Advanced Engineering Math I

Lecture Notes
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(second printing: 2000)

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ENGINEERING MATH I
Math 143 Notes prepared by Stefan Waner

Text: Advanced Engineering Mathematics by Erwin Kreyszig (Ninth ed.)

Part A Introduction to Linear Algebra

1. Matrix Algebra (Kreyszig p.280 §§7.1–7.2)

Definition 1.1 An ***m*×*n* matrix with entries in the reals** is a rectangular array of real numbers.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Examples 1.2 In class

Notation
(1) If A is any $m \times n$ matrix, then the ijth entry of A will sometimes be written as a_{ij} and sometimes as A_{ij}.
(2) We sometimes write $[a_{ij}]$ or $[A_{ij}]$ to refer to the matrix A. Thus, for example, $[a_{ij}]$ means "the matrix whose ijth entry is a_{ij}." Thus,

$A = [a_{ij}] = [A_{ij}]$ = the matrix whose ijth entry is a_{ij}.

(Similarly, the matrix B is written as $[b_{ij}]$, the matrix Γ as $[\gamma_{ij}]$, etc.)

Definitions 1.2
(i) Two matrices are **equal** if they have the same dimensions and their corresponding entries agree (i.e., they are "the same matrix").
(ii) The $m \times n$ matrix A is **square** if $m = n$. In a square matrix, the entries $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ form the **leading diagonal** of A.
(iii) A **column vector** is an $n \times 1$ matrix for some n; a **row vector** is a $1 \times n$ matrix fro some n.

We now turn to the algebra of matrices.

Definitions 1.3
(a) If $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $m \times n$ matrices, then their **sum** is the matrix

$A + B = [a_{ij} + b_{ij}]$.

In words, "$A+B$ is the matrix whose ijth entry is $a_{ij} + b_{ij}$."
Thus, we obtain the ij$^{\text{th}}$ entry of $A+B$ by adding the ij$^{\text{th}}$ entries of A and B.
(b) The **zero *m*×*n* matrix O** is the $m \times n$ matrix all of whose entries are 0.
(c) If $A = [a_{ij}]$, then $-A$ is the matrix $[-a_{ij}]$.
(d) More generally, if $A = [a_{ij}]$ and if r is a real number, then rA is the matrix $[ra_{ij}]$. This is called **scalar multiplication by *r*.**
(e) If $A = [a_{ij}]$ is any matrix, then its **transpose**, A^T, is the matrix given by

$(A^T)_{ij} = A_{ji}$.

That is, A^T is obtained from A by writing the rows as columns.

(f) The matrix A is **symmetric** if $A^T = A$. In other words, $A_{ji} = A_{ij}$. for every i and j. A is **skew-symmetric** if $A^T = -A$; that is, $A_{ji} = -A_{ij}$ for every i and j.

Examples 1.4 In class

The following rules follow from the counterparts in arithmetic.

Proposition 1.5 (Algebra of Addition and Scalar Multiplication)
One has, for all real numbers r, s and $m\times n$ matrices A, B, C,

(a) $(A+B)+C = A+(B+C)$	(associativity)
(b) $A+B = B+A$	(commutativity)
(c) $A+O = O+A = A$	(additive identity)
(d) $A+(-A) = (-A)+A = O$	(additive inversion)
(e) $r(A+B) = rA + rB$	(left distributativity)
(f) $(r+s)A = rA + sA$	(right distributativity)
(g) $(rs)A = r(sA)$	(associativity of scalar multiplication)
(h) $1.A = A$	(scalar multiplicative identity)
(i) $0.A = O$	(annihilation by zero)
(j) $(-1).A = -A$	(no name)
(k) $(A+B)^T = A^T + B^T$	(transpose of a sum)

Definition 1.6 Let $A = [a_{ij}]$ be an $m\times n$ matrix, and let $B = [b_{ij}]$ be an $n\times l$ matrix. Then their *product*, $A\cdot B$ is the $m\times l$ matrix whose ij^{th} entry is given by

$$\boxed{(A.B)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}}$$ **Matrix Product**

Diagramatically, this entry is obtained by "dotting" the i^{th} row of A with the j^{th} column of B.

Examples in class

Definition 1.7. If $n \geq 1$, then the $n\times n$ identity matrix I_n is the $n\times n$ matrix whose ij^{th} entry is given by 1 if $i = j$, and 0 if $i \neq j$.

Thus,

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

When there is no danger of confusion, we just write I and drop the subscript n.

Proposition 1.8 (Multiplicative Properties)
Assuming that the sizes of the matrices are such that the operations can be performed, one has for all A, B, C, λ, μ:

(a) $(AB)C = A(BC)$ (associativity)
(b) $A(B+C) = AB + AC$ (left distributativity)
(c) $(A+B)C = AC + BC$ (right distributativity)
(d) $AI = IA = A$ (additive identity)
(e) $\lambda(AB) = (\lambda A)B$ (associativity of scalar multiplication)
(f) $O.A = A.O = O$ (actually follows from (c) and Proposition 1.5.)
(g) $(AB)^T = B^T A^T$ (transpose of a product)

What seems to be missing is the *commutative rule:* $AB = BA$. Unfortunately (or perhaps fortunately), however, it fails to hold in general.

Example of failure of commutativity

Examples 1.9
(a) There exist matrices $A \neq O$, $B \neq O$, but with $A.B \neq O$, shown in class. Such matrices A and B are called "zero divisors."
(b) As a result of (a), the cancellation law fails in general. Indeed, let A and B be as in (i). Then

$$A.B = O = A.O,$$

but "canceling" the A would yield

$$B = O,$$

which it isn't.

Exercise Set 1
p. 277, #1–15 odd
p. 286, #1–9 odd,
Hand-In (Value = 15 points)
1. Show that any square A matrix can be written as a sum of symmetric matrix and a skew-symmetric matrix.
3. If the point (x, y) in the plane is rotated counterclockwise through an angle θ, then the new coordinates of the point are:

$$\bar{x} = x\cos\theta - y\sin\theta$$
$$\bar{y} = x\sin\theta + y\cos\theta$$

(a) Show that

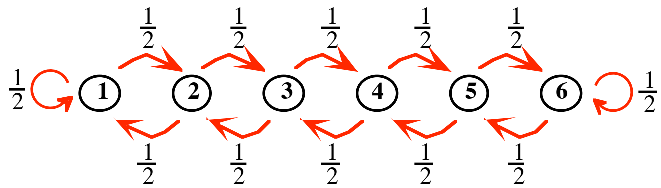
$$\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = R \begin{bmatrix} x \\ y \end{bmatrix},$$

where R is the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ (called a "rotation matrix.")
(b) Show that

$$R^2 = \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}.$$

Interpret this equation geometrically by saying what the left-hand side and right-hand side are doing geometrically.
3. Read Example 13 on p. 285 and do the following:

Random Walks A random walk with "partially reflecting barriers" may be modeled by the following Markov chain.



This is a crude model for the motion of a particle along a straight line where the particle moves at random either to the left or the right at each time step. When it hits either end (State 1 or 6) it either remains there the next time step or is reflected back. This process is also known as a **discrete model of one-dimensional dissipation with partially reflecting barriers**.

(a) Write down the transition matrix A for the process.

(b) Compute the two-step transition matrix A^2 and use it to calculate the distribution vector after two steps if the initial distribution vector is $[1 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0]$.

(c) Find the probability that a particle starting in state 5 will reach state 2 after four steps.

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2. Systems of Linear Equations (Kreyszig, p. 287, §7.3)

Definition 2.1 A **linear equation** in the n variables $x_1, x_2, \ldots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b,$$

where the a_i and b are real constants, and the x_i are the unknowns.

Examples 2.2

$3x - y/2 = -11$ is a linear equation in x and y;

$y = x/2 + 67y - z$ can be rearranged to give a linear equation in x, y and z.

$x_1 + x_2 + \ldots + x_n = 1$ is a linear equation in $x_1, x_2, \ldots, x_n$.

$x + 3y^2 = 5$, $3x + 2y - z + xz = 4$, and $y = \sin x$ are not linear.

Definition 2.3 A **solution vector** of the linear equation

$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$

is a sequence of numbers $(s_1, s_2, \ldots, s_n)$, each of which is in $\mathbb{R}$, such that, if we replace each x_i by s_i, the expression becomes true. The set consisting of all solutions to the equation is called the **solution set**.

Examples 2.4

(a) $(1, -3)$ is a solution of $3x + y = 0$, since $3(1) + (-3)$ does equal 0.

(b) $(1, 3)$ is not a solution of $3x + y = 0$, since $3(1) + (3)$ does not equal 0.

(c) The solution set of $3x + y = 0$ consists of all pairs $(t, -3t)$, $t \in \mathbb{R}$. We write:

Solution Set = $\{(t, -3t) : t \in \mathbb{R}\}$.

(d) Find the solution set of $x_1 - x_2 + 4x_3 = 6$.

Answer: Choosing x_2 and x_3 arbitrarily, $x_2 = \alpha$, $x_3 = \beta$, say, we can now solve for x_1, getting $x_1 = 6+\alpha-4\beta$. Thus, solutions are all triples of the form $(6+\alpha-4\beta, \alpha, \beta)$, whence

Solution Set = { $(6+\alpha-4\beta, \alpha, \beta) : \alpha, \beta \in \mathbb{R}$}.
(e) The solution set of the linear equation $0x - 0y = 1$ is empty , since there are no solutions. Thus,

Solution Set = Ø

in this case.

Definitions 2.5 A **system of linear equations** in the variables $x_1, x_2, \ldots, x_n$ is a finite set of linear equations in $x_1, x_2, \ldots, x_n$. Thus, a system of m equations in $x_1, x_2, \ldots, x_n$ has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

The system is **homogeneous** if $b_1 = b_2 = .. = b_m = 0$. A **solution** to a system of equations is a sequence of numbers $(s_1, s_2, \ldots, s_n)$ each of which is in $\mathbb{R}$ such that, if we replace each x_i by s_i, all the expressions become true. As before, the set consisting of all solutions to the system of equations is called the solution set.

Examples 2.6
(a) The system

$$4x_1 - x_2 + 3x_3 = -1;$$
$$3x_1 + x_2 + 9x_3 = -4$$

has a solution $(1, 2, -1)$. However, $(1, 8, 1)$ is not a solution, since it only satisfies the first equation.
(b) The system

$$x + y = 1$$
$$2x + 2y = 3$$

has no solutions.
(c) Geometric interpretation of systems with two unknowns, and the possible outcomes—discussed in class.

Definition 2.7 A system of equations which has at least one solution is called **consistent**. Otherwise, it is called **inconsistent**. Thus, for example, 2.6(a) is consistent, while 2.6(b) is inconsistent.

Matrix Form of System of Equations
Consider the matrix equation

$$AX = B,$$

where $A = [a_{ij}]$ is an $m\times n$ matrix, and where X is $n\times 1$, so that B must be $m\times 1$. Diagramatically, we have

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}$$

Since the left hand side, when multiplied out, gives a $m \times 1$ matrix, we have the following equation of $m \times 1$ matrices:

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}$$

But, for these matrices to be equal, their corresponding entries must agree. That is, we get the following (rather familiar) system of m linear equations in n unknowns.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \dots\dots\dots\dots\dots\dots\dots& \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}\right\} \dots\dots (*)$$

Thus, we can represent the system (*) by the matrix equation $AX = B$, where X is the column matrix of unknowns, and A is the **coefficient matrix**. Now, to solve for the matrix X, it seems that all we need do is multiply both sides by A^{-1}, getting $X = A^{-1}B$. But this is no simple matter! For starters, we don't yet have a way of getting A^{-1}, and in fact it may not even *exist*. Instead, we'll use a different approach to solve a system:

Definition 2.8 If

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \dots\dots\dots\dots\dots\dots\dots& \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

is a system of linear equations, then the $m \times (n+1)$ matrix

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & b_m \end{array}\right]$$

is called the **augmented matrix** of the system.

Example 2.9 The augmented matrix for the system

$$\begin{aligned} 2x_1 - 4x_2 + x_3/2 &= -1 \\ x_1 \qquad - 11x_3 &= 0 \\ x_2 + x_3/2 &= 3/5 \end{aligned}$$

We find solutions to systems by replacing them by ”equivalent” systems, obtained by the following operations.

Multiply an equation by a constant
Interchange two equations
Add a multiple of one equation to another.

Note that we don't lose any information by these operations; we can always recover the original set of equations by doing the reverse operations—e.g. the reverse of adding twice equation 3 to equation 1 is subtracting twice equation 3 from equation 1.

Since a system of equations is ”fully represented” by the augmented matrix, we make the following definition.

Definition 2.10 An **elementary row operation** is one of the following:

1. Multiplication of a row by a nonzero* constant.
2. The interchanging of two rows.
3. Addition of a multiple of one row to another.

They are called *elementary* because any kind of legitimate manipulation you do may be obtained by doing a sequence of elementary row operations. Thus they are the ”building blocks” for getting ”equivalent” matrices.

Example of use of elementary row operations to solve a system

Definition 2.11. A matrix is in **row-reduced echelon form** if:

1. The first nonzero entry in each row (the leading entry) is a 1.
2. Each column which contains a leading entry has zeros everywhere else.
3. The leading entry of any row is further to the right than those of the rows above it.
4. The rows consisting entirely of zeros are at the bottom.

The **rank** of a matrix A is the number of non-zero rows left after row-reduction.

Examples 2.12

(a) $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

is not.

Examples 2.13 Solve the systems represented by **(a)**, **(b)** and **(c)** in 2.12.

Illustration of procedure to row-reduce an arbitrary matrix
(In class)

> **Fundamental Theorem for Linear Systems**
> **(a)** Any homogeneous system of equations either has exactly one solution (the trivial one), or else it has infinitely many solutions.

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* If you multiply by zero, you've lost the corresponding equation.

