ look like?

(b) What does $S^1\wedge S^1$ look like (where S^1 is based at $(1, 0)$) ? (Draw pictures.)

5. Give examples of relations on $\mathbb{Z}$ which are:

(a) reflexive and symmetric but not transitive;

(b) transitive and reflexive but not symmetric;

(c) transitive and symmetric but not reflexive.

6. Disjoint Union If A_1 and A_2 are sets, then their **disjoint union $A_1 \coprod A_2$** consists of $A_1\times\{1\} \cup A_2\times\{2\}$. More generally, the disjoint union of an indexed collection A_α ($\alpha \in \Omega$) is given by

$$\coprod_{\alpha\in\Omega} A_\alpha = \bigcup_{\alpha\in\Omega}(A_\alpha\times\{\alpha\})$$

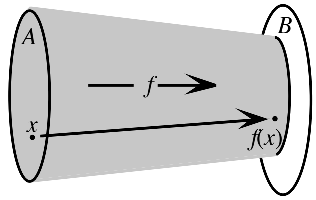
Show that disjoint unions are always disjoint collections.

2. Functions

Definition 2.1 Let A and B be sets. A **map** or **function** f: $A\to B$ is a triple (A, B, f) where f is a subset of $A\times B$ such that for every $a \in A$, there exists a unique $b \in B$ (that is, one and only one $b \in B$) with $(a, b) \in f$. We refer to this element b as $f(a)$. A is called the **domain or source** of f and B is called the **codomain** or **target** of f.

Notes

1. We think of f a rule which assigns to every element of A a unique element $f(a)$ of B, and we can picture a function f: $A\to B$ as follows:



2. The codomain of f is *not* the "range" of f; that is, not every element of B need be of the form $f(x)$.
3. The sets A and B are part of the information of f; specifying f by saying, for instance, $f(x) = 2x-1$ is not sufficient. We should instead say something like this: "Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x-1$."
4. The "x" in $f(x) = \ldots$ is a dummy variable; saying $f(x) = 2x-1$ and $f(a) = 2a-1$ is the *same thing*.

Examples 2.2
Some in class, plus:

A. The **identity map** $1_A\colon A\to A$; $1_A(a) = a$ for every $a \in A$, for any set A

B. If $B \subset A$, then we have the **inclusion map** $\iota\colon B\to A$; $\iota(b) = b$ for all $b \in B$

C. The **empty map** $\varnothing\colon \varnothing\to A$ for any set A.

D. Let $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$ the **unit circle**. Define $f\colon \mathbb{R} \to S^1$ by

$$f(x) = (\cos x, \sin x).$$

In complex form, this is $f(x) = e^{ix}$. The map f is called the **simply connected cover** of the circle (for reasons that will emerge later in the course).

E. Let A and B be sets, and consider the set of *all* functions

$$f\colon \{1, 2\}\to A\cup B$$

with the property that $f(1) \in A$ and $f(2) \in B$. Then this set "looks like" $A\times B$; that is, its elements can be used to represent elements of $A\times B$.

Definition 2.3 Let $f\colon A\to B$ be a map. Then f is **injective** (or **one-to-one**) if

$$f(x) = f(y) \Rightarrow x = y$$

In other words, *if* $x \neq y$, then $f(x)$ cannot equal $f(y)$.
Note This definition gives us the following procedure.

Proving that a Function is Injective To prove that $f\colon X\to Y$ is injective, assume $f(x) = f(y)$, and prove that $x = y$. To prove that f is *not* injective, produce two elements x_1 and x_2 *of* X with $x_1 \neq x_2$ but $f(x_1) = f(x_2)$.

Examples 2.4

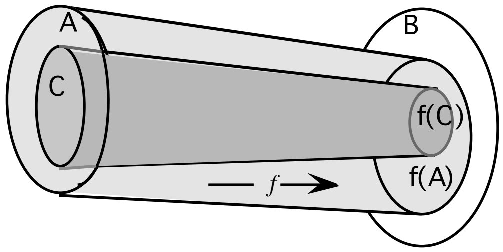
A. $f\colon \mathbb{R} \to \mathbb{R}$; $f(x) = 2x-1$ is injective

B. $f\colon \mathbb{R} \to \mathbb{R}$; $f(x) = x^2 + 1$ is not

C. identity 1_A: $A \to A$ is always injective **D.** inclusion ι: $B \to A$ is always injective
E. The simply connected cover f: $\mathbb{R} \to S^1$ is not injective

Definitions 2.5 Let f: $A \to B$ be a map, and let $C \subset A$. Then the **image of C under f** is the subset

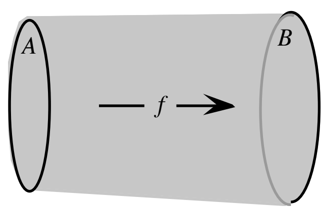
$$f(C) = \{f(c) \mid c \in C\}$$



f is **surjective** (or **onto**) if $f(A) = B$. In other words, given $b \in B$, there exists an $a \in A$ such that $f(a) = b$. That is

$$b \in B \Rightarrow \exists\ a \in A \text{ such that } f(a) = b$$

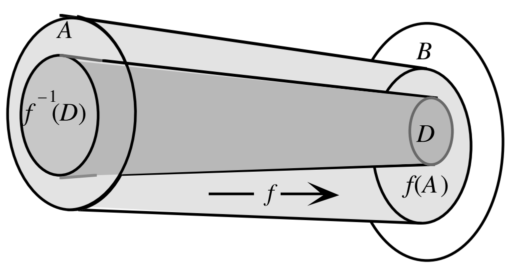
Thus, f "hits" every element in B.



f is surjective iff $f(A) = B$

If $D \subset B$, then its **preimage under f** is the subset

$$f^{-1}(D) = \{a \in A \mid f(a) \in D\}$$



If D consists of a single point $b \in B$, we write $f^{-1}(\{b\})$ as $f^{-1}(b)$.

Notes

1. If f: $A \to B$, then $f(A)$ is sometimes called the **range of f,** or the **image of f,** and denoted by Imf.
2. f: $A \to B$ is injective if and only if, for every point $b \in B$, $f^{-1}(b)$ contains at most one element.
3. The definition gives us the following procedure.

Proving that f: $A \to B$ is Surjective
Choose a general element $b \in B$, and then prove that b is of the form $f(a)$ for some $a \in A$.

Examples 2.6
A. f: $\mathbb{R} \to \mathbb{R}$; $f(x) = x^2 + 1$. Find $f(\mathbb{R})$ and $f[0,+\infty)$
B. Identity maps are always surjective.
C. The inclusion ι: $C \to B$ is surjective iff $C = B$.
D. The canonical projections of a (possibly infinite) product.
E. The simply connected cover f: $\mathbb{R} \to S^1$ is surjective
F. *Very important example* Let S be any set and let $\approx$ be an equivalence relation on S. Denote the set of equivalence classes in S by $S/\approx$. Then there is a natural surjection ν: $S \to S/\approx$..

Lemma 2.7 Let f: $A \to B$. Then:
(a) $f^{-1}(f(C)) \supset C$ for all $C \subset A$ with equality iff f is injective.
(b) $f(f^{-1}(D)) \subset D$ for all $D \subset B$, with equality iff f is surjective.

Proof: Exercise Set 1. We'll prove (a) in class.

Definition 2.8 f: $A \to B$ is **bijective** if it is both injective and surjective.

Examples 2.9
A. Exponential map $\mathbb{R} \to \mathbb{R}^+$
B. f: $[0, +\infty) \to [0, +\infty)$; $f(x) = \sqrt{x}$.
C. f: $[0, +\infty) \to [0, +\infty)$; $f(x) = x^2$
D. f: $\mathbb{R} \to [0, +\infty)$; $f(x) = x^2$ is not.
E. Restricted trig functions, e.g. g: $[-\pi, \pi] \to [-1, 1]$; $g(x) = \sin x$.
F. Inverse Trig functions
G. The identity map on any set
H. Any linear map f: $\mathbb{R} \to \mathbb{R}$; $f(x) = mx+b$ with $m \neq 0$.

Definition 2.10 If f: $A \to B$ and g: $B \to C$, then the **composite** $g \circ f$: $A \to C$ is the function $g \circ f(a) = g(f(a))$

Example in class

Lemma 2.11
Let f: $A \to B$ and g: $B \to C$,. Then:
(a) If f and g are injective, then so is $g \circ f$.
(b) If f and g are surjective, then so is $g \circ f$.
(c) If $g \circ f$ is injective, then so is f.
(d) If $g \circ f$ is surjective, then so is g.

Definition 2.12 $f\colon A \to B$ and $g\colon B \to A$ are called **inverse maps** if $g \circ f = 1_A$ and $f \circ g = 1_B$. In this event, we write $g = f^{-1}$ (and say that g is the inverse of f) and $f = g^{-1}$. If f has an inverse, we say that f *is* **invertible**.

Form the definition, we have:

Showing that f and g are Inverses
(1) Check that the domain of f = codomain of g and codomain of f = domain of g.
(2) Check that $g(f(x)) = x$ for every x in the domain of f.
(3) Check that $f(g(y)) = y$ for every y in the domain of g.

Theorem 2.13 (Inverse of a Function)
(a) $f\colon A \to B$ is invertible iff f is bijective
(b) The inverse of an invertible map is unique

We prove (a) in class and leave (b) as an exercise.

Function Sets and Indexed Products
Using the notion of a function, we can cook up new kinds of sets.

Definition 2.14 If A and X are sets, define
$$A^X = \{f\colon X \to A),$$
the set of all functions from A to B.

In the Exercises, you will prove an "exponential rule:"

$$(A^X)^Y \approx A^{X \times Y}$$

where $\approx$ indicates the existence of a bijection between the two sets.

Turning to products, recall that we can think of $A \times B$ as the set of functions

$$f\colon \{1, 2\} \to A \cup B$$

with the property that $f(1) \in A$ and $f(2) \in B$.

Definition 2.15 Given an indexed collection A_α ($\alpha \in \Omega$), define their **product** by

$$\prod_{\alpha \in \Omega} A_\alpha = \{f\colon \Omega \to \bigcup_{\alpha \in \Omega} A_\alpha \mid f(\alpha) \in A_\alpha\}.$$

Axiom of Choice
If A_α ($\alpha \in \Omega$) is any collection of non-empty sets, then $\prod_{\alpha \in \Omega} A_\alpha$ is non-empty.

Exercise Set 2

1 (a) Let $M(n)$ be the set of $n\times n$ matrices, let P be some fixed $n\times n$ matrix, and define f: $M(n)\to M(n)$ by $f(A) = PA$. (f is called "left translation by P.") Show that f is injective iff P is invertible.

(b) Let f be as in **(a)**. Show that f is surjective iff P is invertible.

2. Prove Lemma 2.7.

3 (a) Give an example of maps f: $A\to B$ and g: $B\to C$ with $g\circ f$ injective but g not injective. (See Lemma 2.11.)

(b) Give an example of maps f: $A\to B$ and g: $B\to C$ with $g\circ f$ surjective but f not surjective. (See Lemma 2.11.)

4. Prove Theorem 2.13(b).

5. Prove that composition of functions is associative: $(f\circ g)\circ h = f\circ(g\circ h)$ and unital: $f\circ 1_A =$ $= f$ for all f: $A\to B$.

6. Let f: $A\to B$, and define a relation on A by $a\approx a'$ if $f(a) = f(a')$. Show that $\approx$ is an equivalence relation on A.

7 (a) Show that if f: $A\to B$ and g: $B\to C$ are bijective, then so is $g\circ f$.

(b) Give an example of maps f: $A\to B$ and g: $B\to C$ with $g\circ f$ bijective, but with neither f nor g bijective.

8. **(a)** Prove that $(A^X)^Y \approx A^{X\times Y}$ for all sets A, X, Y.

(b) What is $A^{\emptyset}$?

9.* Prove by induction that the axiom of choice is valid for products of countably many sets.

10.* **(a)** Give a complete set of exponential rules extending $(A^X)^Y \approx A^{X\times Y}$ to the [addition" rule $a^{x+y} = a^x a^y$, $a^0 = 1$.

(b) What is A^{X-Y} if $Y \subset X$?

3. Metric Spaces

The concept of distance is fundamental to the study of the real continuum ($\mathbb{R}$), the n-dimensional continuum ($\mathbb{R}^n$) and the notion of "continuity." In discussing properties of the continuum, we tend to use only certain key facts about the properties of distance.

Let $\|\text{-}\|$ be the standard Euclidean norm on $\mathbb{R}^n$. Thus, when n = 1, we have $\|x\| = |x|$, the absolute value of x, and in general, if $x = (x_1 , x_2 , \ldots, x_n) \in \mathbb{R}^n$, then

$$\|x\| = \sqrt{x_1{}^2 + \ldots + z_n{}^2}$$

The distance from a point x to a point y, both in $\mathbb{R}^n$, is then given by $\|y - x\|$. (In particular, if x and y happen to be in $\mathbb{R}$, then their distance apart is $|y - x|$.) This distance function satisfies the following important properties:

1. $\|y - x\| \geq 0$ for every pair of elements (x, y) of $\mathbb{R}^n$; furthermore

$\|y - x\| = 0$ iff $x = y$. **Positive definite property**

2. For every pair of elements (x, y) of $\mathbb{R}^n$

$\|x - y\| = \|y - x\|$. **Symmetry**

3. For every triple of elements (x, y, z) of $\mathbb{R}^n$, one has

$\|z - x\| \leq \|z - y\| + \|y - x\|$ **Triangle Inequality**

We shall see that these three simple properties alone are sufficient to enable one to build up the whole theory of limits and of continuous functions. To whit:

Definition 3.1 A **metric space** is a pair (X, d) where X is a set and where d: $X \times X \to R$ is a function satisfying the following properties:

1. $d(x, y) \geq 0$ for every pair of elements (x, y) of X; furthermore

$d(x, y) = 0$ iff $x = y$. **Positive definite property**

2. For every pair of elements (x, y) of X,

$d(x, y) = d(y, x)$ **Symmetry**

3. For every triple of elements (x, y, z) of X, one has

$d(x, z) \leq d(x, y) + d(y, z)$ **Triangle Inequality**

The quantity $d(x, y)$ is referred to as the **distance from *x* to *y*** and the function d is referred to as the **metric on *X***.