(b) A homogeneous system has a unique solution if and only if the rank of the coefficient matrix equals the number of unknowns n.
(c) Any system of equations either has no solution, exactly one solution, or else it has infinitely many solutions.
(d) If $m \le n$ (fewer equations than unknowns) there can not be a unique solution. In fact, if the *rank* of the coefficient matrix is less than n, there can be no unique solution.
(e) There is a unique solution if an only if the rank of the coefficient matrix is equal to n.
(f) The general solution to any system = general solution to the associated homogeneous system + a particular solution

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Exercise Set 2
Kreyszig p. 295, #1–13 odd (use row-echelon reduction instead), 22
Hand In (Value = 15 points)
1. *Analysis of Circuits* Kreyszig p. 295 #18 (see p. 290 for discussion)
2. (10 *points*) *Analysis of a Random Walk*
(a) Referring to Exercise Set 1 #3, we say that a vector $v = [x_1, x_2, \dots, x_6]$ is **invariant** if $vP = v$, where P is the transition matrix of the Markov system. Find the unique invariant vector $v = [x_1, x_2, \dots, x_6]$ with the property that $x_1 + x_2 + \dots + x_6 = 1$.
(b) If S is the 6×6 matrix whose rows are equal to each other and to the solution vector in part (a), what can you say about the matrix SP?
(c) Now calculate (with technology $P^2, P^3, \dots P^{10}, \dots$ What do you notice?
(d) Interpret your result in terms of a random walk.

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3. Inverses and Determinants

Definition 3.1 An $n\times n$ matrix A is said to be **invertible** if there is an $n\times n$ matrix B such that $AB = BA = I$. We then call B an **inverse of A.** If A has no inverse, then we say that A is **singular.**

Proposition 3.2 An invertible matrix A has exactly one inverse, (and we therefore refer to it as A^{-1}, so that $AA^{-1} = A^{-1}A = I$).

Proof There are many ways to prove this. S'pose B and C were both inverses of A. Then one has
$$B = (CA)B = C(AB) = C \text{ ✻}$$

Example 3.3 The inverse of any 2×2 matrix is given by the formula:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ **Inverse of 2×2 Matrix**

If $ad-bc = 0$, then in fact the inverse does not exist (the matrix is singular) as we shall find out later.

Proposition 3.4 (Properties of Inverses)
If A &B are invertible, then:
(a) AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

(b) A^{-1} is invertible, and $(A^{-1})^{-1} = A$.
(c) If $\lambda \neq 0$, then λA is invertible, and $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$.

Remark It follows from part (a), that, if $A_1, A_2, \ldots, A_r$ are all invertible matrices, then so is $A_1A_2 \ldots A_r$, and its inverse is $A_r^{-1} A_{r-1}^{-1} \ldots A_2^{-1} A_1^{-1}$.

Determining the Inverse by Row Reduction (Kreyszig, §6.7)
We want to find a B such that $AB = I$. Looking at each column of B in turn, this says that

A(ith column of B) = (ith column of I)

so we can obtain the ith column of B by row reducing the associated augmented matrix and then taking the rightmost column as the solution. Putting this together for all the columns of B gives us the following method for finding the inverse of A:

1. Augment A using the $n \times n$ identity.
2. Row reduce to get the $n \times n$ identity on the left
3. A^{-1} then appears on the right.

Example
We calculate A^{-1} if $A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$.

Using the Inverse to Solve a System

Example We solve several systems at once using the same inverse.

Note there is another way to determine the inverse: using *determinants*.

Definition 3.5 We define the **determinant**, $\det A$ or $|A|$, of an $n \times n$ matrix A recursively as follows:
If $n = 2$, then

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

For larger n, we define

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{bmatrix} = a_{11}|M_{11}| - a_{12}|M^{12}| + \ldots + (-1)^{1+n}a_{1n}|M^{1n}|$$

where M_{ij} is the **minor** of a_{ij}; the submatrix obtained from A by deleting the row and column through a_{ij}.
We can also write this as

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n},$$

where $C_{ij} = (-1)^{i+j}|M_{ij}|$ is called the **cofactor** of a_{ij}.

Examples 3.6
(a) determinant of a general 3×3 matrix
(b) expansion by other rows or columns; a 4×4 matrix
(c) determinant of a triangular matrix

Proposition 3.7 (Properties of the Determinant)
Let A' be obtained from A by a row operation of the type e. then:
(a) If e is multiplication of some row by λ, then $\det(A') = \lambda\det(A)$.
(b) If e interchanges two rows, then $\det(A') = -\det(A)$.
(c) If e is addition of μ times one row to another, then $\det(A') = \det(A)$.
Also,
(d) $\det A^T = \det A$.
(e) $\det(AB) = \det(A)\det(B)$
(f) $\det A \neq 0$ if and only if A is invertible.
(g) $\det A = 0$ if and only if the homogeneous system $AX = O$ has non-zero solutions.

Notes on proof: **(a)** expand by that row **(b)** it works for 2×2 matrices, and the result follows by induction **(c)** expand on that row **(d)** follows from the fact that we can expand on a row or column. (see Kreyszig, p. 343 for details -- he refers to a proof in the Appendix) To prove **(e)**, we usually need an argument using "elementary matrices." **(f)** will be proved in the homework.

Remarks 3.7
1. It follows from the above that we can calculate the determinant of any (square) matrix by keeping track of the row operations required to reduce it. In fact, we see that we need only keep track of the number of times we switch two rows as well as the numbers by which we multiply rows. If the matrix does not reduce to I, then the determinant vanishes.
2. It follows from part **(b)** of 3.7 that, if A has two identical rows, then $\det(A) = 0$. More generally, it follows from the above remark that if A does not row-reduce to the identity, then $\det(A) = 0$.

Cramer's Rule for Computing the Inverse
Let A be an $n\times n$ invertible matrix with cofactors C_{ij}. Then, if C is the $n\times n$ matrix of cofactors,

$$\begin{aligned} A^{-1} &= \frac{1}{\det A} C^T \\ &= \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & C_{31} & \dots & C_{n1} \\ C_{12} & C_{22} & C_{32} & \dots & C_{n2} \\ \dots & \dots & \dots & \dots & \dots \\ C_{1n} & C_{2n} & C_{3n} & \dots & C_{nn} \end{bmatrix} \end{aligned}$$

Example in class

One more definition:

Definition 3.8 the **rank** of an $m\times n$ matrix is the dimension of the largest square submatrix (obtained by deleting rows and/or columns) with non-zero determinant.

Example 3.9 We find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & -3 & -4 \\ 2 & 4 & 6 & 9 \end{bmatrix}$ by systematically eliminating columns and rows.

Fact The rank is uneffected by row and/or column operations. This fact gives us an easier way of finding the rank.

Exercise Set 3
Kreyszig p. 314, #5–19 odd, p. 322, 1–7 odd (use both Cramer's rule and row reduction to find these inverses.)

Hand In (Value = 25 points)
1. (10 *points*) Prove:

(a) If a square matrix is invertible, its determinant must be non-zero.
(b) if a square matrix does not reduce to the identity, its determinant must be zero.
(c) if a square matrix is not invertible, it cannot reduce to the identity (see the discussion after Proposition 3.4)

2. (5 *points*) Refer again to the random walk example in Exercise Sets 1 and 2. What can you say about the determinant of $P-I$ if the system represented by P has an invariant vector?
3. (10 *points*) *Random Walk with Traps* The following diagram models a random walk with two "traps" (one at each end). The traps are called **absorbing states**, and the Markov system is called an **absorbing system**.

1 3 5 4 2

The **fundamental matrix** $\boldsymbol{Q}$ of an absorbing system is defined by

$$Q = (I-S)^{-1},$$

where S is the matrix obtained from the transition matrix P by deleting the rows and columns corresponding to the absorbing states. The ij entry of Q is the expected (average) number of times a particle presently in the state corresponding to (row) i will be in the state corresponding to (column) j prior to absorption at one of the traps.
(a) Calculate the fundamental matrix Q of the system, and use it to answer parts (b), (c), (d).
(b) How long, on average, will it take for a particle starting in State 5 to reach one of the traps?
(c) If a particle starts in State 5, how many subsequent times should one expect it to return there before entrapment?
(d) Enumerate three or four possible routes from State 5 to State 1, showing the probability of each.

Part B Introduction to the Complex Plane

4. Algebra and Geometry of Complex Numbers (based on §§13.1–13.2 of Kreyszig)

Definition 4.1 A **complex number** has the form $z = (x, y)$, where x and y are real numbers. x is referred to as the **real part** of z, and y is referred to as the **imaginary part** of z. We write

$$\mathrm{Re}(z) = x, \mathrm{Im}(z) = y.$$

Denote the set of complex numbers by $\mathbb{C}$. Think of the set of real numbers as a subset of $\mathbb{C}$ by writing the real number x as $(x, 0)$. The complex number $(0, 1)$ is called $\boldsymbol{i}$.

Examples
$3 = (3, 0)$, $(0, 5)$, $(-1, -\pi)$, $i = (0, 1)$.

Geometric Representation of a Complex Number- in class.

Definition 4.2 Addition and multiplication of complex numbers, and also multiplication by reals are given by:

$$(x, y) + (x', y') = ((x+x'), (y+y'))$$
$$(x, y)(x', y') = ((xx'-yy'), (xy'+x'y))$$
$$\lambda(x, y) = (\lambda x, \lambda y).$$

Geometric Representation of Addition- in class. (Multiplication later)

Examples 4.3
(a) $3+4 = (3, 0)+(4, 0) = (7, 0) = 7$ **(b)** $3\times4 = (3, 0)(4, 0) = (12-0, 0) = (12, 0) = 12$
(c) $(0, y) = y(0, 1) = yi$ (which we also write as iy).
(d) In general, $z = (x, y) = (x, 0) + (0, y) = x + iy$. $\boxed{z = x + iy}$
(e) Also, $i^2 = (0, 1)(0, 1) = (-1, 0) = -1$. $\boxed{i^2 = -1}$
(g) $4 - 3i = (4, -3)$.

Note In view of (d) above, from now on we shall write the complex number (x, y) as $x+iy$.

Definitions 4.4 The **complex conjugate**, $\bar{z}$, of the complex number $z = x+iy$ given by
$\bar{z} = x - iy$.
The **magnitude**, $|z|$ of $z = x+iy$ is given by
$|z| = \sqrt{x^2+y^2}$.

Examples and Geometric Representation of Conjugation and Magnitude - in class.

Notes

1. $z + \bar{z} = (x+iy) + (x-iy) = 2x = 2\mathrm{Re}(z)$. Therefore, $\boxed{\mathrm{Re}(z) = \frac{1}{2}(z+\bar{z})}$

$z - \bar{z} = (x+iy) - (x-iy) = 2iy = 2i\mathrm{Im}(z)$. Therefore, $\boxed{\mathrm{Im}(z) = \frac{1}{2\mathrm{i}}(z-\bar{z})}$

2. Note that $z\bar{z} = (x+iy)(x-iy) = x^2-i^2y^2 = x^2+y^2 = |z|^2$ $\boxed{z\bar{z} = |z|^2}$

3. If $z \neq 0$, then z has a multiplicative inverse. Why? because:

$$z\cdot\frac{\bar{z}}{|z|^2} = \frac{z\bar{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1. \text{ Hence,} \qquad \boxed{z^{-1} = \frac{\bar{z}}{|z|^2}}$$

Examples

(a) $\dfrac{1}{i} = -i$ **(b)** $\dfrac{1}{3+4i} = \dfrac{3-4i}{25}$

(c) $\dfrac{1}{\frac{1}{\sqrt{2}}(1+i)} = \frac{1}{\sqrt{2}}(1-i)$ **(d)** $\dfrac{1}{\cos\theta + i\sin\theta} = \cos(-\theta) + i\sin(-\theta)$

4. There is also the **Triangle Inequality:**

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Proof We square both sides and compare them. Write $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$\begin{aligned}|z_1 + z_2|^2 &= (x_1+x_2)^2 + (y_1+y_2)^2 \\ &= x_1^2 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 + 2y_1y_2.\end{aligned}$$

On the other hand,

$$\begin{aligned}(|z_1| + |z_2|)^2 &= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \\ &= x_1^2 + x_2^2 + y_1^2 + y_2^2 + 2|z_1||z_2|.\end{aligned}$$

Subtracting,

$$\begin{aligned}(|z_1| + |z_2|)^2 - |z_1 + z_2|^2 &= 2|z_1||z_2| - 2(x_1x_2 + y_1y_2) \\ &= 2\big[|(x_1,y_1)||(x_2,y_2)| - (x_1,y_1).(x_2,y_2)\big] \quad \text{(in vector form)} \\ &= 2\big[|(x_1,y_1)||(x_2,y_2)| - |(x_1,y_1)||(x_2,y_2)| \cos\alpha\big] \\ &= 2\ |(x_1,y_1)||(x_2,y_2)|\ (1 - \cos\alpha) \\ &\geq 0,\end{aligned}$$

giving the result.

Note The triangle inequality can also be seen by drawing a picture of $z_1 + z_2$.

5. We now consider the **polar form** of these things: If $z = x+iy$, we can write $x = r\cos\theta$ and $y = r\sin\theta$, getting $z = r\cos\theta + ir\sin\theta$, so

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

This is called the polar form of z. It is important to draw pictures in order to feel comfortable with the polar representation. Here r is the magnitude of z, $r = |z|$, and ø is called the **argument of** z, denoted $\arg(z)$. To calculate θ, we can use the fact that $\tan\theta = y/x$. Thus θ is not $\arctan(y/x)$

as claimed in the book, but by:

$$\boxed{\theta = \begin{cases} \arctan(y/x) & \text{if } x \geq 0 \\ \arctan(y/x)+\pi & \text{if } x \leq 0 \end{cases}}$$

since the *arctan* function takes values between $-\pi/2$ and $\pi/2$. The **principal value** of arg(z) is the unique choice of θ such that $-\pi < \theta \leq \pi$. We write this as Arg(z)

$$\boxed{-\pi \leq \mathrm{Arg}(z) \leq \pi}$$

Examples
(a) Express $z = 1+i$ in polar form, using the principal value
(b) Same for R3 + 3R3 i
(c) $6 = 6(\cos 0 + i\sin 0)$

6. Multiplication in Polar Coordinates

If $z_1 = r_1(\cos\theta_1 + i \sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i \sin\theta_2)$, then

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i \sin\theta_1)(\cos\theta_2 + i \sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i (\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2). \end{aligned}$$

Thus $$\boxed{z_1 z_2 = r_1 r_2[[\cos(\theta_1+\theta_2) + i \sin(\theta_1+\theta_2)]}$$

That is, we *multiply the magnitudes and add the arguments.*

Examples In class.

7. Multiplicative Inverses in Polar Coordinates

Once we know how to do multiplication, division follows formally: Let $z = r(\cos\theta + i\sin\theta)$ be given. We want to find z^{-1}. So let $z^{-1} = s(\cos\phi + i\sin\phi)$. Then, since $zz^{-1} = 1$, we have

$$rs(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = 1$$

ie., $$rs(\cos(\theta+\phi) + i\sin(\theta+\phi)) = 1 = 1(\cos 0 + i\sin 0).$$

Thus, we can take $s = 1/r$ and $\phi = -\theta$. In other words, $\boxed{z^{-1} = \text{--}r^{-1}(\cos(-\theta) + i \sin(-\theta))}$

Examples In class.

8. Division in Polar Coordinates

Finally, since $\dfrac{z_1}{z_2} = z_1 z_2^{-1}$, we have: $$\boxed{\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i \sin(\theta_1-\theta_2)]}$$

That is, we *divide the magnitudes and subtract the arguments.*
Examples
(a) $z_1 = -2 + 2i$, $z_2 = 3i$

(b) Formula for z^n **De Moivre's formula** $\boxed{z^n = r^n(\cos n\theta + i \sin n\theta)}$

In words, to take the nth power, *we take the nth power of the magnitude and multiply the argument by n.*

Examples Powers of unit complex numbers.

9. *n*th Roots of Complex Numbers

Write

$$z = r(\cos(\theta+2k\pi) + i\ \sin(\theta+2k\pi)),$$

even though different values of k give the same answer.

Then

$$\boxed{z^{1/n} = r^{1/n}[\ \cos(\theta/n+2k\pi/n) + i\ \sin(\theta/n+2k\pi/n)]}$$

Note that we get *different* answers for $k = 0, 1, 2,..., n-1$. Thus there are ***n*** **distinct** ***nth*** **roots of** z.

Examples

(a) $\sqrt{i}$ **(b)** $\sqrt{4i}$ **(c)** Solve $z^2 - (5+i)z + 8 + i = 0$

(d) *n*th roots of unity: Since $1 = \cos 0 + i\sin 0$, the distinct nth roots of unity are:

$$\boxed{\gamma_k = \cos(2k\pi/n) + i\ \sin(2k\pi/n), \quad (k = 0, 1, 2, \ldots, n-1)}$$

More examples In class.

10. Exponential Notation

We know what e raised to a real number is. We now define what e raised to an *imaginary* number is:

Definition: $e^{i\theta} = \cos\theta + i\sin\theta$.

Thus, the typical complex number is

Exponential Form of a Complex Number

$$\boxed{re^{i\theta} = r[\cos\theta + i\sin\theta]}$$

De Moivre's Theorem now implies that $e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$, so that the exponential rule for addition works, and the inverse rule shows that $1/e^{i\theta} = e^{-i\theta}$, so that the inverse exponent law also works. Similarly, the other laws also work. Duly emboldened, we now define

$$e^{x+iy} = e^x e^{iy} = e^x[\cos y + i\sin y] \qquad \boxed{e^{x+iy} = e^x[\cos y + i\sin y]}$$

Examples in class

Exercise Set 4

Kreyszig p.606 #1–17 odd, #18 — very important!

p. 611. 3, 5, 7, 11, 13, 15, 19, 21, 23, 27, 29, 31

Hand In (Value = 10 points)

1 (a) One of the quantum mechanics wave functions of a particle of unit mass trapped in an infinite potential square well of width 1 unit is given by

$$\Psi(x,t) = \sin(\pi x)\ e^{-i(\pi^2\hbar/2)\ t} + \sin(2\pi x)e^{-i(4\pi^2\hbar/2)\ t},$$

where $\hbar$ is a certain constant. Calculate $|\Psi(x,t)|^2$. (This is the probability density function for the position of the particle at time t.)

(b) The expected position of the particle referred to in part (a) is given by

$$\langle x\rangle = \int_0^1 x\ |\Psi(x,t)|^2\ dx\ .$$

Calculate $\langle x\rangle$ and compute its amplitude of oscillation.

---

5. Functions of a Complex Variable

Definition 5.1 Let $S \subset \mathbb{C}$ A **complex valued function on *S*** is a function

$$f\colon S \longrightarrow \mathbb{C}.$$

S is called the **domain** of f.

Examples 5.2
(a) Define $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ by $f(z) = z^2$;
(b) Define $g\colon \mathbb{C}-\{0\} \longrightarrow \mathbb{C}$ by $g(z) = -\frac{1}{z} + -z$. Find $g(1+i)$.
(c) Define $h\colon \mathbb{C} \longrightarrow \mathbb{C}$ by $h(x+iy) = x + i(xy)$.

Notes
(a) In general, a complex valued function is completely specified by its real and imaginary parts. For example, in **(a)** above,

$$f(x+iy) = (x+iy)^2 = (x^2-y^2) + i(2xy).$$

Write this as $u(x,y) + iv(x,y)$,
where $u(x,y)$ and $v(x,y)$ are a pair of *real−valued* functions.
(b) An important way to picture a function $f\colon S \longrightarrow \mathbb{C}$ is as a "mapping" − picture in class.

Examples 5.3
(a) Look at the action of the functions $z + z_0$ and αz for fixed $z_0 \in \mathbb{C}$ and α real.
(b) Let S be the unit circle; $S = S^1 = \{z : |z| = 1\}$. Then the functions

$$f\colon S \longrightarrow S;\quad f(z) = z^n$$

are "winding" maps.
(c) The function $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ given by $f(z) = 1/z = z^{-1}$ is a special case of (a) above, and "winds" the unit circle backwards. It maps the circle of radius r backwards around the circle of radius $-r$.
(d) The function $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ given by $f(z) = \bar{z}$ agrees with $1/z$ on the unit circle, but not elsewhere.

Transcendental Functions

Definition 5.4. The e**xponential complex function** exp: $\mathbb{C} \longrightarrow \mathbb{C}$ is given by

$$\exp(z) = e^x(\cos y + i \sin y),$$

for $z = x+iy$. This is also written as e^z, for reasons we saw in the last section.

Properties of the Exponential Function
1. For x and y real, $e^{iy} = \cos y + i \sin y$ and e^x is the usual thing.
2. $e^z e^w = e^{z+w}$
3. $e^z / e^w = e^{z-w}$
4. $(e^z)^w = e^{zw}$
5. $|e^{iy}| = 1$

Examples 5.5
(a) We compute e^{3+2i}, and e^{3+ai} for varying a.
(b) The geometric action of the exponential function: it transforms the complex plane. Vertical lines go into circles. The vertical line with x-coordinate a is mapped onto the circle with radius e^a. Thus the whole plane is mapped onto the punctured plane.
(c) The action of the function $g(z) = e^{-z}$.

Definition 5.6 Define the trigonometric **sine and cosine functions** by

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

(Reason for this: check it with z real.) Similarly, we define

$$\tan z = \frac{\sin z}{\cos z},$$

etc.

Example 5.7 We compute the sine and cosine of $\pi/3 + 4i$

Properties of Trig Functions

1. Adding $\cos z$ to $i \sin z$ gives **Euler's Formula** $\boxed{e^{iz} = \cos z + i \sin z}$

2. The traditional identities work as usual

$$\sin(z+w) = \sin z \cos w + \cos z \sin w$$

$$\cos(z+w) = \cos z \cos w - \sin z \sin w$$

$$\cos^2 z + \sin^2 z = 1$$

Definition 5.8 We also have the **hyperbolic sine and cosine,**

$$\cosh z = \frac{1}{2}(e^{z} + e^{-z})$$

$$\sinh z = \frac{1}{2}(e^{z} - e^{-z})$$

Note that $\cosh(iz) = \cos z$, $\sinh(iz) = i \sin z$.

Problems. p.752 #7-13 odd.

Definition 5.9 A **natural logarithm,** $\ln z$, of z is defined to be a complex number w such that $e^w = z$.

Notes
1. There are *many* such numbers w; For example, we know that $e^{i\pi} = -1$. Therefore,

$$\ln(-1) = i\pi.$$

But, $e^{i\pi + i2\pi} = -1$ as well, therefore,

$$\ln(-1) = i\pi + i.(2\pi)$$

Similarly,

$$\ln(-1) = i\pi +$$

In general, if

$\ln z = w,$

then

$\ln z = w + i(2n\pi)$

2. We calculate $w = \ln z$ as follows: First write z in the form $z = re^{i\theta}$. Now let $w = u+iv$. Then

$e^w = z$

gives $e^{u+iv} = z = re^{i\theta}$.

Thus, $e^u e^{iv} = re^{i\theta}$.

Equating magnitudes and arguments,

$e^u = r,\ v = \theta,$

or $u = \ln r,\ v = \theta.$

Thus,

Formula for ln z

$$\boxed{\ln z = \ln r + i\theta,\ \ r = |z|,\ \theta = \arg(z)}$$

3. If θ is chosen as the *principal value* of arg(z), that is, $-\pi < \theta \le \pi$, then we get the principal value of $\ln z$, called Ln z. Thus,

Thus,

Formula for Ln z

$$\boxed{\mathrm{Ln}\, z = \ln r + i\theta,\ r = |z|,\ \theta = \mathrm{Arg}(z)}$$

Also

$$\boxed{\ln z = \mathrm{Ln}\, z + i(2n\pi);\ n = 0, \pm 1, \pm 2, \ldots}$$

4. $\ln 0$ is still undefined, as there is no complex number w such that $e^w = 0$.

Examples 5.10

(a) $\ln 1 = 0 + 2n\pi i = 2n\pi i$; $\mathrm{Ln}\ 1 = 0$;

(b) $\ln 4 = 1.386... + 2n\pi i$; $\mathrm{Ln}\ 4 = 1.386...$

(c) If r is real, then
$\ln\ r =$ the usual value of $\ln r + 2n\pi i$; $\mathrm{Ln}\ r = \ln r$

(d) $\ln i = \pi i/2 + 2n\pi i$; $\mathrm{Ln}\ i = \pi i/2$;

(e) $\ln(-1) = \pi i + 2n\pi i$; $\mathrm{Ln}\ (-1) = \pi i$;

(f) $\ln(3-4i) = ln 5 + i\arg(3-4i) + 2n\pi i$
$= \ln 5 + i\arctan(4/3) + 2n\pi i$; $\mathrm{Ln}(3-4i) = \ln 5 + i\ \arctan(4/3)$.

More Properties

1. $\ln(z\ w) = \ln z + \ln w$; $\ln(z/w) = \ln(z) - \ln(w)$.

This doesn't work for Ln; eg., $z = w = -1$ gives

$\mathrm{Ln}\, z + \mathrm{Ln}\, w = \pi i + \pi i = 2\pi i,$

but $\mathrm{Ln}(zw) = \mathrm{Ln}(1) = 0.$

2. Ln z jumps every time you cross the negative $x-$axis, but is continuous everywhere else (except zero of course). If you want it to remain continuous, you must switch to another **branch** of the logarithm. (Lnz is called the **principal branch** of the logarithm.)

3. $e^{\ln z} = z$, and $\ln(e^z) = z + 2n\pi i$;

$e^{\mathrm{Ln}\, z} = z$, and $\mathrm{Ln}(e^z) = z + 2n\pi i$;

(For example, $z = 3\pi i$ gives $e^z = -1$, and $\mathrm{Ln}(e^z) = \pi i \ne z$.)

Exercise Set 5
p.616 #1–19 odd, p. 629 #5–15 odd, p. 633 #5–23 odd

Hand In (Value = 10)
1. Find functions f that do the following:
(a) Map the region $\{z \mid 0 \le \arg(z) \le \pi/2\}$ onto the whole plane
(b) Map the upper half plane to the lower half plane
(c) Maps the second quadrant onto the right-half plane
(d) What happens to the strip $\{x+iy \mid 0 \le y \le 1,\ x \ge 0\}$ under the map $f(z) = ie^{-z}$?
2. A **Möbius transformation** is a complex function of the form

$$f(z) = \frac{az + b}{cz + d}.$$

(a) Find a Möbius transformation f with the property that $f(1) = 1$, $f(0) = i$, and $f(-1) = -1$.
(b) Prove that your function is the only possible Möbius transformation with this property. (It is suggested you do some research in the Section 17.2 of the textbook.)

Part C Introduction to Differential Equations and Fourier Series

6. Ordinary Linear Differential Equations

What is a differential equation? In words, a **differential equation** (DE) is an equation involving some unknown function $y(x)$ (of possibly more than one variable) as well as derivatives of y.

Example A $\dfrac{dy}{dx} = e^x + x^2$.

This is an equation involving the derivative of y, and is called a **first order** DE, since it involves only the first derivative of the unknown function y. A **solution** of a differential equation is a function $y(x)$ that satisfies the given equation. Thus the solution to the above equation is:

$$y = \int (e^x + x^2)\, dx = e^x + \frac{x^3}{3} + C,$$

where C is any constant. Thus it seems that a DE can have many solutions. Eg. here, $y = e^x + \frac{x^3}{3} + 9$, $e^x + \frac{x^3}{3} + 1{,}003$, $e^x + \frac{x^3}{3} + \pi$ are all **particular solutions**. $e^x + \frac{x^3}{3} + C$ is called the **general solution**, since any solution is of this form. (You can check that it is, in fact, a soln. by substituting y back in the original eqn.)

Example B Let $u(t)$ be the position of a particle at time t. Find $u(t)$ if

$$m\frac{d^2u}{dt^2} = ku + \frac{du}{dt}.$$

Here, m and k are constants. Our job is to find a function $u(t)$ which satisfies the eqn. There is no obvious answer, but we will be able to solve it later in the course. This equation is called a **second order** DE, since it involves the second derivative of the unknown function u.

The DE's in Examples A and B are collectively referred to as **ordinary DE's**, since they deal with ordinary—as opposed to partial—derivatives. Thus, the DE in Example A is a first order ordinary DE, while the one in Example B is a second order ordinary DE.

Example C Let $g(x, t)$ be a function of x and t. Then

$$\frac{\partial g}{\partial x} = \frac{\partial^2 g}{\partial t^2} + 4\sin(xt)$$

is a diff. eqn. in the unknown function $g(x,t)$ of two variables x and t. This is called a **partial** DE, since it involves partial derivatives of the unknown function y. These are harder to solve, and we won't know how 'till later.