Examples 3.2

A. Let $X = \mathbb{R}$, with

$d(x, y) = |y - x|$.

B. Let $X = \mathbb{R}^n$, with

$d(x, y) = \|y - x\|$.

The metric $\|\text{-}\|$ is referred to as the **standard metric** or the **Euclidean metric on $\mathbb{R}^n$.**

C. The unit n-sphere S^n is defined as $S^n = \{x \in \mathbb{R}^n{+}1 : \|x\| = 1\}$—a subset of $\mathbb{R}^n{+}1$. If we define $d(x, y) = \|y - x\|$ for all x and y in S^n, then (S^n, d) is a metric space. (The properties hold in S^n precisely because they hold in $\mathbb{R}^n{+}1$. We say that S^n **inherits the structure of a metric space** from $\mathbb{R}^n{+}1$.

D. More generally, if X is any subset of $\mathbb{R}^n$, then we can define $d(x, y) = \|y - x\|$ for all x and y in X, so that X inherits the structure of a metric space from $\mathbb{R}^n$.

E. If X is any set, define

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} .$$

This metric d is called the **discrete metric on *X***, and (X, d) is referred to as a **discrete metric space.**

F. If (X, μ) and (Y, ρ) are two metric spaces, define a metric d on $X \times Y$ by the formula

$$d((x_1, y_1), \ (x_2, y_2)) = \max\{\mu(x_1, x_2) \ , \rho(y_1, y_2)\}.$$

The metric d is called the **product metric** on $X \times Y$.

G. If (X_1, d_1) , . . . , (X_n, d_n) are metric spaces, define a metric d on $X_1 \times \ . \ . \ . \ \times X_n$ by

$$d((x_1, . . . ,x_n), (y_1 \ , . . ., y_n) = \max\{d_1(x_1, y_1), \ . \ . \ . \ , d_n(x_n, y_n)\}.$$

This is again called the product metric on $X_1 \times \ . \ . \ . \ \times X_n$.

H. The product metric on $\mathbb{R}^n = \mathbb{R} \times \ . \ . \ . \ \times \mathbb{R}$ (n times).

I. If $p > 0$, define a metric d on $\mathbb{R}^n$ by

$$d(x, y) = (|y_1 - x_1|^p + \ . \ . \ . + |y_n - x_n|^p)^{1/p}.$$

Note that, when $p = 2$, this coincides with the Euclidean metric.

Definition 3.3 Let (X, d) be a metric space, let $x \in X$ and let $r > 0$. The **open ball of radius *r* centered at *x*** is defined by

$$B(x, r) = \{y \in X : d(y, x) < r\}.$$

Intuitively, $B(x, r)$ consists of all point in X sufficiently close to x.

Examples 3.4 Look at various open balls in each of the above examples.

Definition 3.5 A subset $U \subset X$ of a metric space X is **open** if for every $u \in U$, there is an associated $r > 0$ with $B(u, r) \subset U$. (Note that the choice of r usually depends on the choice of u.)

In order to provide examples of open sets, we show:

Proposition 3.6 ***Are Open Balls Open?***
Open balls are open.

Proposition 3.7 ***Properties of Open Sets***
The collection of open subsets of X satisfies the following properties:
1. X and ϕ are open subsets of X.
2. If $\{U_\alpha\}_{\alpha \in A}$ is a collection of open subsets of X, then their union, $\cup_\alpha U_\alpha$ is open.
3. If U_1 and U_2 are open, then $U_1 \cap U_2$ is open.

Remark
Actually, Property 3 above immediately implies that finite intersections of open sets are open.

We now characterize the open sets in X.

> **Proposition 3.8** ***Open Sets Are Nothing But Unions of Open Balls***
> U is open in the metric space X iff U is a union of open balls.

Exercise Set 3
1. Are the following metric spaces? (Prove or give a counterexample in each case)
(a) $X = \mathbb{R}$; $d(x , y) = 0$ for every x and y.
(b) $X = \mathbb{R}$; $d(x, y) = \min\{|y - x|, 666\}$.
(c) $X = \mathbb{R}^{\infty}$, the set of all infinite sequences $(x_1, x_2, \ldots, x_n, \ldots)$ of reals such that all but finitely many x_i are 0. (Thus $\mathbb{R}^{\infty} = \mathbb{R} \cup \mathbb{R}^2 \cup \mathbb{R}^3 \cup \ldots$, where we regard $\mathbb{R}^n$ as the set of all sequences $(x_1, x_2, \ldots, x_n\ 0, 0, \ldots)$.) The function d is given by

$$d((x_1, x_2, \ldots, x_n, \ldots), (y_1, y_2, \ldots, y_n, \ldots)) = \max_i \{ |x_i - y_i|\}.$$

(d) $X = C[0,1]$, the set of continuous functions f: $[0, 1] \to \mathbb{R}$;

$$d(f, g) = \int_0^1 |f(x) - g(x)|\ dx$$

(e) $X = [0 , 1] \cup [2, 3]$;

$$d(x, y) = \begin{cases} 1 & \text{if } x \text{ and } y \text{ are in different intervals} \\ |x - y| & \text{otherwise} \end{cases}$$

2. Show that the following subsets of $\mathbb{R}^2$ are open under both the standard metric and the product metric:
(a) $\{(x , y) : y > 0\}$;
(b) $(a, b) \times (c, d)$ (product of open intervals) where $a < b$ and $c < d$;
(c) The open unit ball $\{(x, y) : x^2 + y^2 < 1\}$
(d) $\{(x, y) : x > 0 \text{ and } y > 0\}$.
3. S'pose that X is a metric space and let $Y \subset X$ have the metric inherited from X. Show that $W \subset Y$ is open in Y iff $W = U \cap Y$ for some open subset U of X.
4. Use induction to prove that the following statements are equivalent:
(a) If U_1 and U_2 have property P, then $U_1 \cap U_2$ has property P;
(b) If each U_i $(i = 1, \ldots, n)$ has property P, then $U_1 \cap U_2 \cap \ldots \cap U_n$ has property P.
Conclude that the remark after Proposition 1.6 holds true.
5. Let (X, d) be a metric space, and let x and y be distinct elements of X. show that there exist disjoint open sets U_x and U_y with $x \in U_x$ and $y \in U_y$. (This is referred to as the **Hausdorff** property.)
6. Let $\hat{\mathcal{E}}$ be the collection of open sets in $\mathbb{R}^n$ under the Euclidean metric, and let $\hat{\mathcal{E}}'$ be the collection of open sets in $\mathbb{R}^n$ under the discrete metric. Show that $\hat{\mathcal{E}} \subset \hat{\mathcal{E}}'$, but $\hat{\mathcal{E}} \neq \hat{\mathcal{E}}'$

7. Let d denote the Euclidean metric on $\mathbb{R}^2$ and let ρ denote the product metric on $\mathbb{R}^2$. Show that U is open under d iff U is open under ρ. (We say that these metrics *give rise to the same topology on* $\mathbb{R}^2$.)
8*. Verify the triangle inequality for the Euclidean metric on $\mathbb{R}^n$. (It is called **Cauchy's inequality.**)
9*. Verify that Example **I** gives a metric on $\mathbb{R}^n$. (The triangle inequality here is referred to as **Minkowski's inequality.**)

4. Closed Sets

Notation 4.1 If x is a point in the metric space X, then let $B(x, r)^- = B(x, r) - \{x\}$. We call $B(x, r)^-$ the **deleted open ball center x radius r.**

Definition 4.2 Let A be a subset of the metric space X. A **limit point of A** is a point $x \in X$ such that each open disc centered at x contains at least one point of A other than x. That is, x is a limit point of A if $\forall r>0$,

$$B(x, r)^- \cap A \neq \emptyset.$$

Examples 4.3
A. 1 and 1/2 are limit points of $(0, 1) \subset \mathbb{R}$, whereas 2 is not.
B. Any point $x \in \mathbb{R}^2$ with $\|x\| = 1$ is a limit point of $B(0, 1)$.
C. $[0,1]$ contains all its limit points.
D. 3 is not a limit point of $\{1, 2, 3\} \subset \mathbb{R}$.
E. $\{1, 2, 3\} \subset \mathbb{R}$ has no limit points.
F. The subset $\{1, 1/2, 1/3, \dots\}$ of $\mathbb{R}$ has 0 as (its only) limit point.

Definition 4.4 A subset C of the metric space X is **closed** if it contains all its limit points.

Examples 4.5
A. $[0,1] \subset \mathbb{R}$
B. $\{1, 2, 3\} \subset \mathbb{R}$
C. $\overline{B}(0, 1) \subset \mathbb{R}^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, the **closed ball** center 0 radius 1.
D. $\{0, 1, 1/2, 1/3, \dots\} \subset \mathbb{R}$.

Proposition 4.6 ***Open vs. Closed***
The subset C of the metric space X is closed iff its complement, $X-C$, is open.

Corollary 4.7 ***Properties of Closed Sets***
The collection of closed subsets of X satisfies the following properties:
1. X and $\emptyset$ are closed subsets of X.
2. If $\{C_\alpha\}_{\alpha \in A}$ is a collection of closed subsets of X, then their intersection, $\cap_\alpha C_\alpha$ is closed.
3. If C_1 and C_2 are closed, then $C_1 \cup C_2$ is closed.

Proposition & Definition 4.8 ***Open vs. Closed***
Let X be any metric space. Then the **closed ball,**

$\overline{B}(x, r) = \{y \in X : d(y, x) \le r\}$

is a closed subset of X.

Definitions 4.9 If X is any metric space and if $A \subset X$, then the **closure** $\overline{A}$ of A is the union of A and the set of all its limit points.

Note The closure of a set is automatically closed (why?)

Exercise Set 4

1. S'pose that X is a metric space and let $Y \subset X$ have the metric inherited from X. Show that $D \subset Y$ is closed in Y iff $D = C \cap Y$ for some closed subset C of X.

2. Prove that A is closed iff $A = \overline{A}$.

3. Prove that $\overline{A}$ is the intersection of all closed subsets of X containing A.

4. Prove that if C is any closed subset of X containing A, then $C \supset \overline{A}$.

5. Prove that C is closed iff $C \supset \partial C$. (See Exercise 9.)

6. Prove that, if C and D are closed subsets of X with $C \cap D = \emptyset$, then there exist open sets $U \supset C$ and $V \supset D$ with $U \cap V = \emptyset$.

7. Show that all finite subsets of every metric space X are closed.

8. Describe the closure of each of the following subsets of $\mathbb{R}$:

- **(a)** the integers;
- **(b)** the rationals;
- **(c)** the irrationals;
- **(d)** the Cantor set.

9. If A is any subset of the metric space X, then its **boundary** ∂A is given by

$\partial A = \overline{A} \cap \overline{A'}$.

Give examples of sets A with

- **(a)** $A = \partial A$
- **(b)** $\partial A = \emptyset$.

10. If A is any subset of the metric space X, then its **interior** A° is given by

$A^{\circ} = A - \partial A$

Show that A° is the maximal open subset of A and can be defined as the union of all open sets contained in A.

5. Continuous Mappings

Definition 5.1 If X is a metric space and $x_0 \in X$, then $N \subset X$ is a **neighborhood** of x_0 if there exists an open set U with

$$x_0 \in U \subset N.$$

Examples 5.2

A. Open sets are neighborhoods of all their points. (class)

B. Any set that is a neighborhood of all its points is open. (exercises)

Definition 5.3 Let f: $X \rightarrow Y$ be any map and let $x_0 \in X$. Then f is said to be **continuous at x_0** if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x, x_0) < \delta \text{ implies } d_Y(f(x), f(x_0)) < \varepsilon.$$

Examples 5.4
A. If f: $\mathbb{R} \rightarrow \mathbb{R}$, under the usual metric, then this is the usual definition: f is continuous at x_0 if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$.
B. If R is replaced by $\mathbb{R}^n$, then we again have the customary definition.

Very Important Remark 5.5
Definition 5.3 is equivalent to the following:

For each $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_X(x_0, \delta)) \subset B_Y(f(x_0), \varepsilon)$
$\Leftrightarrow$ For each $\varepsilon > 0$, there exists $\delta > 0$ such that $B_X(x_0, \delta) \subset f^{-1}(B_Y(f(x_0), \varepsilon))$
$\Leftrightarrow$ For each $\varepsilon > 0$, $f^{-1}(B_Y(f(x_0), \varepsilon))$ is a neighborhood of x_0.
$\Leftrightarrow$ For each open set U containing $f(x_0)$, $f^{-1}(U)$ is a neighborhood of x_0.

Notice that the final formulation is free of the cumbersome metrics.

Definition 5.6 f: $X \rightarrow Y$ is continuous if it is continuous at each point $x_0 \in X$.

More Very Important Remarks 5.7
1. In view of Very Important Remarks 5.5, f: $X \rightarrow Y$ is continuous if for each open subset U of Y, $f^{-1}(U)$ is open in X. (*Y*ou'll be proving this yet again in the problems.)
2. In view of this, we seem to really require only the notion of open sets to talk about continuity, rather than the metric, so we shall abstract the notion of open sets & throw away the metric from now on. This is what topology is all about.

The Problem with Metric Spaces
1. No matter which of the several metrics described above we use on $\mathbb{R}^n$, it seems that you get the same open sets, and hence the same continuous functions. Thus the metric is a bit of a bugaboo.
2. S'pose we want to talk about huge spaces, such as $\mathbb{R} \times \mathbb{R} \times \ . \ . \ . \ ,$ the set of all infinite sequences of real numbers. Just what metric do we use?
3. S'pose we want to "glue" two metric spaces along specified subsets together to form a larger one. Then specifying a suitable metric on the resulting space can be a nightmare. (The open sets, however, are quite easy to describe, as we shall see.)

Exercise Set 5
1. Prove directly from the definitions (without using the Very Important Remarks):
The following are equivalent:

(a) f: $X \rightarrow Y$ is continuous.
(b) For each open subset U of Y, $f^{-1}(U)$ is open in X.
(c) For each closed subset C of Y, $f^{-1}(C)$ is closed in X.

2. Prove that the following functions are continuous:

(a) The identity function f: $X \to X$ for every metric space X.

(b) The inclusion function ι: $A \to X$ for every subset $A \subset X$.