Finally, there are special types of ordinary DE's called **linear** DE's. These have the form

Linear Ordinary Differential Equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x) = g(x) \qquad (*)$$

This is a linear eqn. of order n (since the highest derivative is the nth). The $a_n(x)$ are called the **coefficients** and may be constants or even horrible functions.

Examples of linear DE's

1. Equations A and B above are both linear DE's, since they can be rewritten with all the derivative terms on the left, making them look like (*).

2. The equation

$$\frac{d^2 y}{dx^2} + 3x^2\frac{dy}{dx} + 4xy = e^x \sin x$$

has $a_2(x) = 1$, $a_1(x) = 3x^2$, $a_0(x) = 4x$

3. The equation

$$\frac{d^2 y}{dx^2} + 3y\frac{dy}{dx} + 4xy = e^x \sin x$$

is not linear, since the coefficient of $\frac{dy}{dx}$ is not a function of x.

4. Here is a nice one: $y'' + y = 0$. This in fact has solutions $y = \sin x$ and also $y = \cos x$. To check that they are in fact solutions, plug them in & see! Moreover, if A and B are any constants, then

$$y = A \sin x + B \cos x$$

is also a solution—again, check by plugging in. (In fact, this is the *general solution*.).

Methods of Solving Two broad methods:

I Analytical This is the way we solved the first one above. It means giving an exact solution in terms of known functions by using various methods we'll study.

II Numerical Many DE's are impervious to analytical methods, so there are ways of getting approximate solutions via the use, usually, of computers. These give solutions as accurately as you like, but in terms of numbers rather than known functions. For example, it might say that the solution to B above has the property that:

$$u(1) = 4.0985, \ u(1.3) = 5.69584, \text{ etc.}$$

This does allow one to draw its graph, but doesn't tell one what $u(t)$ is, but it is useful in applications where you couldn't care whether it's a known function, and only need $u(1)$, $u(1.3)$ etc. We'll do mainly analytical methods, as they provide understanding. Currently available computer software does numerical solving.

Solving First Order Linear Equations (Section 1.7 in text)

Recall that these involve only the first derivative, and are, in addition, linear. Here, y will always be an (unknown) function of x, and we are required to find y. Recalling that the general form of a 1st order linear DE is:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x),$$

we divide both sides by $a_1(x)$ to get an equation of the form

Standard Form of First Order Linear DE $\boxed{y' + p(x)y = f(x)}$

We look at two specific types of these:

A. Easy $(p(x) = 0)$ Thus they have the form $y' = f(x)$.

These are like the first one we looked at. One solves them by doing a simple integration, and the solution is $y = \int f(x)\,dx$.

Examples of Easy Ones

A. Solve $y' = \sin x$. *Solution* $y = \int f(x)\,dx = \int \sin x\,dx = -\cos x + C$. (general soln.)

B. Solve $y' = \frac{1}{1+x^2}$. *Solution* $y = \int f(x)\,dx = \int \frac{1}{1+x^2}\,dx = \arctan x + C$. (general soln.)

B. General First Order Linear As noted above, these have the form $y' + p(x)y = f(x)$.

Here, $p(x)$ and $f(x)$ are given functions of x.

Example $y' + xy = x$; $y' - 2xy = x$; $xy' + 2y = \sin x$ (because dividing the last by x gives: $y' + (2/x)y = (1/x)\sin x$).

Method of Solution of $y' + p(x)y = f(x)$.

Look at the left hand side. It looks a little like the product rule. . . Multiply both sides by a "certain" function $\mu(x)$. Get:

$$\mu(x)y' + p(x)\mu(x)y = \mu(x)f(x) \qquad \text{(a)}$$

If only

$$p(x)\mu(x) \text{ was equal to } \mu'(x) \qquad \text{(b)}$$

Then we would have:

$$\mu(x)y' + \mu'(x)y = \mu(x)f(x),$$

ie. $$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x),$$

so $$\mu(x)y = \int \mu(x)f(x)\,dx + C,$$

giving

General Solution for First Order Linear DE $\boxed{y = \frac{1}{\mu(x)}\left[\int \mu(x)f(x)\,dx + C\right]}$

That's all well & good, but we still don't know what $\mu(x)$ is!
But, by (b), we must have $p(x)\mu(x) = \mu'(x)$. That is,

$$\mu'(x)/\mu(x) = p(x)$$

ie. $$\frac{d}{dx}[\ln \mu(x)] = p(x).$$

ie. $$\ln \mu(x) = \int p(x)\, dx,$$

or **Integrating Factor** $\boxed{\mu(x) = e^{\int p(x)\, dx}}$

Thus our method of solution is given as follows:

Solving a Linear First Order DE
(1) Make sure that the coefficient of y' is a 1 (by dividing if necessary)
(2) Set $\mu(x) = e^{\int p(x)\, dx}$
(3) Then the general solution is $y = \dfrac{1}{\mu(x)}\left[\int \mu(x)f(x)\, dx + C\right]$

Note $\mu(x)$ is called an **integrating factor**. C is an arbitrary constant of integration.

Examples 6.1
A. We solve $y' + 2y = e^{-x}$, subject to the **initial condition** $y(0) = 3$.
B. Solve $y' - 2xy = x$, subject to $y(0) = 1$.
C. Solve $xy' + 2y = \sin x$.

Second Order Linear DE's
A general 2nd order DE looks like this:

$$F(y'', y', y, x) = 0$$

We seek solutions to this equation with (possibly) given initial conditions of the form:

$$y = y_0 \text{ when } x = x_0$$

and $$y' = z_0 \text{ when } x = x_0$$

The reason for *two* boundary conditions is that we expect *two* arbitrary constants. For example, if $y'' = 1$, then, integrating twice, we get: $y' = x + C$, and so $y = \dfrac{x^2}{2} + Cx + D$ (C & D both arb. constants.)

Second order linear DE's have the form

Second Order Linear DE $\boxed{\dfrac{d^2y}{dx^2} + p(x)\dfrac{dy}{dx} + q(x)y = f(x)}$

Examples 6.2
A. A mass m on a spring subjected to a varying force $F(t)$ has displacement $u(t)$ satisfying an equation of the form:

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = F(t)$$

for given constants k and c. (We'll see why later.) This kind has *constant coefficients*.
B. **Bessel's Equation**

$$x^2y'' + x\,y' + (x^2 - \alpha^2)y = 0$$

for a constant α This pops up in some applications.

General Strategy for Solving 2nd Order DE's

Definition A 2nd order linear DE is **homogeneous** if it has the form

$$y'' + p(x)y' + q(x)y = 0,$$

so that here, $f(x) = 0$.

Given *any* linear DE $y'' + p(x)y' + q(x)y = f(x)$, its **associated homogeneous equation** is then obtained by replacing $f(x)$ by 0. Thus, eg. the associated homog. eqn. of

$$y'' + 6y = 77x^2$$

is $\quad y'' + 6y = 0.$

What's so special about these? Well…

> **Theorem 6.3** (General form of Solution to 2nd Order Linear DE)
> The general solution to a linear DE has the form
> $$y = y_h(x) + y_p(x)$$
> where $y_p(x)$ is any one particular solution and $y_h(x)$ is the general solution to the associated homogenous equation.

Examples

A. same example as above B. $y'' + y = e^x$

Thus it is important to learn two things:

(1) Finding **complementary functions** (that is, general solns. to the Hom. Eqn.)

(2) Finding particular solutions (to the whole equation). One is enough.

First, we look at the general solution to homogenous equations, and we begin with the following.

> **Theorem 6.4 (General Solution of a Second Order Homogeneous Linear DE)**
> If u and v are any two **linearly independent** solutions (that is, one is not a constant multiple of the other) of a second order homogeneous DE on an interval, then every solution $y(x)$ of the DE on the given interval can be expressed as a linear combination,
> $$y(x) = Au(x) + Bv(x)$$
> for some constants A and B. Thus $Au(x) + Bv(x)$ is the general solution of the given DE in the specified interval.

Solving Homogeneous Second Order Linear DE's with Constant Coefficients (§2.2 in book)

In other words, we see how to cook up the u's and v's. First, a linear 2nd order DE with *constant coefficients* has the form

2nd Order Homog. Linear DE with Const. Coeffs. $\boxed{ay'' + by' + cy = 0}$ … (*)

We wish to find the two independent solutions $u(x)$ and $v(x)$. We've seen that, no matter how we cook these up, (even by guessing), we will get the general solution this way, namely:

$$y(x) = Au(x) + Bv(x) \qquad (A, B \text{ arb. constants})$$

We try as a good guess, a solution of the form:

$$y = e^{rx}$$

for some suitable constant r. In class, we see how this leads to the **characteristic equation** $ar^2 + br + c = 0$. We then look at case of real solutions to the characteristic quadratic, and conclude:

Theorem 6.5 (Solution of 2nd Order Homogeneous Constant Coeffs)
(a) (Distinct Real Roots) If the discriminant $b^2-4ac > 0$, then the general solution to equation (*) is $y(x) = Ae^{r_1 x} + Be^{r_2 x}$, where r_1 and r_2 are the roots of the characteristic equation.
(b) (Repeated Real Roots) If the discriminant $b^2-4ac = 0$, then the general solution to equation (*) is $y(x) = Ae^{rx} + Bxe^{rx}$, where r is the repeated root of the characteristic equation.
(c) (Complex Roots) If the discriminant $b^2-4ac < 0$, then the general solution to equation (*) is $y(x) = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$, where the roots of the characteristic equation are $\alpha \pm i\beta$.

Examples 6.6
(a) Solve $y'' - y' - 2y = 0$.
(b) Find the solution of $y'' + 2y' + 1 = 0$ subject to $y(0) = 1$; $y'(0) = 2$.
(c) Find the GS of $y'' + y' + y = 0$
(d) Solve $y'' + y = 0$

Nonhomogeneous 2nd Order Linear with Constant Coefficients—Variation of Parameters
We want to solve:

$$\boxed{ay'' + by' + cy = f(x)}$$

To solve this, go back to Theorem 6.3. It states that the general solution is given by

$$\boxed{y = y_h + y_p}$$ **General Solution to Nonhomogeneous Linear DE**

where y_p = any *one particular solution* of the equation (*)
y_h = the *general solution* of the associated homogeneous equation.

We already know how to find CF, so all we have to do is learn how to cook up particular integrals (PI's). Here is a method for getting particular integrals (PI's) for *any* 2nd order linear equation

$$\boxed{y'' + p(x)y' + q(x) = f(x)}$$ **General 2nd Order Linear PE**

whether or not the coefficients are constant. (So, if the coefficients happen to be constant, you can use this instead of undetermined coefficients if you wish).

The Method
(1) First get the two linearly indep. complementary functions (somehow or other). Call them $y_1(x)$ and $y_2(x)$.
(2) Let

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

This is called the **Wronskian** of y_1 and y_2.
(3) The PI is then

$$\boxed{y_p(x) = -\,y_1(x)\int \frac{y_2(x)f(x)}{W[y_1,y_2](x)}\,dx + y_2(x)\int \frac{y_1(x)f(x)}{W[y_1,y_2](x)}\,dx}$$

(You could check by substituting!)

Examples 6.7
Solve

A. $y'' - 3y' - 4y = 9$
B. $ay'' + by' + cy = K$ (K constant; $c \neq 0$)
C. $y'' + 2y' + y = 4x$
D. $y'' + y = 4x^2 - 1$
E. $y'' - 3y' - 4y = e^x$
F. $y'' - 3y' - 4y = e^{-x}$.

Exercise Set 6
p. 32 #5–17 odd, 71 #17, 19, p. 59 #9, 11, 17, 19. p. 83 #1, 3, 9, 11, p. 101, #1, 3, 5

Hand In (Value = 20)

1. Regarding x as an implicit function of y, solve $\dfrac{dy}{dx} = \dfrac{y^2}{e^y - 2xy}$.

2. Error Function

(a) Express the general solution of $y' = 1 + 2xy$ in terms of the **error function**

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt .$$

(b) Find the particular solution to the equation in part (a) given that $y(0) = 2$.

3. (a) Show that, for all constants A and B, there exist constants E and δ such that

$$A\cos(\omega x) + B\sin(\omega x) = E\cos(\omega x + \delta).$$

[Hint: Take $E = \sqrt{A^2 + B^2}$ and force it as a factor of the left-hand side. Be sure to define δ carefully.]

(b) Deduce that the general solution of the DE $y'' + \omega^2 y = 0$ is $E\cos(\omega x + \delta)$ for arbitrary constants E and δ.

(c) Interpret these constants by referring to the graph of the solution.

4. A certain non-homogeneous second order linear DE has the three particular solutions $y_1(x) = x$, $y_2(x) = x - x^2$ and $y_3(x) = x - x^3$. Find its general solution.

---

7. Fourier Series

Definition 7.1 A function f: $\mathbb{R} \longrightarrow \mathbb{R}$ is **periodic** if there is a (positive) number p (called the **period of f**) so that

$$f(x + p) = f(x)$$

for every x.

Graph of a periodic function in class

Examples 7.2

1. $f(x) = \sin x, \cos x, \tan x$ *etc*. have period 2π.
2. $f(x) =$ const. has any old period you like.
3. $f(x) = x, x^2, \mathrm{R}x$, *etc*. are *not* periodic.

4. If f and g are periodic with the same period p, then so are $f+g$, $f-g$ and, more generally, $a_1 f + a_2 g$ for arbitrary constants a_1 and a_2.
5. The functions 1, $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cos 3x$, ..., $\sin nx$, $\cos nx$ ($n = 0, \pm 1, \pm 2, ...$) have period 2π.
6. The functions $\cos\left(\frac{2n\pi x}{T}\right)$, $\sin\left(\frac{2n\pi x}{T}\right)$, $n = 0, \pm 1, \pm 2, \ldots$ have period T. This is the way you can change the period from 2π to T (− just replace x by $\frac{2\pi x}{T}$.

Note If a function has period p, then it also has period $2p$ since:

$$f(x+2p) = f((x+p) + p) = f(x+p) = f(x).$$

Now suppose I have any old function f which is periodic of period 2π. Wouldn't it be nice if:

$$\begin{aligned} f(x) &= a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + ... \\ &= a_0 + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx) \ ? \end{aligned}$$

For this to have a hope in Hades, the series must, of course converge. If this happens to be the case, we refer to the series as the **Fourier series of f.**

Definition 7.3 If f and g are continuous functions of period 2π, let $\langle f, g\rangle$ be given by

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\ dx \ .$$

We think of $\langle f, g\rangle$ as a kind of "dot product", and we say that two functions f and g are **orthogonal** if $\langle f, g\rangle = 0$. Also, we define the **norm** of the function $\|f\|$ to be given by

$$\|f\| = \langle f, f\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 \ \ dx$$

Now, this inner product "-,-' satisfies all kinds of nice laws like:

$$\begin{aligned} &\langle f+g, h\rangle = \langle f, h\rangle + \langle g, h\rangle \\ &\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle \\ &\langle cf, g\rangle = c\langle f, g\rangle = \langle f, cg\rangle \\ &\langle f, g \rangle = \langle g, f\rangle, \end{aligned}$$

in the same way as the usual dot product of vectors.

Now the usual vectors $\boldsymbol{e}_1, \boldsymbol{e}_2$, ... , $\boldsymbol{e}_n$ have the nicest of properties:

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i \neq j \end{cases}$$

and we thus refer to this collection of vectors as being **orthogonal**. Similarly, we say that a collection $f_1, f_2, \dots, f_n, \dots$ of functions is orthogonal if:

$$\langle f_i, f_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i \neq j \end{cases} .$$

Are there such things as orthogonal functions? Yes.

Proposition 7.4 (Fourier Base)
The functions $\{\frac{1}{\sqrt{2}}, \cos x, \sin x, \cos 2x, \sin 2x, \cos 3x, \dots, \sin nx, \cos nx, \dots\}$ are orthogonal.

Proof This we do in class, using the identities

$$\cos nx \cos mx = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x]$$

$$\sin nx \sin mx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$

Finding the Fourier Coefficients

Now suppose that

$$f(x) = a_0 + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx),$$

so that, for each x, the series on the right converges and equals $f(x)$. Our first job is to find the coefficients. Before we do that, we modify the series a tiny bit, and write

$$f(x) = r_0 \frac{1}{\sqrt{2}} + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx),$$

so that this is an infinite linear combination of functions we know to be orthogonal, ie.,

$$f = d_1 f_1 + d_2 f_2 + d_3 f_3 + \dots + d_n f_n + \dots,$$

where the f_i are all orthogonal. To get the coefficients, do the following: S'pose we want the m-th coefficient, Then take $\langle -, f_m \rangle$ of both sides. This gives:

$$\langle f, f_m \rangle = d_1 \langle f_1, f_m \rangle + d_2 \langle f_2, f_m \rangle + d_3 \langle f_3, f_m \rangle + \dots + d_m \langle f_m, f_m \rangle + \dots,$$

where, by orthogonality, all terms vanish except $d_m \langle f_m, f_m \rangle = d_m.1 = d_m$.
Thus,

$$\langle f, f_m \rangle = d_m,$$

giving us the coefficient d_m. Thus:

The coefficient of f_m *is* $\langle f, f_m \rangle = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x) f_m(x)\, dx$

Turning to our particular orthogonal functions, we now get:

$$r_0 = \text{coeff. of } \frac{1}{\sqrt{2}} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \frac{1}{\sqrt{2}}\, dx = \frac{1}{\sqrt{2}} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx .$$

Therefore,

$$a_0 = r_0\frac{1}{\sqrt{2}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx\ .$$

The other coefficients are

$$a_n = \langle f,\ \cos(nx)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ \ dx$$

$$b_n = \langle f,\ \sin(nx)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ \ dx$$

Fourier Coefficients

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ \ dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ \ dx$$

Example 7.5 Let's do the odd square wave function (p.532)

$$f(x) = c_0(x) - 2c_\pi(x) + 2c_{2\pi}(x) + \ldots;\ f(-x) = -f(x),$$

where $c_a(x) = \begin{cases} 0 & \text{if } x < a \\ 1 & \text{if } x \geq a \end{cases}$.

Exercise Set 7

p. 485 # 1, 2, 3, 5, 7, 9, 11, 15

Hand In (Value = 15)

Analysis of Circuits (Based on Exercise 14 p. 553 of Section 10.6 in Kreyszig)

Read the section carefully and then answer the following question:

The current in an RLC-circuit is governed by the differential equation

$$LI'' + RI' + \frac{1}{C}I = E'(t),$$

where $E(t)$ is the applied voltage. Suppose $R = 100$ ohms, $L = 100$ henrys, $C = 10^{-2}$ farad, and E is periodic with

$$E(t) = \begin{cases} 100(\pi t + t^2) & \text{if } -\pi < t < 0 \\ 100(\pi t - t^2) & \text{if } 0 < t < \pi \end{cases} .$$

(a) Represent E' as a Fourier Series, and find a formula for the n-th coefficient.

(b) Now use the procedure on pp. 551–552 to obtain the steady-state solution (particular solution) for $I(t)$ as a Fourier Series.

(c) Graph the magnitude of the coefficient vs. n. For which value of n is it largest? Compare this with the natural frequency of the circuit obtained by solving the homogeneous equation.

(d) Graph the input and steady state output.

---

8. Other Periods

Recall that the functions $\cos\left(\frac{2n\pi x}{T}\right)$, $\sin\left(\frac{2n\pi x}{T}\right)$, $n = 0, \pm 1, \pm 2, \ldots$ have period T. Thus, to get a period of $2L$ (analogously with 2π), we replace T by $2L$ and use the functions

$$\cos\left(\frac{n\pi x}{L}\right),\ \sin\left(\frac{n\pi x}{L}\right),\ n = 0, \pm 1, \pm 2, \ldots$$

If f and g are continuous functions of period $2L$, let $\langle f, g\rangle$ denote the integral

$$\langle f, g\rangle = \frac{1}{L}\int_{-L}^{L} f(x)g(x)\ dx$$

(this generalizes the previous version for period 2π).Then just as above, we get the following:

Proposition 8.1 (Fourier Base for Period *L*)

(a) The functions

$$\left\{\frac{1}{\sqrt{2}}, \cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \ldots, \sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right), \ldots\right\}$$

are orthogonal.

(b) If $f(x) = a_0 + \sum_{n=1}^{+\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$, then

$$\boxed{a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\ dx}$$

$$\boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\ dx}$$

$$\boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\ dx}$$

The proof, being similar to what we did earlier, is omitted.

Facts

1. One has $f(x)$ equal to its Fourier series of period $2L$ whenever:

1. f itself has period $2L$;
2. f is piecewise continuous with left- and right-derivatives existing everywhere.

2. Odd functions have all the a's equal to zero; even ones have all the b's equal to zero. (Why?)

Examples 8.2

(a) Periodic square wave, period 4 ($L = 2$):

$$f(x) = \begin{cases} 0 & \text{if } -2 < x < -1 \\ k & \text{if } -1 < x < 1 \\ 0 & \text{if } 1 < x < 2 \end{cases}$$

(b) Half-wave rectifier

$$u(t) = \max\{0, E\sin\omega t\},$$

has period $2\pi/\omega$ ($L = \pi/\omega$), and first hits zero when $t = \pi/\omega$.

Another Fact:
The F. Series of a linear combination is that linear combination of the F. Series.

This helps you get F. Series of harder functions in terms of those of simpler ones.

Example 8.3
Raised rectangular pulse of period $2L$: Take Example 7.5 and modify it.

Odd and Even Functions

Definition 8.4 The function f is **even** if $f(x) = f(-x)$ for every x. f is **odd** if $f(x) = -f(-x)$ for every x.

Examples in class

Note:

$$f \text{ even} \Rightarrow \int_{-L}^{L} f(x)\, dx = 2\int_{0}^{L} f(x)\, dx$$

$$f \text{ odd} \Rightarrow \int_{-L}^{L} f(x)\, dx = 0$$

Consequence
If we define a function on $[0, L]$ and then extend it to $[-L, 0]$ as an odd or even function, we have:

Fourier Series of Even Periodic Extension of f: $[0, l] \longrightarrow \mathbb{R}$

$$a_0 = \frac{1}{L}\int_{0}^{L} f(x)\, dx$$

$$a_n = \frac{2}{L}\int_{0}^{L} f(x) \cos\frac{n\pi x}{L}\, dx$$

$$b_n = 0$$

Fourier Series of Odd Periodic Extension of f: $[0, l] \longrightarrow \mathbb{R}$

$$a_n = 0$$

$$b_n = \frac{2}{L}\int_{0}^{L} f(x) \sin\frac{n\pi x}{L}\, dx$$

Examples 8.5
(a) Raised saw-tooth: First drop it to get an odd function, then add a constant to raise it up.

(b) Triangular wave (p. 545): both even and odd extensions

Exercise Set 8
p. 490 #1, 3, 9, 13, 15 p. 546 #1, 3, 5, 11, 13, 15, 17, 18

Extra Credit (Adds up to 5 points to Test 2) Experiencing Maple or MathCad
Go to the computer lab, learn Maple, and generate and graph the first ten terms of the Fourier series for the odd periodic extension of $f(x) = x^2$ with period 4.

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9. Partial Differential Equations; Solving the Wave Equation

Here are some important PDE's

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$ ....one dimensional wave equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$$ ....one dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ ....two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$ ....two-dimensional Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$ ....three-dimensional Laplace equation

Definitions 9.1 A **solution of a PDE in a region *R*** is a function f which has all the partial derivatives required, and is such that it satisfies the given PDE when plugged in. A **homogeneous PDE** is a PDE of the form $D(u) = 0$, where D is some partial differential operator. (For example, all of the above except for Poisson's eqn.)

Examples 9.2 The following are all solutions to the 2-dimensional Laplace equation.:

$$u = x^2 - y^2$$
$$u = e^x\cos y$$
$$u = \ln(x^2 + y^2)$$

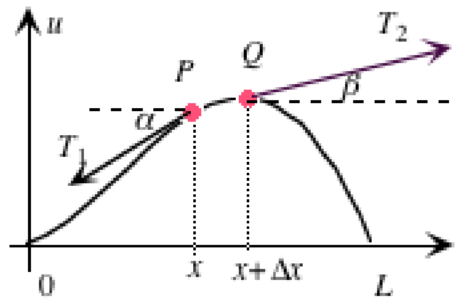
As usual, we have the following general theoretical observations:

Theorem 9.3 (Solution to PDE's)
(a) If u_1 and u_2 are solutions to a homogeneous PDE, then so is $Au_1 + Bu_2$.
(b) If u_1 and u_2 are solutions to a nonhomogeneous PDE, then $u_1 - u_2$ is a solution to the associated homogeneous PDE, whence the general solution to a nonhomogeneous PDE has the form:

$$u = c + p,$$

where c is the GS of the associated homogeneous eqn, and where p is a particular solution of the nonhomog. eqn.

The Wave Equation
This is the equation of transverse motion governing an elastic string of some length L, fixed at the endpoints. (See the diagram.)



Let P and Q be nearby points on the string with difference in $x-$coordinates Δx. Assume the slopes at P and Q make angles α and β with the horizontal as shown. Since the string is elastic, the tension is tangential at P and Q, and let T_1 and T_2 be the magnitude of these forces.

Since there is no horizontal motion,

$$T_1 \cos\beta = T_2 \cos\alpha = T \text{ ....(1)}$$

For vertical motion,

$$T_1 \sin\beta - T_2 \sin\alpha = \rho \Delta x \frac{\partial^2 u}{\partial t^2} \text{ ....(2)}$$

where ρ is the density in mass/unit length. (The length of the *unstretched* string is Δx.) Dividing eqn. (2) by the equal quantities in eqn. (1) gives:

$$\tan\beta - \tan\alpha = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2} \text{ ....(3)}$$

where T is the fixed quantity in (1). Since $\tan\theta$ is just $\partial u/\partial x$, eqn. (3) becomes

$$\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x} = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2} .$$

Dividing by Δx and taking the limit as $\Delta x \longrightarrow 0$, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

Thinking of ρ/T as the reciprocal of the square of a positive number, we now get

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} ,$$

which is the one-dimensional wave equation.

Solution of the One Dimensional Wave Equation
First note that $u(x,t)$ is defined for $0 \le x \le L$ and arbitrary t. Thus the domain of u is an infinite strip in the xt-plane (sketch). The *boundary* of that strip consists of all points of the form $(0,t)$

and (L,t) as t ranges over the real numbers. Since we know that the string is fixed at $x=0$ and $x=L$ at all times, we are in fact specifying a *boundary condition,* that $u = 0$ at all points on the boundary of its domain. (Other boundary conditions, besides $u=0$, are possible.) Besides boundary conditions, we need also to specify what happened an at time $t=0$ (*initial conditions*). Since this is a second order DE, we expect to need both $u(x,0)$ and $\partial u/\partial t(x,0)$. Thus we assume both given. The first refers to the initial displacement of the string, while the second to its initial velocity.

Thus we solve:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \text{, subject to}$$

boundary conditions: $u(0,t)=u(L,t) = 0$ (*all* t)

initial conditions: $u(x,0) = f(x)$;

$$\frac{\partial u}{\partial t}(x, 0) = g(x)$$

We first solve it by the method of **separation of variables.** That is, we seek solutions of the form

$$u(x,t) = F(x)G(t).$$

Under these assumptions,

$$\frac{\partial^2 u}{\partial x^2} = F''G, \text{ and}$$

$$\frac{\partial^2 u}{\partial t^2} = F\ddot{G}$$

Substituting into the PDE gives:

$$F\ddot{G} = c^2F''G.$$

Separating variables,

$$\frac{\ddot{G}}{c^2G} = \frac{F''}{F}.$$

Since the LHS is a function of t only and the RHS is a function of x only, they must both be constant, k say. This gives us two ordinary linear second order DE's:

$$F'' - kF = 0$$

$$\ddot{G} - c^2kG = 0.$$

Before we solve them, we look at what the boundary conditions imply for the functions F and G.