(c) Composites of continuous functions: If f: $X \to Y$ and g: $Y \to Z$ are continuous, then so is the composite $g \circ f$: $X \to Z$.

3. Prove that, if f: $X \to Y$ is continuous on $Z \subset X$, then

(a) For each open subset U of Y, $f^{-1}(U) \cap Z$ is open in Z.

(b) For each closed subset C of Y, $f^{-1}(C) \cap Z$ is closed in Z.

Also show that the converse is false in general. That is, (a) or, equivalently, (b) can hold without f being continuous on Z.

4. **(a)** Suppose $X = C \cup D$, where C and D are closed sets, and suppose f: $X \to Y$ has the property that f is continuous on C and D. Prove that f is continuous on X.

(b) Generalize (a) to a finite union of closed sets.

(c) Use part (b) to prove that the function f: $\mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ 2x+2 & \text{if } 0 \le x < 2 \\ x^2+2 & \text{if } x \ge 2 \end{cases}$$

is continuous at every point. (You may assume continuity of polynomial functions.)

5. Let $X = S^1$ with the induced metric.

(a) Show that the "open segments" $\{e^{ix} \mid a < x < b\}$ are open subsets of S^1.

(b) Show that ever open subset of S^1 is a union of open segments. (Hint: every open ball $B(x, r)$ in $\mathbb{R}^2$ contains a little "wedgelet" obtained by writing x in polar coordinates, and then varying r and θ a little—use the fact that sine and cosine are continuous).

6. Topological Spaces

Definition 6.1 A **topology** on a set X is a collection $\mathscr{T}$ of subsets of X such that:

1. X and $\emptyset$ are in $\mathscr{T}$;
2. The union of any collection of subsets in $\mathscr{T}$ is in $\mathscr{T}$;
3. The intersection of any finite collection of subsets in $\mathscr{T}$ is in $\mathscr{T}$.

The subsets in $\mathscr{T}$ are called the **open subsets** of X and the pair $(X, \mathscr{T})$ is called a **topological space.**

Examples 6.2

A. Let (X, d) be a metric space. Then the collection of its d-open subsets is a topology on X, called the **metric topology**.

B. Find all possible topologies on a two-element set $\{a, b\}$.

C. The **discrete** and **indiscrete** topologies.

D. (Another topology on $\mathbb{R}$). Let U be open if its complement is either closed and bounded or the whole of $\mathbb{R}$.

E. (The reals with two zeros) Let $S = \mathbb{R} \cup \{0'\}$ where $U \subset S$ is open if:

(i) U is an open subset of $\mathbb{R}$ under the usual topology;

(ii) $0' \in U$ and $U - \{0'\} \cup \{0\}$ is an open subset of $\mathbb{R}$ under the usual topology.

Equivalently, let ν: $S \to \mathbb{R}$ be the projection that sends both zeros to 0. Then U is open in S iff $\nu(U)$ is open in $\mathbb{R}$.

Definitions 6.3 If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X, then $\mathcal{T}_1$ is **finer** than $\mathcal{T}_2$ if $\mathcal{T}_1 \supset \mathcal{T}_2$. We also say that $\mathcal{T}_2$ is **coarser** than $\mathcal{T}_1$. Also, as with metric spaces, we say that $N \subset X$ is a **neighborhood** of $x_0 \in X$ if there exists an open set U with $x_0 \in U \subset N$.

Example 6.4 Consider the reals with the usual topology, the discrete and indiscrete topologies, and the topology given in Example 6.2 (D) above.

Definition 6.5 A **basis** for a topology on X is a collection $\mathcal{B}$ of subsets, called **basic subsets,** of X such that:

1. $X = \cup_{B \in \mathcal{B}} B$;
2. If B and C are in $\mathcal{B}$ with $x \in B \cap C$ then there exists a $D \in \mathcal{B}$ with $x \in D \subset B \cap C$.

Examples 6.6

A. The collection of all open balls in a metric space. (Note: not all open sets).

B. The collection of all open rectangles in $\mathbb{R}^2$.

C. The collection of all half-open intervals $(a, \ b] \subset \mathbb{R}$.

Lemma and Definition 6.7 Let $\mathcal{B}$ be a basis for a topology on X. Define $U \subset X$ to be open iff U is a union of basic subsets. Then:

(a) This defines a topology $\mathcal{T}$ on X such that each basic subset is open;

(b) If $\mathcal{T}'$ is any topology which includes $\mathcal{B}$, then $\mathcal{T}' \supset \mathcal{T}$;

(c) U is in $\mathcal{T}$ iff for each $x \in U$, there exists $B \in \mathcal{B}$ with $x \in B \subset U$.

We refer to $\mathcal{T}$ as the **topology generated** by $\mathcal{B}$.

Sketch of Proof:

(a) That Ø and X are open is clear, as is closure under unions. For closure under pairwise intersection, first note that the intersection of any two sets in $\mathcal{B}$ is a union of sets in $\mathcal{B}$. Thus, $(\cup_\alpha U_\alpha) \cap (\cup_\beta V_\beta) = \cup_{(\alpha,\beta)}(U_\alpha \cap V_\beta)$ —a union (unions of) sets in $\mathcal{B}$.

(b) $\mathcal{T}'$ as given above must include unions of basic open sets, and hence must include $\mathcal{T}$.

(c) If $x \in U = (\cup_\alpha B_\alpha) \in \mathcal{T}$, then $x \in B_\alpha$ for some α. Conversely, if U has the property stated, then for each $x \in U$ we can choose B_x with $x \in B_x \subset U$, whence $U = \cup_x B_x \in \mathcal{T}$.

Subspace Topology

In a metric space X, we saw in the exercise set that if $Y \subset X$ has the metric inherited from X, then $W \subset Y$ is open in Y iff $W = U \cap Y$ for some open subset U of X. We generalize this notion to topological spaces.

Proposition and Definition 6.8 Suppose that X is a topological space and $Y \subset X$. Define a topology $\mathcal{T}_Y$ on Y by defining $W \subset Y$ to be open if there exists an open subset U of X with $W = Y \cap U$. Then $\mathcal{T}_Y$ is indeed a topology.

Examples 6.9

A. Look at the open sets in $(a, b\,] \subset \mathbb{R}$,
B. Open subsets of the closed unit disc in $\mathbb{R}^2$.
C. Open subsets of the n-sphere S^n.
D. Discrete subsets of $\mathbb{R}$ are subsets which inherit the discrete topology.

Exercise Set 6

1. Prove: Any set that is a neighborhood of all its points is open.
2. Prove that, if $\mathscr{C}$ is the collection of closed sets in a topological space, then $\mathscr{C}$ contains X and $\emptyset$, and is closed under intersections and finite unions.
3. Find all possible topologies on a three-element set $\{a, b, c\}$.
4. Show that the topology in Example 6.6(B) is the usual topology on $\mathbb{R}^2$.
5. Let $(Y, \mathscr{T}_Y)$ be a topological space, let X be a set, and le f: $X \rightarrow Y$ be any map. Define a topology $\mathscr{T}_X$ on X by taking $\mathscr{T}_X = \{f^{-1}(U) \mid U \in \mathscr{T}_Y\}$. Show that $\mathscr{T}_X$ is indeed a topology on X (called the topology **induced** by f).
6. Let $(X, \mathscr{T}_X)$ be a topological space, let Y be a set, and let f: $X \rightarrow Y$ be any surjection. Define a collection $\mathscr{C}$ of subsets of Y by taking $\mathscr{C} = \{f(U) \mid U \in \mathscr{T}_X\}$.
(a) Give an example to show that $\mathscr{C}$ need not be a topology on Y
(b) Give a condition on f under which $\mathscr{C}$ is always a topology, and justify your assertion.
7. Let $\mathscr{T}$ be a topology on X. A **neighborhood system** for $\mathscr{T}$ is a collection of open subsets $\mathscr{N}$ such that for each open subset U and each $x \in U$, there is an $N \in \mathscr{N}$ with $x \in N \subset U$. Show that $\mathscr{N}$ is a basis for the topology $\mathscr{T}$
8. Show that the set of all open balls of a metric space is a basis for its metric topology.
9. A **sub-basis** $\mathscr{S}$ for a topology on X is any collection of subsets of X whose union is X. Show that the set of all finite intersections of elements of $\mathscr{S}$ forms a basis for a topology on X. Hence describe the open subsets of $\mathscr{T}$ in terms of the sub-basic open sets.

7 Continuous Functions and a Little Bit of Category Theory

Definition 7.1. Let X and Y be topological spaces, and let f: $X \rightarrow Y$. Then f is said to be **continuous** if for each open subset $V \subset Y$, $f^{-1}(V)$ is open in X. In the homework, you will see that this is equivalent to saying the $f^{-1}(C)$ is closed in X for every closed subset $C \subset Y$.

Examples 7.2

A. Continuous functions f: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ which you've encountered in baby calculus.
B. The identity map 1: $X \rightarrow X$ on any topological space X.
C. The inclusion function ι: $A \rightarrow X$ for every subspace $A \subset X$.
D. If X is any space and if $y_0 \in Y$ is any fixed point, then the associated constant map f: $X \rightarrow Y$ given by $f(x) = y_0$, is continuous.
E. If X has the discrete topology and if Y is any space, then all maps $X \rightarrow Y$ are automatically continuous.

F. If Y has the indiscrete topology and if X is any space, then all maps $X \to Y$ are automatically continuous.

G. If X has two topologies $\mathscr{T}_1$ and $\mathscr{T}_2$, then $\mathscr{T}_2 \subset \mathscr{T}_1$ iÎf the identity $(X, \mathscr{T}_1) \to (X, \mathscr{T}_2)$ is continuous.

H. Henceforth, I will denote the closed unit interval [0, 1]. A **path in *X* from *a* to *b*** is a continuous map λ: $I \to X$ with $\lambda(0) = a$ and $\lambda(1) = b$.

The following lemma is fundamental.

Lemma 7.3 ***Continuity of Composites***
If f: $X \to Y$ and g: $Y \to Z$ are continuous, then so is the composite $g \circ f$: $X \to Z$..

Definition 7.4. A **homeomorphism** is an invertible map f: $X \to Y$ such that f^{-1} is also continuous. If there exists a homeomorphism f: $X \to Y$, we write $X \cong Y$.

Examples 7.5

A. $\mathbb{R} \cong (-1, 1)$

B. Any two open intervals in $\mathbb{R}$ are homeomorphic.

C. $S^1 \cong \partial$(Unit square) via f: ∂(Unit square)$\to S^1$ given by

$$f(x, y) = \frac{(x, y)}{\|(x, y)\|} \quad .$$

That f is continuous follows from continuity of algebraic expressions. That f^{-1} is continuous will have to wait for more sophisticated results.

D. $D_2 \cong$ Filled unit square $I \times I$ via f: $I \times I \to D_2$ given by

$$f(x, y) = \begin{cases} \max\{x, y\} \dfrac{(x, y)}{\|(x, y)\|} & \text{if } (x, y) \neq (0, 0) \\ (0, 0) & \text{if } (x, y) = (0, 0) \end{cases}$$

E. $S^1 \not\cong \mathbb{R}$; Open intervals $\not\cong$ Closed intervals.

F. Tori and teacups

Definition 7.6 (Mac Lane) A **category** $\mathscr{C}$ consists of the following:

(1) A class **Ob**(□) of **objects** and, for every pair of objects X and Y in **Ob**, a set $\mathscr{C}(X, Y)$ of **morphisms from *X* to *Y***. If $f \in \mathscr{C}(X, Y)$ we write f: $X \to Y$.

(2) For each triple (X, Y, Z) of objects in $\mathscr{C}$ a **composition**

$$\mathscr{C}(X, Y) \times \mathscr{C}(Y, Z) \to \mathscr{C}(X, Z)$$

written as $(f, g) \mapsto g \circ f$ and satisfying *the following rules*:

Associativity If f: $X \to Y$, g: $Y \to Z$ and h: $Z \to W$, then $(h \circ g) \circ f = h \circ (g \circ f)$

Identity: For each object X in $\mathscr{C}$, there exists a morphism 1_X: $X \to X$ such that, if f: $X \to Y$, then $f \circ 1_X = f$ and $1_Y \circ f = f$. We call 1_X the **identity morphism** on X.

Observe that the following are categories:

Examples 7.7
A. The category **Set** of sets and functions
B. The category of **Vect** vector spaces over $\mathbb{R}$ (or $\mathbb{C}$ or any field) and linear maps (a key point is that the composite of linear maps is linear).
C. The category **Group** of groups and homomorphisms, and also the category **Ab** of abelian groups and homomorphisms.
D. The category **Ring** of rings and ring homomorphisms
E. If G is any group (or any groupoid), then we can regard G as a category with a single object * and set of morphisms $\mathscr{C}(*,*) = G$. Composition is given by the group multiplication.
F. Any set S can be thought of as a category whose objects are the elements of S and whose only morphisms are the identity morphisms.
G. Diff is the category whose objects are the smooth manifolds, and whose morphisms are the smooth maps.
H. Top is the category whose objects are the topological spaces and whose morphisms are the continuous maps.

Definitions 7.8
A category is **small** if its class of objects is a set. Which of the above categories is small?
An **equivalence** in the category $\mathscr{C}$ is a morphism $X \to Y$ in $\mathscr{C}$ that has an inverse. Two objects in $\mathscr{C}$ are **equivalent** if there exists an equivalence from one to the other.

Examples 7.9
We look at the equivalences in the various categories above.

Definition 7.10 Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A **covariant functor** F: $\mathscr{C} \to \mathscr{D}$ assigns to each object X of $\mathscr{C}$ an object $F(X)$ in $\mathscr{D}$., and to each morphism f: $X \to Y$ in $\mathscr{C}$ an associated morphism $F(f)$: $F(X) \to F(Y)$ in $\mathscr{D}$ such that

$F(1_X) = 1_{F(X)}$ for every object X in $\mathscr{C}$, and
$F(f \circ g) = F(f) \circ F(g)$ for every pair of composable morphisms f and g in $\mathscr{C}$.