Since we have $u(0,t) = u(L,t) = 0$ for all t, this means that

$$F(0)G(t) = 0 = F(L)G(t) \text{ for all } t.$$

If $F(0) \neq 0$, then it must mean that $G(t) = 0$ for every t, and hence that $u(x,t) = F(x)G(t) = 0$. But the zero solution is not the one we seek. Thus we cannot assume $F(0) \neq 0$, and hence $F(0) = 0$. A similar argument shows that $F(L) = 0$ a well. Thus we are solving the DE's

$F'' - kF = 0$ subject to $F(0) = 0 = F(L)$

$\ddot{G} - c^2kG = 0$, subject to initial conditions (specified later)

Solving the first equation $F'' - kF = 0$ gives three possible outcomes depending on whether k is positive, zero or negative.

***k* positive:** Write $k = \mu^2$, and get

$$F(x) = Ae^{\mu x} + Be^{-\mu x}.$$

Plugging in the conditions $F(0) = 0 = F(L)$ gives

$$A+B = 0 = Ae^{\mu}L + Be^{-\mu L},$$

implying that $A = B = 0$. (Exercise using a little linear algebra.) Thus $F(x) \equiv 0$ in this case, so we discard it.

***k* zero:** Solving $F''(x) = 0$ gives

$$F(x) = Ax + B$$

by integration, and the boundary conditions again give us $F(x) \equiv 0$.

***k* negative: (Last ditch stand)** Write $k = -p^2$, and obtain

$$F(x) = A \cos px + B \sin px.$$

Plugging in boundary conditions gives:

$$0 = A$$
$$0 = B \sin pL,$$

whence $pL = n\pi;\ n = 0, \pm 1, \pm 2, \ldots$ (unless we take B=0—useless.) Thus we have the following only possible nonzero solutions.

$$F_n(x) = B \sin px = B \sin \frac{n\pi}{L} x \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Since we'll worry about the arbitrary constant B later when we get the solution for $u(x, t)$, we take $B = 1$ for now, so that

$$\boxed{F_n(x) = \sin \frac{n\pi}{L} x \quad (n = 0, \pm 1, \pm 2, \ldots).}$$

Note also that we have shown that the constant k must have the form:

$$k = -p^2,$$

where $p = \dfrac{n\pi}{L}$.

Turning to the equation for $G(t)$, $\ddot{G} - c^2kG = 0$, we have just seen that k must have the form

$k = -p^2 = -\left(\dfrac{n\pi}{L}\right)^2$, so that the equation for G becomes

$$\ddot{G} - \left(\frac{cn\pi}{L}\right)^2 G = 0,$$

or

$\ddot{G} - \lambda_n^2 G = 0$, where $\boxed{\lambda_n = \dfrac{cn\pi}{L}}$

The general solution to this has the form

$$\boxed{G_n(t) = B_n \cos \lambda_n t + B^*_n \sin\lambda_n t.}$$

Combining gives solutions:

$$u_n(x,t) = F_n(x)G_n(t),$$

so that:

$$u_n(x,t) = [[B_n \cos \lambda_n t + B^*_n \sin\lambda_n t] \sin \frac{n\pi}{L} x \qquad (n = \pm 1, \pm 2, ...)$$

Discussion of these solutions Think of the term in square brackets as a sinusoidal function of t (in fact, it has the form $K_n \sin(\lambda_n t + \alpha)$). The sine term at the end is its amplitude, which varies sinusoidally with x, so that, depending on n, sketching the amplitude curve along the string, one has a differing number of **nodes** (= stationary points on the string). These solutions are called the **modes** of the string, with $n=1$ giving the **fundamental mode,** and the other values of n giving the **n th normal mode.**

Also notice that we have infinitely many solutions to this homogeneous equation. By the superposition principle for homogeneous equations, it follows that, provided the series converges, we have a solution

$$\boxed{\sum_{n=1}^{+\infty} [B_n \cos \lambda_n t + B^*_n \sin\lambda_n t] \sin \frac{n\pi}{L} x.}$$

We now substitute the initial conditions:

$u(x, 0) = f(x)$:

$$\sum_{n=1}^{+\infty} B_n \sin \frac{n\pi}{L} x = f(x) \text{..} (1)$$

$\frac{\partial u}{\partial t}(x, 0) = g(x)$:

$$\sum_{n=1}^{+\infty} \lambda_n B^*_n \sin \frac{n\pi}{L} x = g(x) \text{..} (2)$$

Equation (1) says that we are looking at the FS for $f(x)$ given by $x \geq 0$. Extending $f(x)$ to an odd periodic function of period $2L$ gives:

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L} dx ,$$

and similarly,

$$B_n^* = \frac{2}{L\lambda_n}\int_0^L g(x) \sin\frac{n\pi x}{L} dx = \frac{2}{cn\pi}\int_0^L g(x) \sin\frac{n\pi x}{L} dx .$$

u

$$\boxed{\lambda_n = \frac{cn\pi}{L}}$$

To further understand this solution, we assume, to simplify things, that the initial velocity $g(x)$ is zero, so that the B^*_n's are zero. This gives

$$u(x, t) = \sum_{n=1}^{+\infty} B_n \cos \lambda_n t \sin \frac{n\pi}{L} x \; ; \qquad \lambda_n = \frac{cn\pi}{L}$$

$$= \sum_{n=1}^{+\infty} B_n \cos \frac{cn\pi}{L} t \sin \frac{n\pi}{L} x$$

$$= \frac{1}{2} \sum_{n=1}^{+\infty} B_n \left[\sin \frac{n\pi}{L}(x-ct) + \sin \frac{n\pi}{L}(x+ct)\right]$$

$$= \frac{1}{2} \sum_{n=1}^{+\infty} B_n \sin \frac{n\pi}{L}(x-ct) + \frac{1}{2} \sum_{n=1}^{+\infty} B_n \sin \frac{n\pi}{L}(x+ct)].$$

These two series are the FS for the function $f(x)$ with x replaced by $x-ct$ and $x+ct$ respectively. More precisely, we must replace $f(x)$ with its odd periodic extension $f^*(x)$. Thus, *amazingly,* the solution becomes:

Solution of wave eq. with zero initial velocity $\boxed{u(x,t) = \frac{1}{2}\left[f^*(x-ct) + f^*(x+ct)\right]}$

This says that the motion of the string is composed of two traveling waves in the shape of the initial deflection, and going in the opposite direction with speed c.

Example Let's look at what happens when the initial deflection $f(x)$ is triangular by drawing some pictures.

Exercise Set 9
p. 537, # 3-23 odd
p. 546 # 3, 5, 7, 17, 19

Hand In (Value: 20 points) ***Vibrations of Beam***
Kreyszig, p. 547 (§12.3) #15–20

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10. D'Alembert's Solution of the Wave Equation

Now that we've solved the wave equation the hard way, we do it by an easier method:
Start with the original wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

and make the substitutions

$$y = x+ct; \; z = x-ct.$$

To implement these substitutions, we need formulae for the partials u_t and u_x in terms of u_y and u_z. By the chain rule,

$$u_t = u_y y_t + u_z z_t$$
$$= cu_y - cu_z$$

$$= c(u_y - u_z),$$

and so

$$u_{tt} = c\,\frac{\partial}{\partial t}(u_y - u_z) = c(u_{yy}y_t + u_{yz}z_t - (u_{zy}y_t + u_{zz}z_t)\,)$$

But, again, $y_t = c$ and $z_t = -c$, so that

$$u_{tt} = c(cu_{yy} - cu_{yz} - (cu_{zy} - cu_{zz})) = c^2(u_{yy} - 2u_{yz} + u_{zz}).$$

Similarly,

$$u_x = u_y y_x + u_z z_x$$
$$= u_y + u_z,$$

and so

$$u_{xx} = u_{yy} + 2u_{yz} + u_{zz}.$$

Plugging these in to the PDE gives:

$$c^2(u_{yy} - 2u_{yz} + u_{zz}) = c^2(u_{yy} + 2u_{yz} + u_{zz}),$$

whence the wave equation reduces to

$$u_{yz} = 0$$

To solve this silly little equation, all we have to do is integrate twice, getting first,

$$u_y = h(y)$$

for an arbitrary function h of y. Integrating again gives:

$$u = \int h(y)\ dy$$
$$= \phi(y) + \psi(z),$$

where both ϕ and ψ are arbitrary functions of their arguments. Thus,

$$u(x,t) = \phi(x+ct) + \psi(x-ct)$$

is the general solution to the wave equation. Now we plug in the initial conditions to get

$$f(x) = \phi(x) + \psi(x); \ .... \ (1)$$
$$0 = c\phi'(x) - c\psi'(x). \ .... \ (2)$$

The second equation, upon integration, gives:

$$\psi(x) = \phi(x) + K.$$

Substituting in (1) now gives

$$f(x) = \phi(x) + \phi(x) + K = 2\phi(x) + K$$

so that $\phi(x) = \frac{1}{2}(f(x) - K)$.

Thus, $u(x,t) = \phi(x+ct) + \phi(x-ct) + K$

$$= \frac{1}{2}(f(x+ct) - K) + \frac{1}{2}(f(x-ct) - K) + K$$
$$= \frac{1}{2}\,[f(x+ct) + f(x-ct)],$$

which is the solution we so laboriously obtained earlier.

Exercise Set 10
p. 552 # 5, 7, 19

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11. Solution of the Heat Equation

S'pose we have a long insulated bar in the $x-$direction. Then the temperature u depends only on x, so we get the one-dimensional heat eqn:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where u is the temperature at a point in a homogeneous material, and where $c^2 = K/(\sigma\rho)$, and σ = specific heat, and ρ = density (we'll derive this when we do vector calculus).

We first solve this equation with **boundary conditions** the same as those in the wave equation:

$$u(0,t) = u(L,t) = 0 \text{ for all } t.$$

This means that we are keeping the ends of the bar at zero degrees. Also let's put in the **initial condition**

$$u(x,0) = f(x).$$

(No need for $u_t((x,0)$ as this equation is first order with respect to t.)

Since seperation of variables worked before, we might as well try it again, so write

$$u(x,t) = F(x)G(t),$$

and plug it in to the heat equation to get

$$F\dot{G} = c^2 F''G,$$

or

$$\frac{\dot{G}}{c^2 G} = \frac{F''}{F}.$$

As with the wave equation, they must both be constant, k say, giving

$$F'' - kF = 0;$$
$$\dot{G} - c^2 kG = 0.$$

We now go through the ritual of plugging in the ∂ condition and conclude again that $F(0) = F(L) = 0$. Thus we are solving the DE's

$$F'' - kF = 0; \qquad \text{subject to } F(0) = 0 = F(L)$$
$$\dot{G} - c^2 kG = 0$$

Finally, looking at the possible cases for k in solving the first equation give that k must be negative, so we take $k = -p^2$, getting the equations

$$\boxed{\begin{array}{ll} F'' + p^2 F = 0; & \text{subject to } F(0) = 0 = F(L) \\ \dot{G} + c^2 p^2 G = 0 & \end{array}}$$

The solution for F is now a repeat of what we did before:

$$F(x) = A \cos px + B \sin px.$$

The boundary conditions give:

$$A = 0 \text{ (at } x=0)$$
$$B \sin pL = 0,$$

so that

$$p = \frac{n\pi}{L} \qquad (n = 1, 2, \ldots)$$

Thus, as before,

$$F(x) = B \sin px = B \sin \frac{n\pi}{L} x, \quad (n = 1, 2, \ldots).$$

Taking $B = 1$ for now,

$$\boxed{F_n(x) = \sin \frac{n\pi}{L} x \quad (n = 1, 2, \ldots)}$$

Turning to G, we solve the first order D.E.

$$\dot{G} + c^2p^2G = 0,$$

or $$\dot{G} + \left(\frac{cn\pi}{L}\right)^2 G = 0$$

i.e., $$\dot{G} + \lambda_n^{\,2}G = 0,$$

giving solutions

$$\boxed{G_n(t) = B_n e^{-\lambda_n^2 t} \qquad (n = 1, 2, \ldots)}$$

Putting them together gives

$$u_n(x, t) = B_n\, e^{-\lambda_n^2 t} \sin \frac{n\pi}{L} x \qquad (n = 1, 2, \ldots).$$

Summing up gives

$$\boxed{\begin{gathered} u(x, t) = \sum_{n=1}^{+\infty} B_n\, e^{-\lambda_n^2 t} \sin \frac{n\pi}{L} x \\ \lambda_n = \frac{cn\pi}{L} \end{gathered}}$$

Initial Conditions

$$f(x) = \sum_{n=1}^{+\infty} B_n \sin \frac{n\pi}{L} x\,,$$

so again

$$\boxed{B_n = \frac{2}{L} \int_0^L f(x) \sin\frac{n\pi x}{L}\, dx}$$

Remarks

1. Note that the expression for $u(x,t)$ is thus an exponentially damped Fourier series for $f(x)$, with the higher harmonics damped more than the lower ones.
2. In particular, as $t \longrightarrow \infty$, all the terms approach zero, so the bar cools down to zero as expected.
3. If $f(x)$ is sinusoidal, say $f(x) = 100 sin(\pi x/80)$, with $L = 80$, then it is already a FS, and so

$$u(x,t) = 100 sin(\pi x/80)\; e^{-\lambda_1^2 t}\,.$$

We can use this together with the definition of λ_1 to compute the speed of decay. (See p. 648).
4. Also look at Example 3 on p. 648.

Steady State Heat Equation

This is Laplace's equation

$$\nabla^2 u = 0$$

(since the temperature u does not vary with time). In the one-dimensional form, it is

$$\frac{\partial^2 u}{\partial x^2} = 0.$$

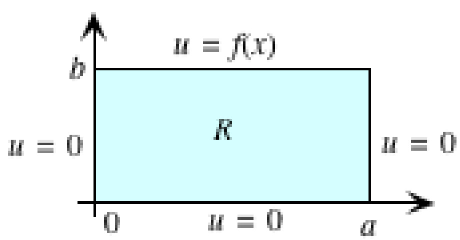
Clearly, we can't use the same boundary conditions (that the ends of the bar are kept at 0 degrees, for then we would get the solution zero.) Also, the one-dimensional form is too simple - you can solve it by integrating twice; if the temp at one end is a and that at the other end is b, then the soln will be:

$$u(x) = b\frac{x}{L} + a\frac{L-x}{L} \quad \text{—not very exciting.}$$

Thus we solve the two-dimensional equation instead:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Solution of this, given prescribed values on the boundary is called the **Dirichlet problem,** and is the subject of ongoing research and indeed of a whole branch of mathematics, called **potential theory**. We now solve it in the case of a rectangular region R given by $0 \le x \le a$ and $0 \le y \le b$, where the temperature is given by $f(x)$ on the upper side and zero on the remaining sides, as shown here.