In other words, F takes commutative diagrams to commutative diagrams (picture in class). A **contravariant functor** F: $\mathscr{C} \to \mathscr{D}$ assigns to each object X of $\mathscr{C}$ an object $F(X)$ in $\mathscr{D}$., and to each morphism f: $X \to Y$ in $\mathscr{C}$ an associated morphism $F(f)$: $F(Y) \to F(X)$ in $\mathscr{D}$ such that

$F(1_X) = 1_{F(X)}$ for every object X in $\mathscr{C}$, and
$F(f \circ g) = F(g) \circ F(f)$ for every pair of composable morphisms f and g in $\mathscr{C}$.

Examples 7.11
A. Let $\mathscr{C}$ be any of the categories **Vect, Group, Ring, Diff,** or **Top**. Then the **forgetful functor** F: $\mathscr{C} \to$ **Set** is given by taking $F(X)$ to be the underlying set X, and $F(f) = f$. In other words, F simply "forgets" the additional structure.
B. More interesting are the "adjoint" functors going the other way, discussed in class.

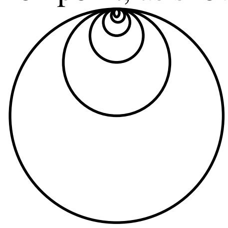
Algebraic Topology is, in essence, the study of functors from **Top** to various algebraic categories like **Group, Ab, Vect** and so on.

Exercise Set 7

1. Which of the following are continuous (justify your assertions).

(a) f: $S^1 \to S^1$ the identity, where the source has the usual topology and the target has the discrete topology.

(b) The *Hawaiian earring H* is the subspace of $\mathbb{R}^2$ given by infinitely many circles (with radii 1, 1/2, 1/4, ...) joined at a common point, as shown:



Let f: $S^1 \to H$ be inclusion of any one of the loops.

(c) As in part (b), but with f: $H \to S^1$ being the function that maps each loop onto the circle.

(d) f: $(0, 1) \cup (1, 2) \to \{0, 1\}$ given by $f(x) = \begin{cases} 0 & \text{if } x \in (0, 1) \\ 1 & \text{if } x \in (1, 2) \end{cases}$.

2. Use the Intermediate Value Theorem from baby calculus to prove that there is no continuous surjection f: $[a, b] \to \{0, 1\}$ where $a < b$ and $\{0, 1\}$ has the discrete topology.

3. Suppose that X is a topological space such that there is no continuous surjection $X \to \{0, 1\}$. Prove that the Intermediate Value Theorem holds for f in the following form: If f: $X \to \mathbb{R}$ is continuous, x_1 and x_2 are in X with $f(x_1) = a$ and $f(x_2) = b$, and $a \le c \le b$, then there exists $y \in X$ with $f(y) = c$.

4. Prove that f: $X \to Y$ is continuous if for each closed subset $C \subset Y$, $f^{-1}(C)$ is closed in X. [Closed sets are defined to be the complements of open ones.]

5. Which of the following define functors? Justify your assertions, and state whether a functor is covariant or contravariant.

(a) F: **Top**$\to$**Top**; F(X) = X with the discrete topology and $F(f) = f$

(b) F: **Top**$\to$**Top**; F(X) = X with the indiscrete topology and $F(f) = f$.

(c) Let Y be a topological space, and define F_Y: **Top**$\to$**Set;** $F_Y(X) =$ **Top**(Y, X), and $F_Y(f\colon X \to X')$ = Composition with f: **Top**$(Y, X) \to$ **Top**(Y, X').

(d) Let Y be a topological space, and define F^Y: **Top**$\to$**Set;** $F^Y(X) =$ **Top**(X, Y), and $F^Y(f\colon X \to X')$ = Composition with f: **Top**$(X', Y) \to$ **Top**(X, Y).

8 Subspaces and Products

We can now consider the subspace topology from a sophisticated vantage point.

Theorem 8.1 ***Subspace Topology from the Categorical Point of View***
Let A be a subset of the space X. Then the subspace topology on A is the unique topology on A such that:

(a) i: $A \to X$ is continuous;

(b) if Y is any space and g: $Y \to A$, then g is continuous iff $\iota \circ g$ is continuous.

$$\begin{tikzcd} Y \arrow[r, "\iota\circ g"] \arrow[dr, "g"'] & X \\ & A \arrow[u, "\iota"'] \end{tikzcd}$$

Another (more fancy) way of stating **(b):**

(b)' Given any commutative diagram of the form

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "g"'] & X \\ & A \arrow[u, "\iota"'] \end{tikzcd}$$

one has f continuous iff g is continuous. In words, if f factors through A, then the factor is continuous iff f is.

(Note: $\iota^{-1}(U) \equiv U \cap A$. Also, **(a)** is a consequence of **(b)**.)

Definition 8.2 More generally, we say that i: $A \to X$ **has the subspace topology**, or **is an embedding** if i is injective, and given any commutative diagram of the form

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "g"'] & X \\ & A \arrow[u, "i"'] \end{tikzcd}$$

one has f continuous iff g is continuous. This is equivalent to saying that i is a homeomorphism of A with $i(A)$, the latter being given the subspace topology inherited from X. Equivalently, U is open in A iff $U = i^{-1}(V)$ for some open subset V of X.

Examples 8.3

A. Let f: $\mathbb{R} \to \mathbb{R}^2$ be given by $f(x) = e^{ix}$. Then f determines a continuous surjection π: $\mathbb{R} \to S^1$.

B. If f: $X \to Y$ is continuous, then the associated map f_{sur}: $X \to f(X)$ given by $f_{\text{sur}}(x) = f(x)$ is automatically continuous.

We now turn to the product topology:

Definition 8.4 Let X and Y be topological spaces. The **product topology** on $X \times Y$ is defined to be the topology with basis $\mathscr{B} = \{U \times V : U \text{ open in } X \text{ and } V \text{ open in } Y\}$.

Noting that $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$, we easily get:

Lemma 8.5 Product Topology
The collection $\mathscr{B}$ described above is indeed a basis for a topology on $X\times Y$, such that to which the canonical projections, π_X: $X\times Y \to X$ and π_Y: $X\times Y \to Y$, are continuous.

Examples 8.6
A. $\mathbb{R} \times \mathbb{R}$.
B. The cylinder $X\times I$ for any space X. Note that, for each $t \in I$, $\{t\}\times X \cong X$.
C. $X \times \{p\} \cong X$.
D. $X \times \{p, q\}$ is two disjoint copies of X.
E. Homotopy Let f and g be (continuous) maps $X \to Y$. A **homotopy** from f to g is a map

$$H\colon\ X \times I \to Y$$

such that $H\circ\iota_0$ = f and $H\circ\iota_1 = g$, where ι_t:$X \to X\times$I denotes the map $x \mapsto (x, t)$. We say that ***f* is homotopic to *g*** if there exists a homotopy from f to g, and write $f \approx g$.

We now characterize the product topology in a most elegant manner:

Theorem 8.7 Universal Property of the Product
The product topology is the unique topology on $X \times Y$ such that:
1) the projections π_X and π_Y are continuous;
2) if Z is any space and f: $Z \to X\times Y$ any map, then f is continuous iff $\pi_X\circ f$ and $\pi_Y\circ f$ are continuous. (See diagram.)

$$\begin{tikzcd}
 & Z \arrow[dl, "\pi_X\circ f"'] \arrow[dr, "\pi_Y\circ f"] \arrow[dd, "f"] & \\
X & & Y \\
 & X\times Y \arrow[ul, "\pi_X"] \arrow[ur, "\pi_Y"'] &
\end{tikzcd}$$

A More Fancy Way of Stating (2):
2)' Given any commutative diagram of the form:

$$\begin{tikzcd}
 & Z \arrow[dl, "g"'] \arrow[dr, "h"] \arrow[dd, "f"] & \\
X & & Y \\
 & X\times Y \arrow[ul, "\pi_X"] \arrow[ur, "\pi_Y"'] &
\end{tikzcd}$$

one has f continuous if g and h are.

Note This says that, to define a continuous map into the product, we need only use a continuous map into X and one into Y.

Examples 8.8
A. The map f: $(0, +\infty) \to \mathbb{R}^2$ given by $f(x) = (\sin x, \ln x)$.
B. The map f: $X \times I \to X \times I$ given by $f(x, t) = (x, t/2)$.

C. The spiral λ: $\mathrm{I} \to S^1 \times I$ given by $\lambda(t) = (e^{it},\ t)$.

Exercise Set 8

1. Prove that the subspace topology on $A \subset X$ is the coarsest topology on A making the inclusion ι: $A \to X$ continuous.

2. Prove that the product topology on $X \times Y$ is the coarsest topology on $X \times Y$ making the projections continuous.

3. Construct continuous surjections f: $S^1 \times S^1 \to S^2$ and g: $\mathbb{R}^2 \to S^1 \times S^1$, justifying your claims concerning continuity.

4. Generalize the definition of the product topology to products of the form $\Pi_i X_i$ where the X_i are given spaces. Hence generalize Theorem 8.7 to this context.

5. (Universality of the Product) A **categorical product** of the spaces X_i ($i = 1$ ..., n) is a space P together with continuous maps π_i: $P \to X_i$ such that, given any collection of continuous maps r_i: $Z \to X_i$, there is a unique continuous map r: $Z \to P$ making the diagrams

$$\begin{tikzcd} & Z \arrow[dl, "r_i"'] \arrow[dr, "r"] & \\ X_i & & P \arrow[ll, "\pi_i"'] \end{tikzcd}$$

commute. Show:

(i) Any two categorical products of the X_i are homeomorphic;

(ii) The product $\Pi_i X_i$ you have defined in Exercise 4 above is a categorical product of the X_i , and hence that any categorical product is homeomorphic to the one you constructed.

9. Closed Sets, Limit Points and the Pasting Lemma

Definition 9.1 A subset $A \subset X$ is **closed** if its complement, $X-A$, is open.

Examples 9.2

A. Closed subsets of metric spaces, in particular, the closed subsets of $\mathbb{R}^2$.

B. Every subset is closed in the discrete topology.

C. Let $Y = (1,\ 2) \cup [3,\ 4]$ have the subspace topology inherited from $\mathbb{R}$. Then $(1,\ 2)$ and $[3, 4]$ are both open and closed.

C. Let $Y = (1,\ 2) \cup (2,\ 3)$ have the subspace topology inherited from $\mathbb{R}$. Then $(1,\ 2)$ and $(2, 3)$ are both open and closed.

As before, we have:

Lemma 9.3 ***Properties of Closed Sets***

The collection of closed subsets of X satisfies the following properties:

1. X and Ø are closed subsets of X.
2. Arbitrary intersections of closed sets are closed..
3. Finite unions of closed sets are closed.

Note: One may as well have defined a topology using the concept of closed sets rather than open ones, with the above as the axioms. (See the exercise set.)

Definition 9.4 Let A $\subset X$. Define the **interior**, A°, of A and the **closure**, $\bar{A}$, of A by:

$A^\circ = \cup\{U\colon\ U \subset A,\ U \text{ open in } X\}$ See Exercise Set 9 # 10

$\bar{A} = \cap\{C\colon\ C \supset A,\ C \text{ closed in } X\}$ See Exercise Set 9 # 3

> **Lemma 9.5**
> **(i)** $x \in \bar{A}$ iff every open set containing x intersects A nontrivially.
> **(ii)** A is closed iff $A = \bar{A}$.

Examples 9.6 Closures of:
A. $(0,1] \subset \mathbb{R}$. (We use Lemma 9.5.)
B. $\{1/n : n = 1, 2, \ldots\} \subset \mathbb{R}$.
C. $B(x, r)$ in a metric space X.

Definition 9.7 The space X is called **Hausdorff** if. for each pair of points x, y in X there exist open sets U and V with $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Example 9.8 All metric spaces. (See Exercise Set 3 #5).

> **Lemma 9.9** ***Equivalent Definition of Continuity***
> $f\colon X \to Y$ is continuous iff preimages of closed sets are closed.

> **Lemma 9.10** ***The Pasting Lemma***
> Suppose $X = C_1 \cup C_2 \cup \ldots \cup C_n$ where are the C_i are closed, and that $f\colon X \to Y$ is a map with the property that $f|C_i$ is continuous for each i. Then f is continuous.

> **Corollary 9.11** ***Piecewise Defined Functions***
> S'pose $X = A \cup B$, where A and B are closed in X, and s'pose given continuous functions $f\colon\ A \to Y$ and $g\colon B \to Y$ which agree on $A \cap B$ (i.e. $f|A \cap B = g|A \cap \boldsymbol{B}$.) Then the map $h\colon\ X \to Y$ given by
>
> $$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$
>
> is continuous.

Examples 9.12
A. Those awful piecewise-defined maps from Calc I
B. Define $f\colon D^2 \times I \to D^2$ by $f(d, t) = 2td$ if $0 \le t \le 1/2$ and $f(d, t) = 0$ if $t \ge 1/2$.

Exercise Set 9

1. S'pose that X is a topological space and let $Y \subset X$ be a subspace Show that $D \subset Y$ is closed in Y iff $D = C \cap Y$ for some closed subset C of X.

2. Show that, if Y is closed in X and Z is closed in Y, then Z is closed in X. (All subspaces are assumed to have the subspace topology.)

3. Show that *homotopic* is an equivalence relation in functions f: $X \to Y$. [Hint: use the pasting lemma to show transitivity.]

4. Let X be a set, and let $\mathscr{C}$ be a collection of subsets such that:

- X and $\emptyset$ are in $\mathscr{C}$.
- The intersection of any family of subsets in $\mathscr{C}$ is in $\mathscr{C}$.
- The union of any finite collection of subsets in $\mathscr{C}$ is in $\mathscr{C}$.

Now define a collection $\mathscr{T}$ of subsets of X by taking $U \in \mathscr{T}$ iff $X - U \in \mathscr{C}$. Prove that $\mathscr{T}$ is a topology on X with $\mathscr{C}$ the collection of closed subsets.

5. If X is a topological space, and $A \subset X$, the **boundary, ∂A** of A, is defined to be

$$\partial A = \overline{A} \cap \overline{A'}.$$

Prove that ∂A is closed, $\partial A = \partial(A')$, $\partial(\partial A) = \partial A$, and $\overline{A} = A \cup \partial A$.

6. Prove that, if $A \subset X$, then $\partial A = \emptyset$ iff A is both open and closed.

7. Let $A \subset X$. Prove that

(a) A is closed iff $\partial A \subset A$

(b) A is open iff $\partial A \subset A'$

8. Prove that, for $A \subset X$, $A^{\circ} = A - \partial A$.

10 Quotient Spaces and Pushouts

We now "dualize" the concept of an inclusion to get an way to "glue" spaces together to form new ones. First recall the definition of a subspace:

We said that ι: $A \to X$ has the subspace topology, if ι is injective, and given any commutative diagram of the form

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "g"'] & X \\ & A \arrow[u, "i"'] \end{tikzcd}$$

then f is continuous if g is continuous. This is equivalent to saying that U is open in A iff $U = \iota^{-1}(V)$ for some open subset V of X.

We now reverse all arrows, and replace "injective" by "surjective" to obtain:

Definition 10.1 We say that π: $X \to B$ **has the quotient topology,** or is a **projection** if π is surjective, and given any commutative diagram of the form

$$\begin{tikzcd} Y & X \arrow[l, "f"'] \arrow[d, "\pi"] \\ & B \arrow[ul, "g"] \end{tikzcd}$$

f is continuous if g is continuous. Intuitively, if f: $X \to Y$ "factors through" B, then the factor is continuous iff f is. We usually just say the ***B* has the quotient topology** under these circumstances.

> **Proposition 10.2** *Quotient Topology*
> **(a)** π: $X \to$ B has the quotient topology iff π is surjective, and U is open in B iff $\pi^{-1}(U)$ is open in X.
> **(b)** If π: $X \to$ B is any surjective function, then part (a) defines a topology on B giving it the quotient topology.

Thus we can create spaces B with the useful property in the definition by simply defining U to be open iff $\pi^{-1}(U)$ is open.

Examples 10.3

A. If X and Y are any spaces, then the "projections" π_X and π_Y are indeed projections.

B. The map π: $\mathbb{R} \to S^1$ given by $\pi(x) = e^{ix}$ is a projection. (Indeed, the open subsets of S^1 have a base consisting of open segments, whose preimages are unions of open segments in $\mathbb{R}$.)

C. π: $\mathbb{R}^2 \to S^1 \times S^1$ given by $\pi(x,y) = (e^{ix}, e^{iy})$.

D. Let $X = [1,2] \cup [3,4]$, $B = [1, 3]$, $\pi(x) = x$ if $x \in [1,2]$ and $\pi(x) = x-1$ otherwise.

More interesting examples will follow the following.

Recall that an **partition** of a set S is a decomposition of S as a union of mutually disjoint subsets. Such subsets may be thought of as the equivalence classes with respect to some equivalence relation on S. If $x \in S$, then $[x]$ will denote its equivalence class. We write $X/\approx$ as the set of equivalence classes given by the equivalence relation $\approx$. Thus we have a natural map ν: $X \to X/\approx$ given by $\nu(x) = [x]$.

Definition 10.4 Let $\approx$ be an equivalence relation on the space X, and let $X/\approx$ be given the quotient topology with respect to ν. Then we call $X/\approx$ **the quotient space of *X* modulo the relation $\approx$.**

Examples 10.5

A. Get a torus as a quotient of the square $I \times I$. Look at its open sets.

B. $S^2 \cong D^2/\partial D^2$.

C. If X is any space, define the (unreduced) **cone** on X, by the formula

$CX = (X \times I)/\approx$,

where $(x,t) \approx (y,s)$ iff $(x, t) = (y, s)$ or $t = 1$.

D. Let $A \subset X$ and f: $A \to Y$. Define an equivalence on $X \coprod Y$ by letting $a \approx f(a)$ for each $a \in$ **A.** (Thus each equivalence class consists of all points in A identified to a common point in Y, together with that point in Y. Drawing in class). We let $X \cup_A Y = X \coprod Y/\approx$ be the identification space which results.

Subexamples.

Ca. $S^2 = D^2 \cup_{S1} D^2$

Cb. $S^2 = D^2 \cup_{S1} \{*\}$ as well.

Cc. $CX = (X \times I) \cup_{X \times \{1\}} *$, the cone on X.

Cd. Mapping cones and cylinders.

Ce. Cell complexes—a little introduction.

Definition 10.6 S'pose given maps f: $A \to X$ and g: $A \to Y$. Then the **pushout,** $P(f,g)$ of f and g is defined to be $X \coprod Y/\approx$, where we make the identifications $f(a) \approx g(a)$ for each $a \in A$.

Note: This is a generalization of $X \cup_A Y$ In fact, $X \cup_A Y = P(i, f)$, where i: $A \to X$ is the inclusion.

Theorem 10.7 ***Universal Property of Pushouts***

The pushout $P(f, g)$ has the following properties:

1. There exist continuous maps $X \to P(f, g)$ and $Y \to P(f, g)$ making the diagram

$$\begin{tikzcd} Y \arrow[r] & P(f,g) \\ A \arrow[u, "g"] \arrow[r, "f"'] & X \arrow[u] \end{tikzcd}$$

commute.

2. Moreover, given any commutative diagram of the form

$$\begin{tikzcd} Y \arrow[r] & Q \\ A \arrow[u, "g"] \arrow[r, "f"'] & X \arrow[u] \end{tikzcd}$$

(where all the maps are continuous) then there exists a unique map u: $P(f, g) \to Q$ making the diagram

$$\begin{tikzcd} & & Q \\ Y \arrow[r] \arrow[urr] & P(f,g) \arrow[ur, dashed] & \\ A \arrow[u, "g"] \arrow[r, "f"'] & X \arrow[u] \arrow[uur] & \end{tikzcd}$$

commute.

Exercise Set 10

1. (*The line with two zeros*) Let $L = \mathbb{R} \coprod \mathbb{R} /\approx$ where we identify every nonzero real number in the first copy of $\mathbb{R}$ with its counterpart in the second copy. Show that L is not Hausdorff. Also describe L as a pushout.

2. Prove or disprove: If X is Hausdorff and $\approx$ is an equivalence relation on X, then $X/\approx$ is Hausdorff.

3. Prove or disprove: If π: $X \to X/\approx$ is projection onto a quotient space, then π is an **open map;** that is, $\pi(U)$ is open whenever U is open in X.

4. Define an equivalence relation on $\mathbb{R}$ by $x \approx y$ iff $x-y \in \mathbb{Q}$, and let $C = \mathbb{R}/\approx$. Show that C (usually referred to as $\mathbb{R}/\mathbb{Q}$) is uncountably infinite as a set. Describe its open sets.

5. Show that $CS^n \cong D^{n+1}$ for $n \geq 0$. ($S^0 = \partial I = \{0, 1\}$.)

6. A continuous map f: $X \to Y$ is **homotopically trivial** if it is homotopic to a constant map (see Example 7.6 (E)). Prove that f is homotopically trivial iff it extends over CX. That is, there exists a (cont.) map F: $CX \to Y$ such that $F(x, 0) = f(x)$ for each $x \in X$. Conclude that, for $n \geq 0$, f: $S^n \to X$ is homotopically trivial iff it extends over D^{n+1}.

7. A space X is **contractible** if there exists $x \in X$ such that the identity map 1_X is homotopically trivial. The subspace $X \subset \mathbb{R}^n$ is **convex** if, for every pair of points x, y in X, the line segment $\{(tb + (1-t)a \ ' \ 0 \leq t \leq 1\}$ is contained in X. Show that convex spaces are contractible.

8. The **homotopy pushout** of f: $A \to X$ and g: $A \to Y$ is defined to be the space

$M(f,g) = (X \coprod Y \coprod (A \times I))/\approx$, where we make the identifications $(a,0) \approx f(a)$ and $(a, 1) \approx g(a)$ for all $a \in A$. (Draw a picture.) We say that the diagram

$$\begin{tikzcd}
Y \arrow[r, "s"] & Q \\
A \arrow[u, "g"] \arrow[r, "f"'] & X \arrow[u, "r"']
\end{tikzcd}$$

homotopy commutes if $r \circ f \simeq s \circ g$. Show that the above diagram homotopy commutes iff there exists a map K: $M(f, g) \to Q$ making the diagram

$$\begin{tikzcd}
 & & Q \\
Y \arrow[rru, "s"] \arrow[r] & M(f,g) \arrow[ru, dashed] & \\
A \arrow[u, "g"] \arrow[r, "f"'] & X \arrow[u] \arrow[ruu, "r"'] &
\end{tikzcd}$$

homotopy commute. Here, the maps $X \to M(f,g)$ and $y \to M(f,g)$ are the evident inclusions.

9. Use Theorem 10.7 to show uniqueness of pushouts up to homeomorphism; that is, if P and Q are both pushouts in that they satisfy the conclusion of Theorem 10.7, then they are necessarily homeomorphic.

11 Connectedness

Definition 11.1. A **separation** of the space X is a pair U, V of disjoint nonempty open subsets whose union is X. We say that X is **connected** if it has no separation.

Lemma 11.2 ***Things Equivalent to Connectedness***
The following are equivalent:
1. X is connected.
2. The only subspaces of X which are both open and closed are X and $\emptyset$.
3. There exists no continuous surjection $X \to \{0, 1\}$.

Scholium 11.3
The unit interval I is connected.

Proof Assume there exists a cont. surjection $f: I \to \{0\ 1\}$. Then composing f with the inclusion $\{0, 1\} \to I$ gives a continuous map $I \to I$ whose image consists of two points, contradicting the intermediate value theorem.

Examples 11.4
A. $[-1,0) \cup (0, 1]$ is disconnected.
B. $\mathbb{Q}$ is not connected.
C. $\{(x, y) : y = 0 \text{ or } y = 1/x\}$ is not connected.
D. $\mathbb{R}$ is connected. In fact:

Definition 11.5 The space X is **path-connected** if, for every pair of points a, b in X, there exists a path from a to b. (See Example 7.2H for the definition of a path.)

Lemma 11.6 ***Path-Connected Implies Connected.***
Every path-connected space is connected.

Corollary 11.7
(a) $\mathbb{R}^n$ is connected; convex subsets of $\mathbb{R}^n$ are connected.
(b) CX is connected for every space X.

Turning back to connectedness:

Proposition 11.8 ***Images and Products of Connected Spaces***
(a) The image of any connected space under a continuous map is connected.
(b) Let X and Y be non-empty sets. Then $X \times Y$ is connected iff X and Y are connected.

(Quick proof of (b):[1]
$\Leftarrow$ S'pose not, so that there exists continuous surjection $f: X \times Y \to \{0, 1\}$, with $f(x, y) = 0$ and $f(\bar{x}, \bar{y}) = 1$. Assume wlog that $f(x, \bar{y}) = 1$. Then the composite

$$f \circ \iota_{\bar{y}}: Y \to X \times Y \to \{0, 1\}$$

gives a continuous surjection from y to $\{0, 1\}$ and hence a contradiction.
$\Rightarrow$ If, wlog, Y is disconnected with continuous surjection $g: Y \to \{0, 1\}$ then the composite

[1] Compare this little gem to the painful hand-waving proof on page 150 in Munkres, who only proves one direction, and forgets to stipulate that X and Y must be non-empty for his proof to work (even though non-emptiness is only needed in the proof of the converse.

$g \circ \pi_Y: X \times Y \to Y \to \{0, 1\}$

is surjective, since X is non-empty, contradicting the assumption.

Corollary 11.9 ***Open & Closed Intervals***
No open interval is homeomorphic to a closed interval.

Exercise Set 11

1. (Munkres) If $\mathcal{T}_1 \supset \mathcal{T}_2$ are two topologies on X, what does connectedness in one topology imply about connectedness in the other?

2. Let A and B be connected spaces such that $A \cap B \neq \emptyset$. Show that their union is connected.

3. X is **totally disconnected** if the only connected subspaces are one-point sets. Show that the following are totally disconnected: **(a)** discrete spaces **(b)** $\mathbb{Q}$ **(c)** $\mathbb{R} - \mathbb{Q}$.

4. Prove that S^1 is not homeomorphic to any interval of the real line.

5. (Munkres) Show that if C is a countable subset of $\mathbb{R}^2$, then $\mathbb{R}^2 - C$ is path-connected. [Hint: how many lines pass through a given point in $\mathbb{R}^2$?]

6. Show that the set $(\{0\} \times I) \cup \langle \{(x, y) \in \mathbb{R}^2 : y = \sin(1/x)\}$ is connected but not path-connected. [Use the continuous surjection trick—it's easiest.]

7. State and prove a version of Proposition 11.8 for path-connected spaces.

8. The **comb space** is the subspace $C = \{I \times \{0\}) \cup \{(1/n) \times I\}$ $(n = 1, 2, \ldots .)$ of $\mathbb{R}^2$. The **broken comb space** B is obtained from C by removing the point at the bottom of the prong at 0:

$$B = C - (0, 1) \times \{0\}.$$

Show that B is connected but not path connected.

9. S'pose $\{Y_\alpha\}_{\alpha \in \Lambda}$ is a collection of connected (resp. path connected) subspaces of X whose intersection is non-empty and connected. Show that $\cup Y_\alpha$ is connected (resp. path connected).

10. Show that contractible spaces are path-connected.

11. Define a relation $\approx$ on a space X by taking $x \approx y$ if there is a connected subspace of X containing both x and y. Show that $\approx$ is an equivalence relation, and that the equivalence class $[x]$ containing $x \in X$ is the union of all connected subspaces of X containing x (i.e. the "largest" connected subspace of X containing x). $[x]$ is called the **connected component of *X* containing *x*.**

12. Define a relation $\approx$ on a space X by taking $x \approx y$ if there is a path from x to y. Show that $\approx$ is an equivalence relation, and that the equivalence class $[x]$ containing $x \in X$ is the union of all path-connected subspaces of X containing x. $[x]$ is called the **path component of *X* containing *x*** or the **path component of *x*** for short. Show that the connected component containing x contains the path component of x, and give an example to show that this inclusion may be strict.

13. X is called locally path-connected if, for each $x \in X$ and each neighborhood U of x, there exists an open path-connected set V with $x \in V \subset U$. Show that, if X is locally path-connected and connected, then it is path-connected.

14. X is called **semilocally simply-connected** if, for each $x \in X$ and each neighborhood U of x, there exists an open simply connected set V with $x \in V \subset U$. Give an example of a path- connected space which fails to be semilocally simply connected. (You may assume that S^1 is not simply connected.)