We again use separation of variables,

$$u(x,y) = F(x)G(y).$$

Left-and right ∂ conditions:

$$0 = F(0)G(y) = F(a)G(y),$$

so $F(0)$ and $F(a)$ must be zero. Plugging in and separating gives:

$$F''(x) + kF(x) = 0; \; F(0) = F(a) = 0$$

$$G''(y) - kG(y) = 0.$$

Solving the first gives, as usual,

$$k = \left(\frac{n\pi}{a}\right)^2,$$

so that we get solutions

$$F_n(x) = \sin\frac{n\pi}{a}x$$

$$G_n(y) = A_n e^{(n\pi/a)y} + B_n e^{-(n\pi/a)y} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(I)}$$

The bottom boundary condition gives

$$0 = A_n + B_n,$$

so $\quad G_n(y) = A_n[e^{(n\pi/a)y} - e^{-(n\pi/a)y}]$

$$= A_n^*\sinh \frac{n\pi}{a} y.$$

Thus, $\quad u_n(x, y) = A_n^* \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a} y,$

and hence

$$\boxed{u(x, y) = \sum_{n=1}^{+\infty} A_n^* \ \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a} y.}$$

Plugging in the top ∂ condition,

$$f(x) = \sum_{n=1}^{+\infty} A_n^* \ \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a} b,$$

so that $\quad A_n^* \sinh \frac{n\pi}{a} b = \frac{2}{a}\int_0^a f(x) \sin\frac{n\pi x}{a}\, dx$

$$\boxed{A_n^* = \frac{2}{a \sinh (n\pi b/a)} \int_0^a f(x) \sin\frac{n\pi x}{a}\, dx \ .}$$

Remarks

In all the above examples, $f(x)$ was assumed to have a periodic extension, since we are dealing with a finite object such as a plate or a bar. We might ask what happens to the accuracy of finite sum approximations to the solution $u(x, y)$ as the length (a in this case) gets larger. This depends on how fast the Fourier coefficients approach zero with increasing n. By drawing pictures for a fixed function $f(x)$, truncated at 0 and a, one can see that, the smaller the period, the faster this will happen, since the oscillations of $\sin(n\pi x/a)$ are smaller for smaller a. Thus Fourier series approximations are less effective for longer objects.

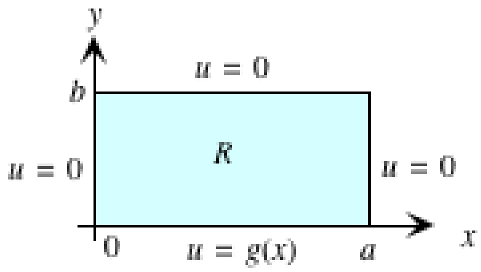
Exercise Set 11

p. 560 # 5, 7, 9, 11, 13.

Hand In (Value = 15 points) Superposition

(a) Show that if $u(x, y)$ and $v(x, y)$ satisfy Laplace's equation with boundary conditions B_1 and B_2 respectively, then $u(x, y) + v(x, y)$ satisfies the equation with boundary conditions B_1+B_2.

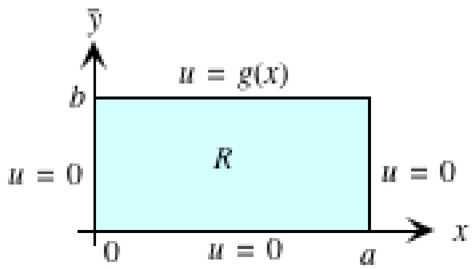
(b) Using the method in the notes, consider the two-dimensional heat equation with the following boundary condition.



Show that, with the change of variables $\bar{y} = b - y$, Laplace's equation with the above boundary conditions transforms to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial \bar{y}^2} = 0$$

with boundary conditions as shown:

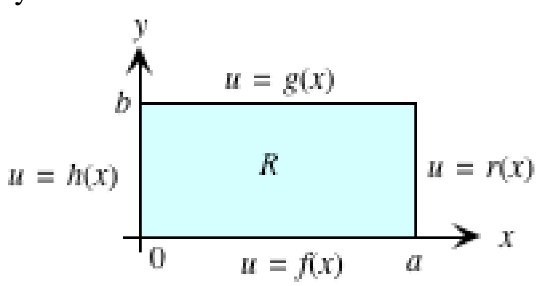


and hence deduce that the solution is

$$u(x, y) = \sum_{n=1}^{+\infty} B_n{}^* \ \sin\frac{n\pi}{a} x \sinh\frac{n\pi}{a}(b-y),$$

where $B_n{}^* = \dfrac{2}{a \sinh\,(n\pi b/a)} \displaystyle\int_0^a g(x) \sin\frac{n\pi x}{a}\, dx$.

(c) Now use parts (a) and (b) to obtain a solution of the equation with the following general boundary condition.



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12. The Two-Dimensional Wave Equation

Consider the forces on a taut membrane stretched over a region R in the xy-plane and vibrating transversally. The transversal displacement (small displacements) satisfies the two-dim wave equation (pp. 616–618):

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

or

$$u_{tt} = c^2\nabla^2 u.$$

Solving the *easier* equation $\nabla^2 u = 0$ is hard enough. The solution to the wave equation is even harder, and we do it in only in two special cases:
1. Rectangular Membrane
2. Circular Membrane

Rectangular Membrane

We assume the rectangle $\{(x, y) \mid 0 \le x \le a,\ 0 \le y \le b\}$ with $u = 0$ on the entire boundary, and with two initial conditions:

$$u(x,y,0) = f(x,y)$$

$$u_t(x,y,0) = g(x,y).$$

We do the separation of variables in two stages. First, write:

$$u(x,y,t) = F(x,y)G(t).$$

Substitution into the wave equation gives:

$$F\ddot{G} = c^2(F_{xx}G + F_{yy}G).$$

Separating variables,

$$\frac{\ddot{G}}{c^2G} = \frac{1}{F}(F_{xx} + F_{yy}),$$

with both now constant, equal to $-\nu^2$, say. (The other cases lead, as usual, to the zero solution, although this will not be apparent until we examine the solutions for F.) This gives the two DE's:

$$\ddot{G} + \lambda^2 G = 0 \quad .... \quad (1) \quad (\text{where } \lambda = c\nu)$$

$$F_{xx} + F_{yy} + \nu^2 F = 0 \ ... \ (2)$$

Here $F(x,y)$ must vanish on the boundary of R. We first solve (2) by again using separation of variables, so set

$$F(x,y) = H(x)Q(y).$$

Here, $H(0) = H(a) = 0$ and $Q(0) = Q(b) = 0$. Plugging into (2) and dividing by HQ gives

$$H''(x)/H(x) + Q''(y)/Q(y) + \nu^2 = 0,$$

so that

$$H''(x)/H(x) = -\,(Q''(y)/Q(y) + \nu^2),$$

and both sides are constant. The boundary conditions on H and Q now give that the constant must have the form $-k^2$, and also that $\nu^2 - k^2 > 0$. (This means that $\nu^2 > k^2$, and this is why the first constant is positive, and we took it as ν^2 in the first place.) Thus we wind up with

$$H''(x) + k^2H(x) = 0,$$

and

$$Q''(y) + p2Q(y) = 0, \text{ with } p^2 = \nu^2 - k^2.$$

Solving, we get

$$H(x) = A\cos kx + B\sin kx,$$

$$Q(y) = C\cos py + D\sin py,$$

subject to

$$H(0) = H(a) = 0 \text{ and } Q(0) = Q(b) = 0.$$

Plugging in $x = y = 0$ gives $A = C = 0$, and we take $B = D = 1$ as usual. Also, plugging in $x = a$ and $y = b$ gives $k = m\pi/a$ and $p = n\pi/b$ (m and n needn't be the same) so that:

$$H_n(x) = \sin\frac{m\pi x}{a}, \quad Q_m(y) = \sin\frac{n\pi y}{b} \qquad (n, m = 1, 2, ...).$$

This gives a whole bunch of solutions

$$F_{mn}(x,y) = \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \quad (n, m = 1, 2, ..).$$

We still have to solve equation (1), $\ddot{G} + \lambda^2 G = 0$, where $\lambda = c\nu$. This means we have to figure out what ν is, given the values for k and p above. Since $p^2 = \nu^2 - k^2$, with $k = m\pi/a$ and $p = n\pi/b$, it follows that

$$\nu = \sqrt{p^2+k^2} = \pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

so that

$$\lambda = c\nu = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.$$

Call this quantity λ_{mn}. This permits us to solve for $G(t)$, getting:

$$G_{mn}(t) = B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t.$$

This gives the solutions

$$u_{mn}(x, t) = (B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t.)\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}.$$

Summing up, we get

$$\boxed{u(x, t) = \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}(B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t.)\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}$$

We have yet to put in the initial conditions, and hence determine the constants B_{mn} and B^*_{mn}.

Plugging in the first initial condition, we get, at $t=0$,

$$f(x, y) = \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} B_{mn}\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

This expression is a FS for a function of two variables. Write this as

$$f(x, y) = \sum_{m=1}^{+\infty} \left(\sum_{n=1}^{+\infty} B_{mn} \sin \frac{n\pi y}{b} \right) \sin \frac{m\pi x}{a}$$

which is, for each fixed y, a FS expansion of the function $f(x,y)$ of x. If follows that the coefficient

$$\sum_{n=1}^{+\infty} B_{mn} \sin \frac{n\pi y}{b} \quad ,$$

is the *nth* Fourier coefficient for the odd periodic extension of $f(x,y)$ (as a function of x). Thus,

$$\sum_{n=1}^{+\infty} B_{mn} \sin \frac{n\pi y}{b} = \frac{2}{a} \int_0^a f(x, y) \sin \frac{m\pi x}{a} dx \ .$$

Now the LHS is a FS expansion of the function of y given on the RHS, and so one has:

$$B_{mn} = \frac{2}{b} \int_0^b \left[\frac{2}{a} \int_0^a f(x, y) \sin \frac{m\pi x}{a} dx \right] \sin \frac{n\pi y}{b} dy \ ,$$

so that

$$\boxed{B_{mn} = \frac{4}{ab} \int_0^b \int_0^a f(x, y) \sin \frac{m\pi x}{a} \ \sin \frac{n\pi y}{b} \ dx\, dy}$$

To get the starred coefficients, take

$$g(x, y) = u_t(x, y, 0) = \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} \lambda_{mn} B^*_{mn} \sin \frac{m\pi x}{a} \ \sin \frac{n\pi y}{b}$$

and get by the above procedure,

$$\boxed{B^*_{mn} = \frac{4}{ab\lambda_{mn}} \int_0^b \int_0^a g(x, y) \sin \frac{m\pi x}{a} \ \sin \frac{n\pi y}{b} \ dx\, dy}$$

Example 12.1 Find the vibration behavior of a square membrane (a=b=1) with given initial displacement $f(x,y)$ and initial velocity zero.

Answer Here, $\lambda_{mn} = c\pi\sqrt{m^2 + n^2}$, and $B^*_{mn} = 0$. Thus,

$$u(x, t) = \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} B_{mn} \cos \lambda_{mn} t \sin m\pi x \sin n\pi y$$

where B_{mn} are given by the above formula. Let us find a few nodal lines, that is lines of zero displacement in the membrane for the various harmonics $u_{mn}(x,y,t)$.

m = n = 1: $u_{11}(x,y,t) = B_{11}\cos \lambda_{11}t \ \sin \pi x \ \sin \pi y$.
This is zero when $\sin\pi x \sin \pi y = 0$, which occurs when $x = 0$ or 1 or when $y = 0$ *or* 1. ie, on the boundary of the membrane. The center of the membrane ($x = y = 1/2$) is then vibrating according with displacement $B_{11}cos\ \lambda_{11}t$.

m = 2, n = 1: $u_{21}(x,y,t) = B_{21}\cos \lambda_{21}t \ \sin \pi x \ \sin 2\pi y$.
This is zero when $x = 0$ or 1 or when $y = 0$, 1/2 or 1. thus the nodal lines consist of the boundary and the line $y = 1/2$.

When we start superimposing the different harmonics, we get more complicated nodal lines.

Circular Membrane
To deal with vibrations of a circular membrane, we need to express things in terms of polar coordinates. We first consider the Laplacian $u_{xx} + u_{yy}$.

Write $\quad x = r\cos \theta,\ \ y = r\sin \theta,$
and get $\quad u_\theta = -u_x r\sin \theta + u_y r\cos \theta,$
and so

$$\begin{aligned} u_{\theta\theta} &= (-u_x r\sin \theta)_\theta + (u_y r\cos \theta)_\theta \\ &= (-u_{x\theta} r\sin \theta - u_x r\cos \theta) + (u_{y\theta} r\cos \theta - u_y r\sin \theta) \\ &= (u_{xx} r\sin \theta r\sin\theta - u_{xy} r\cos \theta r\sin \theta - u_x r\cos \theta) + \\ &\qquad (-u_{yx} r\sin \theta r\cos \theta + u_{yy} r\cos \theta\cos \theta - u_y r\sin \theta), \end{aligned}$$

so

$$u_{\theta\theta} = r^2(u_{xx}\sin 2\theta - 2u_{xy}\cos \theta \sin \theta + u_{yy}\cos 2\ \theta) - r(u_x\cos \theta + u_y\sin \theta).$$

Also,

$$u_r = u_x\cos\theta + u_y\sin\theta,$$

and

$$\begin{aligned} u_{rr} &= u_{xr}\cos\theta + u_{yr}\sin \theta \\ &= u_{xx}\cos 2\theta + u_{xy}\sin\theta\cos\theta + u_{yx}\sin\theta\cos\theta + u_{yy}\sin 2\theta, \end{aligned}$$

so

$$u_{rr} = u_{xx}\cos 2\theta + 2u_{xy}\sin\theta\cos\theta + u_{yy}\sin 2\theta.$$

Now consider

$$\begin{aligned} \frac{1}{r^2} u_{\theta\theta} + u_{rr} &= u_{xx} + u_{yy} - \frac{1}{r}(u_x\cos \theta + u_y\sin \theta) \\ &= u_{xx} + u_{yy} - \frac{1}{r} u_r. \end{aligned}$$

Hence,

Laplacian in Polar Coordinates

$$\boxed{\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}.}$$

Now we can proceed to solve the wave equation for the circular membrane. This has the form:

$$u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right),$$

where we are seeking a solution $u(r, \theta, t)$. We consider only radially symmetric solutions, ie., solutions independent of θ and thus of the form $u(r,t)$. Thus $u_{\theta\theta} = 0$, and

$$u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r\right).$$

is the equation we solve, subject to the boundary conditions and initial conditions:

$$u(R,t) = 0;\ u(r,0) = f(r);\ u_t(r,0) = g(r).$$

Separating variables, put

$$u(r,t) = W(r)G(t),$$

plug in, and get

$$\frac{\ddot{G}}{c^2G} = \frac{1}{W}(W'' + \frac{1}{r}W')$$

and both are constant, say $-k^2$, giving the two DE's:

$$\ddot{G} + \lambda^2 G = 0 \quad (\lambda = ck)$$

and

$$W'' + \frac{1}{r}W + k^2W = 0,$$

subject to $W(R) = 0$; $u(r,0) = f(r)$; $u_t(r,0) = g(r)$. The second equation is a homogeneous linear DE equation with non-constant coefficients. In fact, it is a form of Bessel's equation, and is solved using power series methods, giving solutions in terms of Bessel functions. For this reason, we abandon the problem right here, and continue it in the homework project.