12. Compactness

Definitions 12.1 An **open cover** of X is a collection of open sets $\mathcal{U} = \{U_\alpha\}$ such that $\cup_\alpha U_\alpha = X$. A **subcover** of a given cover $\mathcal{U}$ is a subcollection of $\mathcal{U}$ which constitutes an open cover in its own right. The space C is said to be **compact** if every open cover of C admits a finite subcover.

Examples 12.2
A. Finite subsets of arbitrary spaces.
B. $\mathbb{R}$ is not compact, nor is any open subinterval of $\mathbb{R}$. Similarly, $\mathbb{R}^n$ is not compact, and nor are open balls in $\mathbb{R}^n$.
C. Closed and bounded subsets of $\mathbb{R}^n$. (Borel-Heine)

Proposition 12.3 ***Properties of Compact Spaces***
(a) Closed subspaces of compact spaces are compact.
(b) Compact subspaces of Hausdorff spaces are closed.
(c) The image of a compact space under a continuous map is compact.

This leads to:

Theorem 12.4 ***Some Continuous Bijections are Homeomorphisms***
If f: $X \to Y$ is a continuous bijection with X compact and Y Hausdorff, then f is a homeomorphism.

Joke 12.5 ***(Letterman)***
This relic from Fri, 21 Feb 1997:

Last night on Letterman, he inaugurated a new feature called "Over Our Heads," in which people from different academic fields presented jokes that would only be funny to others in that field ...

In that spirit, Letterman introduced a mathematician from Harvard. Here's his joke ..

"A master's student was taking his final oral exams and the professor
asked him, 'Can you name a compact topological space?' The student
thought for a moment and said, 'The real numbers.'

" There was a long pause until finally the professor said hopefully,

" In what topology?'"

Exercise Set 12

1. Prove that, if X is finite (that is, has finitely many points) then X is compact.

2. Prove that, if $K \subset Y \subset X$, then K is compact in Y iff it is compact in X.

3. Prove or disprove: If K is compact in X and $Y \subset X$, then $K \cap Y$ is compact in Y.

4. Prove that the product of two compact spaces is compact. (Try to draw pictures to help you rather than copying a text's proof.)

5.. The metric space X is sequentially compact if every sequence in X has a convergent subsequence. Prove that X compact implies X sequentially compact. Also prove the following:

The Lebesgue Lemma

If $\mathscr{U} = \{U_\alpha\}$ is an open cover of the compact metric space X, then there exists a $\delta>0$ such that every ball of radius δ is contained in at least one of the Uå

6. A space X is **weak Hausdorff** if every compact subspace of X is closed.

(a) Prove that every finite subset of a weak Hausdorff space is closed.

(b) Show that subspaces of weak Hausdorff spaces are weak Hausdorff. (See Exercise #2.)

(c) Prove that, if f: $X \to Y$ is a continuous bijection with X compact and Y weak Hausdorff, then f is a homeomorphism.

(d) Deduce that, if K is compact and Hausdorff, then its image in a weak Hausdorff space under a continuous injection is Hausdorff.

7. Respond to Joke 12.5. That is, give a topology on $\mathbb{R}$ making it compact. To get more than 60% credit for this question, it should also be Hausdorff.

13. Homotopy of Paths

Recall that a **path from a to b** in a space X is a map λ: $I \to X$ such that $\lambda(0) = a$ and $\lambda(1) = b$.

Definition 13.1 The paths λ and μ from a to b in X are path homotopic if there exists a map H: $I \times I \to X$ such that:

(a) $H(s, 0) = \lambda(s)$ and $H(s, 1) = \mu(s)$ for all $s \in I$;

(b) $H(0, t) = a$ and $H(1, t) = b$ for all $t \in I$.

We call H a **path homotopy** from λ to μ, and we write $\lambda \simeq \mu$.

Remarks 13.2

(a) Thus, for each $t \in I$, we have an intermediate path λ_t: $I \to X$ given by the composite

$$I \xrightarrow{\iota_t} I \times I \xrightarrow{\lambda} X$$

where $\iota_t(s) = (s,t)$. In other words, $\lambda_t(s) = H(s, t)$. Thus, $\lambda_0 = \lambda$ and $\lambda_1 = \mu$.

(b) A path homotopy from λ to μ is just a homotopy with condition (b) ensuring that each stage of the homotopy is a path from a to b.

Lemma 13.3 ***Path Homotopy is an Equivalence Relation***

The relation $\simeq$ on the set of paths from a to b in X is an equivalence relation.

Notation. We write $[\lambda]$ for the path homotopy class (equivalence class) of λ.

Examples 13.4
A. Any two paths in a convex space (such as $\mathbb{R}^2$) are path homotopic.
B. Let $X = \mathbb{R}^2 - \{0\}$. Then the unit circle in X is path homotopic to a circumscribing square.

Definition 13.5 ***(Path Addition)*** If λ is a path in X from a to b, and μ is a path in X from b to c, then defining $\lambda\#\mu$ to be the path from a to c given by

$$\lambda\#\mu(t) = \begin{cases} \lambda(2t) & \text{if } t \leq 1/2 \\ \mu(2t-1) & \text{if } t \geq 1/2 \end{cases}$$

Lemma 13.6 ***Algebra of Path Addition***
If we define $[\lambda][\mu] = [\lambda\#\mu]$, then this is well-defined. Further, addition of path homotopy classes of paths has the following properties:
(a) Associativity: $[\lambda]([\mu][\gamma]) = ([\lambda][\mu])[\gamma]$
(b) Right and Left Identities: $[\lambda][e_a] = [\lambda]$; $[e_b][\mu] = [\mu]$, where e_x denotes the constant path at x;
(c) Inverses: For each class $[\lambda]$ of paths from a to b, there exists a path λ^{-1} from b to a such that $[\lambda][\lambda^{-1}] = [ea]$ and $[\lambda^{-1}][\lambda] = [e_b]$. In fact, $[\lambda^{-1}]$ is unique with these properties.

Exercise Set 13
1. Suppose f: $X \to Y$ is continuous, and $\lambda \simeq \mu$ are two homotopic paths from a to b in X. Show that $f \circ \lambda$ and $f \circ \mu$ are homotopic paths from $f(a)$ to $f(b)$ in Y.
2. Suppose f: $X \to Y$ is continuous, λ is a path in X from a to b, and μ is a path in X from b to c. Show that $[f \circ (\lambda\#\mu)] = [f \circ \lambda][f \circ \mu]$. (In other words, f preserves path addition.)
3. A space X is **contractible** if there exists a point $x_0 \in X$ the identity map 1_X on X is homotopic to the constant map at x_0.
(a) Show that all convex spaces are contractible.
(b) Show that, if X is any space, then CX is contractible.
(c) Show that, if X is contractible and Y is any space, then any two maps $Y \to X$ are homotopic, and that any two maps $X \to Y$ are homotopic.
(d) (An extension property) Show that, if X is contractible, and f: $\partial(I \times I) \to X$ is any continuous map, then f extends continuously over $I \times I$.
(e) Deduce that, if a and b are any two points in the contractible space X, then there is one, and only one, equivalence class of paths from a to b. (In particular, contractible spaces are path-connected.)

4. Give an example of two paths λ and μ in S^1 which are homotopic but not path homotopic. You may assume that the identity map $S^1 \to S^1$ is not null-homotopic. [Hint: Think of $I \times I$ as the cone on its boundary and use a previous result about cones.]
5. Define a category ΠX by taking its objects to be the points in X and its morphisms given by taking $\Pi X(a, b)$ to be the set of path-homotopy classes of paths from a to b. Show that this turns ΠX into a category with composition given by path addition. ΠX is called the **fundamental groupoid** of X.

14 The Fundamental Group of a Space

Definitions 14.1. Let X be a space and let $x_0 \in X$. A **loop based at x_0** is a path in X which begins and ends at x_0. The **fundamental group of X relative to the basepoint x_0**, $\pi_1(X; x_0)$, is the set of path homotopy classes of loops based at x_0. (The definition of the group operation follows below.) When the basepoint is understood, we write $\pi_1(X; x_0)$ as $\pi_1(X)$.

Definitions 14.2. A **based space** is a pair (X, x_0), where $x_0 \in X$ is a distinguished basepoint. A **based map** f: $(X,x_0) \to (Y,y_0)$ from the based space (X,x_0) to the based space (Y,y_0) is a map f: $X \to Y$ such that $f(x_0) = y_0$. Two based maps f and g: $(X,x_0) \to (Y,y_0)$ are **based homotopic** if there is a homotopy H: $f \simeq g$ such that $H(x_0, t) = y_0$ for all $t \in I$. We call H a **basepoint-preserving homotopy** or simply a **based homotopy.**

Remarks 14.3.
1. The relation "based homotopic" among based maps $(X,x_0) \to (Y,y_0)$ is an equivalence relation.
2. Denote the set of based homotopy classes of based maps $(X,x_0) \to (Y,y_0)$ by $[(X,x_0), (Y,y_0)]$. This is the set of equivalence classes of based maps $(X,x_0) \to (Y,y_0)$). Then one has a canonical bijection

$$[(S^1, *), (X,x_0)] \cong \pi 1(X; x_0),$$

where $*$ is the point $(1, 0) \in S^1$.
3. The **homotopy groups** of X are given by defining

$$\pi_n(X,x_0) = [(S^n, *), (X,x_0)]$$

for $n \geq 1$, where S^n is based at $(1,0, \ldots ,0)$. The higher homotopy groups ($n > 1$) are beyond the scope of this course.

Lemma 14.4 ***Group Operation in $\pi_1(X)$***
$\pi_1(X; x_0)$ is a group under the operation $[\lambda][\mu] = [\lambda\#\mu]$ defined above.

Proof: This follows from Lemma **13.6**. The identity is $[e^{x_0}]$.

Examples 14.5

A. $\pi_1(\mathbb{R}^n;a) = \{[e_a]\}$, the trivial group, where e_a is the trivial loop at a.

B. The same applies to $\pi_1(X; x_0)$ if X is any contractible space (such as a convex subspace or $\mathbb{R}^n$ or the cone CY on some space Y). (See Exercise Set 13 #3.)

Definition 14.6. If f: $(X,x_0)\to(Y,y_0)$ is a based map, then define an associated homomorphism f_*: $\pi_1(X,x_0)\to\pi_1(Y,y_0)$ by the formula

$$f^*[\lambda] = [f\circ\lambda].$$

> **Theorem 14.7** ***The Functor*** $\boldsymbol{\pi_1}$
>
> The assignments $(X,x_0) \mapsto \pi_1(X,x_0)$ and $f \mapsto f_*$ define a covariant functor from the category $\mathscr{T}$ of based spaces and maps of based spaces to the category **Group.** In other words:
>
> 1. If f: $(X,x_0)\to(Y,y_0)$ is a map of based spaces, then f_* is a well-defined group homomorphism
> 2. $(g\circ f)_* = g_*\circ f_*$
> 3. $(1_X)_* = 1_{\pi_1(X)}$, the identity group homomorphism.

Proof (1) is essentially an application of Exercise Set 13 #2, while (2) and (3) follow formally from the definitions.

Remarks 14.8

1. Some people write f_* as $\pi_1(f)$ in keeping with functor notation.
2. It follows from the theorem that:

(a) Applying $(-)_*$ to the based maps and $\pi_1(-;-)$ to any commutative diagram of based spaces yields a commutative diagram of groups and homomorphisms.

(b) If f is a homeomorphism, then f_* is an isomorphism of groups, and $(f^{-1})_* = f_*^{-1}$. Thus, *the fundamental groups of homeomorphic spaces are isomorphic*. Put another way, *if two spaces have .non-isomorphic fundamental groups, they cannot be homeomorphic*. This result is the cornerstone of the methodology of algebraic topology

> **Proposition 14.9** ***Change of Basepoints***
>
> If x_0 and x_1 are two basepoints in X, and if α is a path from x_0 to x_1, define $\alpha^\#$: $\pi_1(X; x_1)\to\pi_1(X; x_0)$ by $\alpha^\#[\lambda] = [\alpha\#\lambda\#\alpha^{-1}]$. Then $\alpha^\#$: is a well-defined group isomorphism.

> **Corollary 14.10** ***Independence of Basepoint up to Isomorphism***

> If X is any path connected space and if x_0 and x_1 are any two points in X, then $\pi_1(X;x_0) \cong \pi_1(X, x_1)$.

Note. This isomorphism depends heavily on the choice of a path from one basepoint to the other. More precisely, it depends on the path homotopy class of the chosen path.

Definition 14.11 The path-connected space X is **simply connected** if $\pi_1(X,x_0) = 0$ for some (and hence every) basepoint x_0.

Exercise Set 14

1. Show that contractible spaces are simply connected.

2. The space $X \subset \mathbb{R}^n$ is **star convex** if there exists a point $x_0 \in X$ such that the line segment from each point in X to x_0 is contained in X. Show that star convex spaces are contractible (and hence simply connected).

3. Show that X is simply connected iff it is path connected, and every map $S^1 \to X$ is homotopically trivial.

4. Suppose that r: $X \to A$ is a retraction, and let $a_0 \in A$. Prove that

$$r_*: \pi_1(X, a_0) \to \pi_1(A, a_0)$$

is a surjection.

5. (Munkres) Let x_0 and x_1 be points in the path-connected space X. Show that $\pi_1(X, x_0)$ is abelian iff $\alpha^\# = \beta^\#$ for every pair of paths α, β from x_0 to x_1.

6. Show that if f and g are based homotopic maps $(X;x_0) \to (X, x_1)$, then $f_* = g_*$. Thus the induced homomorphism depends only on the based homotopy class of the map.

7. Let $\mathscr{T}$ be the category of based topological spaces and continuous basepoint-preserving maps as above. Define the associated **homotopy category,** $h\mathscr{T}$ as follows: For the objects of $h\mathscr{T}$, take the objects of $\mathscr{T}$ (that is, the based spaces). A morphism in $h\mathscr{T}$ is defined to be a basepoint-preserving homotopy class of maps in $\mathscr{T}$.

(a) Show that $h\mathscr{T}$ is indeed a category, and construct a natural covariant functor h: $\mathscr{T} \to h\mathscr{T}$.

(b) Prove that the functor π_1 "factors through" $h\mathscr{T}$. In other words, there exists a unique functor $h\mathscr{T} \to$ **Group** such that the diagram

$$\begin{tikzcd} \mathscr{T} \arrow[r, "\pi_1"] \arrow[d, "h"'] & \textbf{Group} \\ h\mathscr{T} \arrow[ur] & \end{tikzcd}$$

of functors commutes. In other words, we think of the functor π_1 as a functor defined on the homotopy category of based spaces.

8.[2] Prove that $X = \mathbb{R}^n - \{0\}$ is simply connected if $n > 2$ as follows:

(a) If λ is a loop in X, argue that it is homotopic to a piecewise linear loop (cover its image by little open discs not intersecting the origin. Their preimages form an open cover of the unit interval. Use the Lebesgue number of that cover to partition the interval. The

---

[2] This exercise is pretty ugly. A far more elegant proof can be gotten from the *Seifert-van Kampen* theorem

image of each subinterval is thus inside a little disc, and hence homotopic to a line segment...)
(b) Since there are finitely many line segments in this piecewise linear path, show there exists a half-line through the origin missing all of the vertex points.
(c) If that half-line L intersects the piecewise linear path, it must do so in a single point. Argue that that path segment is homotopic to one that misses L entirely. [Hint: the crossing lines define a two-dimensional plane. There is at least one extra dimension to move in..]
(d) Having altered the path so that it misses L entirely, argue that $\mathbb{R}^n - L$ is contractible.

15. Covering Spaces

Definition 15.1 Let p: $E \to$B be a continuous surjection with the following property: for each $b \in B$ there is an open set $U \subset B$ containing b such that $p^{-1}(U)$ is a disjoint union of open sets $V_\alpha \subset E$ with $p|V_\alpha$: $V_\alpha \to U$ a homeomorphism for each α. We then call p a **covering map** and E a **covering space of B.** We sometimes call the open sets U **canonical open sets** in B.

Remarks 15.2
1. If b and U are as above, then $p^{-1}(U)$ may be an infinite disjoint union. If it is finite, we refer to E as a **finite cover** of B.
2. For each $b \in B$ one has $p^{-1}(b) = \coprod_\alpha b_\alpha$ where $b_\alpha = p^{-1}(b) \cap V_\alpha$. Thus $p^{-1}(b)$ is a discrete subspace of E.
3. Any covering projection is necessarily an open surjection, and thus a projection in the usual sense. (See the exercises.)

Examples 15.3
A. The identity 1_X: $X \to X$ for any space X.
B. The projection p: $\coprod_\alpha X_\alpha \to X$ for any space X, where each X_α is a copy of X.
C. The map p: $\mathbb{R} \to S^1$, given by $p(x) = e^{ix}$.
D. The exponential exp: $\mathbb{C} \to C-\{0\}$ given by $\exp(z) = e^z$.
E. The projection p: $\mathbb{R}\times\mathbb{R}$ □ $S^1\times S^1$ given by $p(x, y) = (e^{ix}, e^{iy})$.

Definition 15.4 Let p: $E \to B$ be a covering map. If f: $X \to$B is any map, then a **lift of f** is a map $\tilde{f}$: $X \to E$ such that the diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

commutes.

Examples 15.5
A. (Trivial example) The identity $E \to E$ is a lift of p: $E \to B$.

B. One has the following lifting diagram

$$\begin{tikzcd} & S^1 \arrow[d, "g"] \\ S^1 \arrow[ur, "g"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

where g is the 2-fold winding map and f is the 4-fold winding map.

C. (A generic Example) Let p: $E \to B$ be a covering space, and let $F = p^{-1}(b)$ for some fixed point $b \in B$, and let S be any space (think of S as some discrete space perhaps). Then the trivial map $S \to B$ taking everything to b has as lifts, all possible maps $S \to F \subset E$.

Lemma 15.6 ***Path Lifting***
Let p: $E \to B$ be a covering projection such that $p(e) = b$. Then any path in B beginning at b has a unique lift to a path in E beginning at e.

Lemma 15.7 ***Homotopy Lifting***
Let p: $E \to B$ be a covering projection with $p(e) = b$, and let H: $I \times I \to B$ be a homotopy with $H(0, 0) = b$. Then there is a unique lifting of H to a map
$\tilde{H}$: $I \times I \to \mathrm{E}$ with $\tilde{H}(0, 0) = e$. If H is a path homotopy, then so is $\tilde{H}$.

Proof Cover B with a union of canonical open sets U_α and let $W_\alpha = H^{-1}(U_\alpha)$, so that the W_α are an open cover of $I \times I$. Using the product metric on $I \times I$, let δ be the Lebesgue number of the cover, so that each square of width δ is contained in some W_α. Subdivide $I \times I$ into rectangles of diameter $\leq \delta$. If the restriction to H any connected subset of its boundary has been lifted to E, then it must, by virtue of its connectivity, land in some $V_{\alpha\beta}$ (one of the pieces of $p^{-1}(U_\alpha)$ that is homeomorphic to U_α, and so we can use the homeomorphism to extend the lift uniquely to the whole rectangle. ♣

Theorem 15.8 ***Path Homotopy Lifting***
Let p: $E \to B$ be a covering projection with $p(e_0) = b_0$, and let λ and μ be any two paths in B from b_0 to b_1. Denote their (unique) lifts to paths in E starting at e_0 by $\tilde{\lambda}$ and $\tilde{\mu}$ respectively. If λ and μ are path homotopic, then $\tilde{\lambda}$ and $\tilde{\mu}$ end at the same point in E and are also path homotopic.

Proof If λ and μ are path homotopic, then lift the path-homotopy H: $I \times I \to B$ to E starting at e_0. Note that one has, by definition of path homotopy

(a) $H(s, 0) = \lambda(s)$ and $H(s, 1) = \mu(s)$ for all $s \in I$;
(b) $H(0, t) = a$ and $H(1, t) = b$ for all $t \in I$.

What does the lift look like? By uniqueness of path lifts, $\tilde{H}|(I \times 0)$, being a lift of λ, must equal $\tilde{\lambda}$, and $\tilde{H}|(I \times 1)$, being a lift of μ, must equal $\tilde{\mu}$. I now claim that $\tilde{H}$ is constant on $0 \times I$ and $1 \times I$, meaning that it is a path homotopy. But, $\tilde{H}|(0 \times I)$ is a path in E covering the single point b_0. Since there can only be one such path (uniqueness again), and since the constant path at e_0 is such a path, the first part of the claim follows. The second part of the claim is similar. ♣

> **Corollary 15.9 p_* *Is Injective***
> If p: $E \to B$ is a covering space and $e_0 \in E$, then
> $$p_*: \pi_1(E; e_0) \to \pi_1(B; p(e_0))$$
> is injective. Hence, $\pi_1(E; e_0)$ is isomorphic to a subgroup of $\pi_1(B; p(e_0))$.

Proof If $p_*[\lambda] = p_*[\mu]$, then the loops $p\circ\lambda$ and $p\circ\mu$ are path homotopic in B. Since λ and μ are the lifts of these two paths to E, it follows that they are also path homotopic in E, as required. ♣

Remarks
Corollary 15.9 asserts that the fundamental group pf the total space "is" a subgroup of the fundamental groups of the base space. We shall see that the conjugacy class of this subgroup uniquely determines the topological structure of the covering space!

Exercise Set 15

1. Prove that covering projections are open surjections. Deduce that B has the quotient topology induced by p.

2. Let p: $E \to B$ be a covering space with B connected and $p^{-1}(b) = \{e_1, \dots, e_k\}$ for some integer k. Show that the preimage of every point consists of k elements. We call such a cover a ***k*-fold cover**. A **finite cover** of B is any cover with $p^{-1}(b)$ finite for every b.

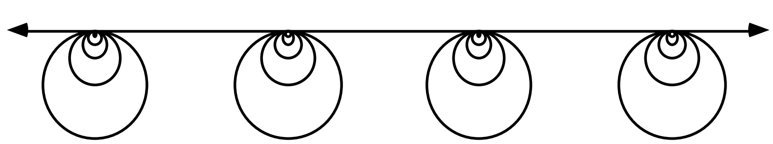
3. Show that, if p: $E \to B$ is a finite cover (see Exercise 2) and q: $F \to E$ is any cover, then $p\circ q$ is a cover.

4. Show that, if p: $E \to B$ is a cover and $C \subset B$ is a subspace, then $p|p^{-1}(C)$: $p^{-1}(C) \to C$ is also a cover.

5. **(a)** Show that the projection f: $\mathbb{R} \times \mathbb{R} \to S^1 \times S^1 = T$ $f(x, y) = (e^{2\pi ix}, e^{2\pi iy})$ is a cover of the torus T.

(b) Let $B \subset T$ consist of $(S^1 \times \{1\}) \cup (\{1\} \times S^1)$ (the figure-eight). Use part (a) to find a cover of the figure-eight.

6. Find infinitely many connected covers of the "Hawaiian earring." [Hint: Here is one of them:



The line on top wraps around the outer circle.]

7. Show that the finiteness example in Exercise 3 is necessary as follows: Let H be the Hawaiian earring, and define p: $H \times \mathbb{N}^* \to H$ as the projection, and q: $\coprod E_n \to H \times \mathbb{N}^*$ be, on the nth copy, the nth cover of H you constructed in Exercise 6. Show that $q\circ p$ is not a covering space.

16. The Fundamental Group of the Circle and Other Applications

Theorem 16.1 ***Fundamental Group of the Circle***

$$\pi_1(S^1) \cong \mathbb{Z}$$

Proof Choose the basepoint $s_0 = (1, 0) \in S^1$, and define φ: $\pi_1(S^1;s_0) \to \mathbb{Z}$ as follows. Let exp $\mathbb{R} \to S^1$ be given by $\exp(r) = e^{2\pi ir}$. Notice that $\exp(0) = s_0 \in S^1$. If λ is a loop in S^1 based at (0, 1), then define $\varphi[\lambda]$ = endpoint of lift of λ starting at 0. By Theorem 15.8, this is well-defined (independent of the representative).

Additivity: Suppose λ_1 and λ_2 are two loops at s_0, with $\psi[\lambda_1] = n_1$ and $\psi[\lambda_2] = n_2$. Then we can lift $\lambda_1\#\lambda_2$ by successively lifting λ_1 and then λ_2. The lift of λ_1 ends at n_1, and the lift of λ_2 starting at n_1, is a translate of its lift starting at 0, and so ends at n_1+n_2, showing additivity. (Note that we are using uniqueness of lifts heavily here...)

Surjectivity: Let $n \in \mathbb{Z}$. The path p in $\mathbb{R}$ from 0 to n maps onto a loop λ which lifts to p, so that $\psi[\lambda] = n$.

Injectivity: If $[\lambda]$ is in the kernel of ψ, then λ lifts to a loop at 0 in $\mathbb{R}$. Since $\mathbb{R}$ is contractible, this loop is null-homotopic, whence so is its image λ. ♥

Lemma 16.2 ***Homotopies that Change Basepoints***

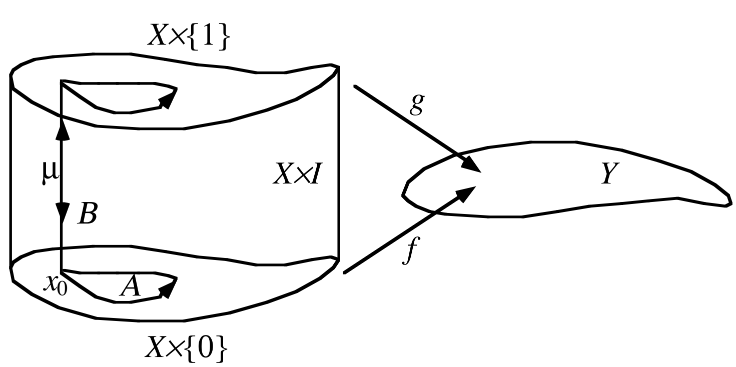
S'pose that (X, x_0) and (Y, y_0) are based spaces, f: $(X, x_0) \to (Y, y_0)$ is any map and H: $X\times I \to Y$ is a homotopy (that does not necessarily preserve the basepoint) from f to g: $X \to Y$, with $g(x_0) = y_1$. Let λ be the path in Y from y_0 to y_1 given by $H|\{x_0\}\times I$. Then

$$f_* = \lambda^{\#}\, g_* : \pi_1(X; x_0) \to \pi_1(Y; y_1) \to \pi_1(Y; y_0)$$

Proof First, let us base $X\times I$ at $(x_0, 0)$. Since $H(x_0, 0) = f(x_0) = y_0$, we have a homomorphism

$$H_*: \pi_1(X\times I; (x_0,0)) \to \pi_1(Y; y_0).$$

Let α be a loop in X based at x_0, and let μ be the path in $X\times I$ from $(x_0, 0)$ to $(x_0, 1)$ given by $\mu(t) = (x_0, t)$. Consider what happens under H_* to the two loops $A(t) = (\alpha(t), 0)$ and $B(t) = \mu \# (\alpha(t), 1) \# \mu^{-1}$ based at $(x_0, 0)$. (See figure.)



Visibly, $H_*[A] = f_*[\alpha]$, while $H_*[B] = \lambda^{\#}[g\circ\alpha] = \lambda^{\#}\ g_*[\alpha]$. Thus, it suffices to show that A and B are homotopic loops in $X\times I$ based at $(x_0, 0)$.

For each $s \in I$ let μ_s: $I\to X\times I$ be the path $\mu_s(t) = \mu(st)$. Then μ_s is a path from $(x_0, 0)$ to (x_0, s) following μ part-way up. Moreover, μ_0 is the constant path at $(x_0, 0)$ while $\mu_1 = \mu$. Now define a path homotopy K: $I\times I\to X\times I$ by

$$K(s, t) = \mu_s \# (\alpha(t), s) \# \mu_s^{-1}$$

Then $K(s, 0) = \mu_0 \# (\alpha(t), 0) \# \mu_0^{-1} \simeq A$ and $K(s, 1) = \mu \# (\alpha(t), 1) \# \mu^{-1} = B$, showing that the paths are homotopic, modulo believing that K is continuous. But the definition of path addition allows us to explicitly write down a formula for K showing this to be true. ♥

Theorem 16.3 ***Homotopy Equivalences Give Group Isomorphisms***

(a) S'pose f: $(X,x)\to(Y,y)$ is a homotopy equivalence of based spaces

or

(b) S'pose f: $X\to Y$ is a homotopy equivalence of spaces, and $x \in X$, with $f(x) = y$. Then

$$f_* : \pi_1(X; x)\to\pi_1(Y; y)$$

is an isomorphism.

Proof

(a) By Exercise Set 14 #7, $f \simeq f'$ implies $f_* = f'_*$. Now, s'pose f is a homotopy equivalence of based spaces. Then there exists g with $fg \simeq 1_Y$ and $gf \simeq 1_X$ (all through based maps). It now follows that $f_*g_* = (1_Y)_* = 1_{\pi_1(Y)}$ and $g_*f_* = 1_{\pi_1(X)}$, as required.

(b) Let g be a homotopy inverse of f, and suppose $g(y) = x'$. Then one has a composite

$$\pi_1(X; x) \xrightarrow{f} \pi_1(Y; y) \xrightarrow{g} \pi_1(X; x')$$

with $g\circ f$ homotopic to the identity on X. By the lemma $1_* = \lambda^{\#}\ (g\circ f)_*$, so that $(g\circ f)_*$ is an isomorphism. Thus, f_* is injective. A similar argument using $f\circ g$ now shows that $(f\circ g)_*$ is also an isomorphism, whence f_* is surjective too. ♥

Examples 16.4

A. $(R^n-\{0\},(1,0,...,0)) \cong (S^{n-1},1)$ and so their fundamental groups are isomorphic (and hence zero if $n \geq 3$ – see below).

B. The plane with two holes $\cong S^1\vee S^1$.

C. S^n is not homotopy equivalent to S^1 if $n > 1$.

Theorem 16.5 ***Homotopy Groups of Higher Spheres***

$$\pi_1(S^n) = 0 \text{ if } n \geq 2.$$

Elegant Proof later

Definition 16.6 A subspace $A \subset X$ is a **retract of X** if there exists a (continuous) map r: $X \to A$ (called a **retraction**) such that $r(a) = a$ for every A. (Note that this is equivalent to saying that $r \circ \iota = 1_A$, where ι is the inclusion $A \to X$.)

Lemma 16.7 ***Fundamental Group of a Retract***
If A is a retract of X, then $\pi_1(A)$ is isomorphic with a subgroup of $\pi_1(X)$.

Proof One has $r \circ \iota = 1_A$. Hence, $r_* \iota_* = 1_{\pi_1(A)}$ (where we are basing everything at some point in A). It follows that ι_*: $\pi_1(A) \to \pi_1(X)$ must be injective, so that $\pi_1(A)$ is isomorphic to its image, which is a subgroup of $\pi_1(X)$, as required. ♦

Theorem 16.8 ***No-Retraction Theorem & Brouwer Fixed-Point Theorem***
(a) S^1 is not a retract of D^2.
(b) Every map $D^2 \to D^2$ has at least one fixed point.

Proof
(a) This follows from the lemma, since $\pi_1(S^1) \cong \mathbb{Z}$ cannot be a subgroup of $\pi_1(D^2) = 0$. Note that what we have proved is that S^1 is not a *homotopy* retract of D^2. That is, there is no map $D^2 \to S^1$ whose restriction to S^1 is even *homotopic* to the identity on S^1.
(b) Suppose, on the contrary, that f: $D^2 \to D^2$ possessed no fixed points. Let

$$v: D^2 \to D^2 - \{0\}$$

be given by $v(x) = x - f(x)$. Normalizing this gives us a map w: $D^2 \to S^1$. Then the restriction of w to S^1 is homotopic to the identity via the normalization of

$$(1-t)v(x) + tx.$$

Indeed, it suffices to show that $(1-t)v(x) + tx$ is nowhere zero (so that the homotopy just defined takes values on S^1). But

$$(1-t)v(x) + x = (1-t)(x - f(x)) + tx = x - (1-t)f(x).$$

Now $f(x)$ is somewhere in the disc, and so, $(1-t)f(x)$, having length $\le (1-t)$, cannot be on the sphere (and thus equal x) unless $t = 0$. Thus, the only way that $x - (1-t)f(x)$ can be zero is if $t = 0$, giving $f(x) = x$, a contradiction.

Thus, what we have produced is a homotopy retraction $D^2 \to S^1$; that is, a map $D^2 \to S^1$ whose restriction to S^1 is homotopic to the identity, contradicting (a).♣

Definition 16.9 A covering p: $E \to B$ is called a **universal cover** if E is simply connected.

Examples 16.10
A. exp: $\mathbb{R} \to S^1$.
B. The cover $\mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ considered earlier.

Theorem 16.11 ***The Universal Cover***
Any two universal covers E and E' of B are homeomorphic over B; that is, there exists a homeomorphism $E \cong E'$ covering B.

Exercise Set 16

1. (Munkres) Show that, if A is a deformation retract of X, and B is a deformation retract of A, then B is a deformation retract of X.

2. ((a)–(e) are from Munkres) for each of the following, the fundamental group is either trivial, infinite cyclic, or isomorphic to the fundamental group of the figure eight. Determine which one:

(a) The solid torus $D^2 \times S^1$ **(b)** The torus with a point removed **(c)** $S^1 \times I$ **(d)** $S^1 \times \mathbb{R}$ **(e)** $\mathbb{R}^3$ with the non-negative x, y, and z-axes deleted. **(f)** S^2 with a handle attached (remove two disjoint little discs and paste in a copy of $S^1 \times I$ **(g)** The solid torus with a solid handle attached (**h**) The "theta space" $S^1 \cup ([-1, 1] \times 0)$

3. (Munkres) Show that a space is contractible iff it has the homotopy type of a single point space.

4. (Munkres) Show that a retract of a contractible space is contractible.

5. Suppose x_1 and x_2 are any two basepoints in the path-connected space X, and that $\pi_1(X; x_1)$ is abelian. Show that:

(a) $\pi_1(X; x_2)$ is abelian

(b) If λ_1 and λ_2 are any two paths from x_1 to x_2, then $\lambda_1{}^{\#} = \lambda_2{}^{\#}$.

6. (Munkres; The degree of a self-map of S^1). If f: $S^1 \to S^1$ is any map Define its **degree** as

$$\mathrm{Deg}(f) = \phi(\lambda^{\#} f_*(e)) \in \mathbb{Z},$$

where e is the identity loop $S^1 \to S^1$ based at $(1, 0)$, λ is any path in S^1 from $f(1, 0)$ to $(1, 0)$ (see Exercise 5) and ϕ identifies $\pi_1(S^1)$ with $\mathbb{Z}$.

(a) Show that homotopic maps have the same degree.

(b) Show that $\mathrm{Deg}(f \circ g) = \mathrm{Deg}(f)\mathrm{Deg}(g)$

(c) Find the degrees of

(i) the constant map

(ii) the identity map

(iii) the reflection map $(x, y) \mapsto (x, -y)$

(iv) the antipodal map $(x, y) \mapsto (-x, -y)$

(v) the map $z \mapsto z^n$

(d) Show that maps of the same degree are homotopic.

7. The covering space p: $E \to B$ satisfies the **covering homotopy property** (CHP) for maps $X \to B$ if, given any map f: $X \to B$ which lifts to a map $\tilde{f}$: $X \to E$, and a homotopy h: $X \times I \to B$ from f to g, there exists a lift of h to a map H: $X \times I \to E$ covering h with $H(x, 0) = \tilde{f}(x)$.

(a) Show that the covering homotopy property holds for $X = S^1$.

(b) Prove that, if the CHP holds for X, then it holds for $Y = X \cup_\phi D^2$, where ϕ: $\partial D^2 \to X$ is any given map.

(c) Prove that, if the CHP holds for X, then it holds for $Y = X \cup_\phi D^n$, where ϕ: $\partial D^n \to X$ is any given map.

8. An ***n*-dimensional CW complex** is a space constructed inductively as follows:

X^1 is a wedge of circles;

$X^r = X^{r-1} \cup_{\phi_å} (\coprod_\alpha D^r)$, where ϕ: $\partial D^r \to X^{r-1}$ is arbitrary (called an **attaching map**; the D^rs are called the ***r*-cells** of X).

$X = X^n$.

Prove that the CHP holds for all n-dimensional CW complexes.

17 Fundamental Group of S^n, T and Certain Other Spaces

Here we shall show, among other things, that the sphere and the torus are not homeomorphic, or even homotopy equivalent.

Theorem 17.1 ***Fundamental Group of a Union***
S'pose $X = U \cup V$ where $U \cap V$ is path-connected and non-empty. Then, for any $x_0 \in U \cap V$, $\pi_1(X; x_0)$ is generated by the images of $\pi_1(U; x_0)$ and $\pi_1(V; x_0)$ under the homomorphisms induced by respective inclusions.

Proof Let λ: $I \to X$ be any loop in X based at x_0. It suffices, to prove the theorem, to write λ, up to path homotopy in X, as a sum of loops in U and V based at x_0. First, I claim there is a subdivision of I, $0 = b_0 < b_1, \ldots, < b_n = 1$, such that λ maps each subinterval $[b_{i-1}, b_i]$ into either U or V. This claim follows by the Lebesgue lemma using the cover $\{\lambda^{-1}(U), \lambda^{-1}(V)\}$ *of I*.

S'pose without loss of generality that $\lambda[b_0, b_1] \subset U$, and let $[b_{k_1}, b_{k_1+1}]$ be the first subinterval not landing in U, Then $\lambda[b_0, b_{k_1}] \subset U$. Proceeding in this way, we choose integers $0 = b_{k_0} < b_{k_2} < b_{k_3} < \ldots < b_{k_m} = 1$ with the property that $\lambda[b_{k_2}, b_{k_3}] \subset V$, $\lambda[b_{k_3}, b_{k_4}] \subset U$, and so on. Now choose paths μ_i in $U \cap V$ from each b_{k_i} to x_0. *Let* λ_i be the restriction of λ to $[b_{k_{i-1}}, b_{k_i}]$, so that $\lambda = \lambda_1 \# \lambda_2 \# \ldots \# \lambda_n$. Then we can write

$$\lambda \simeq (\lambda_1 \# \mu_1) \# (\mu_1^{-1} \# \lambda_2 \# \mu_2) \# \ldots \# (\mu_{n-1}^{-1} \# \lambda_{n-1} \# \mu_n) \# (\mu_n^{-1} \# \lambda_n)$$

which is a sum of loops alternatively in U and V, as required. ❥

Corollary 17.2 ***Fundamental Group of a Union of Simply Connected Spaces***
S'pose $X = U \cup V$ where $U \cap V$ is path-connected and non-empty, with U and V simply connected. Then $\pi_1(X) = 0$.

In particular, $\pi_1(S^2) = 0$. What about $\pi_1(T)$? We use the following interesting results:

Theorem 17.3 ***Fundamental Group of a Product***

$$\pi_1(X \times Y; (x_0, y_0)) \cong \pi_1(X; x_0) \times \pi_1(Y; y_0)$$

Proof First of all, a loop in $X \times Y$ is a homotopy class of maps $I \to X \times Y$. But maps from $I \to X \times Y$ are in 1-1 correspondence with pairs λ_1: $I \to X$ and λ_2: $I \to Y$. Further, loops correspond to pairs of loops, homotopies to pairs of homotopies (for the same reason) and sums of loops to sums of pairs. ➧

Corollary 17.4 ***Fundamental Group of T***

$\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$

Hence the torus and the 2-sphere are homotopically distinct. Now what about some other interesting spaces.

Definition 17.5 The **projective plane** $\boldsymbol{P^2}$ is defined as $S^2/\approx$ where we identify pairs of antipodal points.

Proposition 17.6 ***The Sphere Covers the Projective Plane***

P^2 is a compact surface, and the canonical projection p: $S^2 \to P^2$ is a covering map.

Proof That P^2 is compact follows from the fact that it is the image of the compact space S^2 under a continuous map, That it is a surface will follow from the fact that it is covered by S^2 (whence it is locally homeomorphic to 2-dimensional Euclidean space).

To show that p is a covering map, choose, for each $x \in P^2$ an open neighborhood U of any point in its preimage such that its closure $\bar{U}$ contains no pair of antipodal points. Then $p|\bar{U}$ is injective and continuous. Further, since $\bar{U}$ is compact and its image is Hausdorff (see the exercises) it is a homeomorphism onto its image. Therefore, the same is true of its restriction, $p|U$. Further, $p^{-1}(p(U) = U \coprod -U$, with the restriction of p to both U and $-U$ a homeomorphism, completing the proof. ✣

Corollary 17.7 ***The Projective Plane has Fundamental Group*** $\mathbb{Z}/2$

Proof We know that p: $S^2 \to P^2$ is a covering space with S^2 simply connected. Choose a basepoint $x_0 \in P^2$. I claim there is a 1-1 correspondence between points in $p^{-1}(x_0)$ and elements of $\pi_1(P^2; x_0)$ (showing the result, since $\mathbb{Z}/2$ is the only group of order 2). The desired correspondence ϕ is given as follows: Write $p^{-1}(x_0) = \{a, b\}$. Define $\phi(a)$ = class of any loop based at a, at x_0 and $\phi(b)$ = Class of the image of any path from a to b. (Since S^2 is simply connected, any two such are path homotopic.) Now any loop in P^2 lifts to one or the other (by simple connectivity of S^2 again) and so ϕ is surjective. Further, Theorem 15.8 tells us that ϕ is injective. ☛

Exercise Set 17

1. Show that P^2 is a Hausdorff space.

2. Generalize the argument in Corollary 17.7 to prove the following: If p: $E \to B$ is a covering space with E simply connected, then elements of $\pi_1(B)$ are in 1-1 correspondence with elements of $p^{-1}(x)$ for any $x \in B$.

3. Prove that, if p: $E \to B$ is a covering space with E path connected and B simply connected, then p is a homeomorphism.

4. Use the preceding exercise to classify all covering spaces of a simply connected space B.

5. Prove the fundamental Theorem of Algebra: Any polynomial equation of degree ≥ 1 with complex coefficients has at least one (real or complex) root. Deduce that any polynomial equation of odd degree ≥ 1 with real coefficients has at least one real root. [Hint: For the first assertion, consult a text. For the second assertion, show that the complex roots must occur in complex conjugate pairs.]