Exercise Set 12
Problems p. 578 # 19, 20

Hand In (Value = 15)
Write a summary of the complete solution of the circular membrane problem, up to formula (19) on p. 583. Your summary should include a brief discussion of Bessel Functions, and the Fourier-Bessel series.

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13. Review of Vectors; Vector Fields

A *vector* in 3-space is an arrow. Two vectors are regarded as **the same** if they are of the same length and point in the same direction. Illustration in class. Moving a vector to the origin gives description in terms of **coordinates**.

Examples 13.1

A. $\boldsymbol{u} = \langle 1, 2, 3\rangle$, $\boldsymbol{v} = \langle 0, 2, -1\rangle$, $\boldsymbol{w} = \langle 0, -1, 0\rangle$.

B. The position vector of the point $P(x, y, z)$ is $\boldsymbol{OP} = \langle x, y, z\rangle$. Example: the position vector of $P(1, -1, 0)$ is $\boldsymbol{OP} = \langle 1, -1, 0\rangle$.

C. Given $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, obtain the vector

$$\boldsymbol{PQ} = \langle (x_2 - x_1),\ (y_2 - y_1),\ (z_2 - z_1)\rangle.$$

Example: Given $P(1, 2, 3)$, $Q(2, 5, -1)$, find the vector ’$\boldsymbol{PQ}$

Facts About Vectors

1. Algebra of Vectors: Let $\boldsymbol{u} = \langle u_1, u_2, u_3\rangle$, $\boldsymbol{v} = \langle v_1, v_2, v_3\rangle$. Then:

$\boldsymbol{u} + \boldsymbol{v} = \langle u_1+v_1\ ,\ u_2+v_2\ ,\ u_3+v_3\rangle$;

$c\boldsymbol{u} = \langle cu_1,\ cu_2\ ,\ cu_3\rangle$

$\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-1)\boldsymbol{v} = \langle u_1-v_1\ ,\ u_2 - v_2\ ,\ u_3-v_3\rangle$.

Example Let $\boldsymbol{u} = \langle 1,\ 2,\ 3\rangle$, $\boldsymbol{v} = \langle 0,\ 2,\ -1\rangle$, $\boldsymbol{w} = \langle 0, -1, 0\rangle$. Then find: $4\boldsymbol{u} - 2\boldsymbol{v}$.

2. Geometric Interpretation of the algebra of vectors. (In class)

3. Some Special Vectors:

$\boldsymbol{i} = \langle 1, 0, 0\rangle$, $\boldsymbol{j} = \langle 0, 1, 0\rangle$, $\boldsymbol{k} = \langle 0, 0, 1\rangle$

4. Decomposition of General Vector: We see that $\langle a, b, c\rangle = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$.

5. The length of a vector is given by

Length of a Vector
$$|\langle a, b, c\rangle| = \sqrt{a^2 + b^2 + c^2}$$

A **unit vector** is a vector of length 1.

Examples

A. Find the length of $\boldsymbol{u} = \langle -1, 2, 3\rangle$

B. Find a unit vector in the direction of $\boldsymbol{u}$ above.

C. Find a vector of length 3 in the direction of $\langle -1,\ 1,\ 2\rangle$.

6. The Dot Product

Let $\boldsymbol{u} = \langle u_1, u_2, u_3\rangle$, $\boldsymbol{v} = \langle v_1, v_2, v_3\rangle$. Then:

Dot Product
$$\boldsymbol{u}.\boldsymbol{v} = u_1v_1 + u_2v_2 + u_3v_3$$

Examples

A. $\langle -2, 3, 1\rangle.\langle 2, -1, -1\rangle$. B. $\boldsymbol{i}.\boldsymbol{j}$, C. $(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}\ ).\ (3\boldsymbol{i} + \boldsymbol{k}\ )$.

Geometric Interpretation Using the law of cosines, we get:

$$|\boldsymbol{u} - \boldsymbol{v}|^2 = |\boldsymbol{u}|^2 + |\boldsymbol{v}|^2 - 2\ |\boldsymbol{u}||\boldsymbol{v}|\cos\theta.$$

Plugging in coordinates gives:

Geometric Form of Dot Product
$$\boldsymbol{u}.\boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}|\cos\theta$$

Solving also gives

Angle between two Vectors
$$\cos\theta = \frac{\boldsymbol{u}.\boldsymbol{v}}{|\boldsymbol{u}|\,|\boldsymbol{v}|}$$

Consequence: The two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal if and only if $\boldsymbol{u}.\boldsymbol{v} = 0 \longleftarrow$ <u>a test for orthogonality.</u>

Examples

A. Find the angle between the vectors $\langle -2,\ 3,\ 1\rangle$ and $\langle 2,\ -1,\ -1\rangle$.

B. Show that $\langle 2, 2, -1\rangle$ and $\langle 5, -4, 2\rangle$ are orthogonal.

7. The Cross Product

Let $\boldsymbol{u} = \langle u_1, \ u_2 \ , \ u_3\rangle$, $\boldsymbol{v} = \langle v_1, \ v_2 \ , \ v_3\rangle$. Then:

Cross Product
$\boldsymbol{u} \times \boldsymbol{v} = \langle u_2v_3 - u_3v_2 \ , \ u_3v_1 - u_1v_3 \ , \ u_1v_2 - u_2v_1\rangle$

In class, we set it up as a determinant.

Examples

A. $\langle 1, -1, 2\rangle \times \langle 0, 2, 3\rangle$

Graphing Calculator Program for Cross Product In class

Properties

1. $\boldsymbol{u} \times \boldsymbol{v}$ is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$.
   (Prove by taking $\boldsymbol{u}.(\boldsymbol{u} \times \boldsymbol{v})$ and $\boldsymbol{v}.(\boldsymbol{u} \times \boldsymbol{v})$.)
2. $|\boldsymbol{u} \times \boldsymbol{v}| = |\boldsymbol{u}||\boldsymbol{v}| \sin\theta$.
   (Prove by calculating $|\boldsymbol{u} \times \boldsymbol{v}|^2 = |\boldsymbol{u}|^2|\boldsymbol{v}|^2 - (\boldsymbol{u}.\boldsymbol{v})^2 = |\boldsymbol{u}|^2|\boldsymbol{v}|^2(1-\sin 2\theta)$.)
3. $\boldsymbol{u} \times \boldsymbol{v} = 0$ if and only if the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel.
4. $|\boldsymbol{u} \times \boldsymbol{v}|$ = area of parallelogram with adjacent sides $\boldsymbol{u}$ and $\boldsymbol{v}$.
5. $\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u}$.
6. $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$; $\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$; $\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$.
7. Volume of the parallelepiped spanned by $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ is $|(\boldsymbol{u} \times \boldsymbol{v}).\boldsymbol{w}|$.

Examples

A. Find a vector perpendicular to both $\langle 1, -1, 2\rangle$ and $\langle 0, 2, 3\rangle$.

B. Find the volume of the parallelepiped spanned by $\langle 1, -1, 2\rangle$, $\langle 1, 1, 1\rangle$ and $\langle 0, 2, 3\rangle$.

Additional Properties:

1. $(c\boldsymbol{u}) \times \boldsymbol{v} = c(\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{u} \times (c\boldsymbol{v})$.
2. $\boldsymbol{w} \times (\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{w} \times \boldsymbol{u} + \boldsymbol{w} \times \boldsymbol{v}$, and others. (See text.)

Vector and Scalar Fields

Definitions 13.1 A **scalar** field is a function ϕ which assigns to each point in $\mathbb{R}^3$ a scalar. In other words, it is just a real-valued function of three variables. A **vector field in** $\mathbb{R}^3$ is a function F that assigns to each point (x, y, z) in $\mathbb{R}^3$ a vector $\boldsymbol{F}(x, y, z)$.

Examples 13.2

Scalar Fields:

A. $\phi(x,y,z) = x^2 + yz$.

B. $\phi(x, y, z)$ = temperature at (x, y, z)

C. $\phi(x, y, z)$ = density of material at (x, y, z)

Vector Fields:

A. $\boldsymbol{F}(x, y, z) = \langle x, y, 0\rangle$

B. $\boldsymbol{F}(x, y, z) = \langle xy, y, xz\rangle$

C. $\boldsymbol{F}(x, y, z) = \langle x, y, z\rangle/|\langle x, y, z\rangle|$, or $\boldsymbol{F}(\boldsymbol{r}) = \dfrac{\boldsymbol{r}}{|\boldsymbol{r}|}$.

D. Gravitational force exerted by a point mass M at the origin on a point mass at $\boldsymbol{r}$:

$$\boldsymbol{F}(\boldsymbol{r}) = -\frac{GmM\boldsymbol{r}}{|\boldsymbol{r}|^3}$$

E. Grav. Field due to a point mass M at the origin:

$$\boldsymbol{E}(\boldsymbol{r}) = -\frac{GM\boldsymbol{r}}{|\boldsymbol{r}|^3}$$

F. Grav. Field due to a point mass at $\boldsymbol{r}_0$:

$$\boldsymbol{E}(\boldsymbol{r}) = -\frac{GM(\boldsymbol{r}-\boldsymbol{r}_0)}{|\boldsymbol{r}-\boldsymbol{r}_0|^3}$$

G. Electric Field due to a charge q at $\boldsymbol{r}_0$:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{4\pi\varepsilon_0 q(\boldsymbol{r}-\boldsymbol{r}_0)}{|\boldsymbol{r}-\boldsymbol{r}_0|^3}$$

H. Given a scalar field, we can get its **gradient**, $\langle\phi_x , \phi_y, \phi_z\rangle$, which is now a vector field.

I. Show that the gravitational field in (E) above is the gradient of

$$\phi(\boldsymbol{r}) = \frac{MG}{|\boldsymbol{r}|} ,$$

called the **gravitational potential.**

Exercise Set 13

p. 370 Every second odd number 1-33, also p. 376 1-21 every 2nd odd number, p. 383, same thing 1-37, p. 389 #1-29 odd

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14. Gradient, Divergence, and Curl

Definitions 14.1 Define the **differential vector operator ∇** by

$$\nabla = \langle\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle.$$

(a) If ϕ is a scalar field, define its **gradient** by

$$\boxed{\textbf{grad}\phi = \nabla\phi = \langle\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\rangle}$$

(b) If $\boldsymbol{F}$ is a vector field, define its **divergence** by

$$\boxed{\mathrm{div}\boldsymbol{F} = \nabla.\boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

(c) If $\boldsymbol{F}$ is a vector field, define its **curl** by

$$\boxed{\textbf{curl}\boldsymbol{F} = \nabla\times\boldsymbol{F} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}}$$

(d) If ϕ is a scalar field, define its **Laplacian** by

$$\boxed{\nabla^2\phi = \nabla.\nabla\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}}$$

Gradient

Example: See (H) above.

Facts About the Gradient

Recall that the partial derivative ϕ_x of the scalar field ϕ measures the rate at which ϕ is increasing in the x-direction. In other words, it is the rate of increase of ϕ per unit increase in the direction of x. The **directional derivative in the direction of the vector *u*** measures the rate of increase of ϕ per unit increase in the direction of $\boldsymbol{u}$, and is given by

Directional Derivative in Direction of Unit Vector *u*

$$D_u\phi = \nabla\phi.\boldsymbol{u}$$

Note $\boldsymbol{u}$ must be a unit vector for the formula to work. Also note that that $D_i\phi = \partial\phi/\partial x$, and similarly $Dj\phi = \partial\phi/\partial y$. In view of this, we sometimes write $D_u\phi$ as $\partial\phi/\partial u$.

Example

Find the directional derivative of $\phi(x, y, z) = x^3y^4$ in the direction of the vector $\boldsymbol{v} = \langle 2, 3, 4\rangle$ at the point (0, 1, 1).

Summary of Facts about the Gradient

1. $\nabla\phi$ is normal to the level surfaces $\phi = const$.

2. $\nabla\phi$ is the direction in which ϕ increases most rapidly.

3. $|\nabla\phi|$ is the magnitude of this most rapid rate of increase.

4. $\nabla\phi$ is normal to the tangent plane of the surface $\phi(x, y, z) = \text{const}$.

Examples

A. The temperature of a region in space is given by $T(x, y, z) = 1/(x^2 + (y-3)^2 + 2z^2)$. The heat-seeking missile Triple-Six X is situated at (2, 1, 2) and always travels in the direction of fastest increase of temperature. In what direction will it travel? Assuming it is moving at 1 unit per second in that direction, how fast, in degrees/second, is the temperature increasing?

B. Verify that the gravitational potential $\phi(\boldsymbol{r}) = \dfrac{MG}{|\boldsymbol{r}-\boldsymbol{r}_0|}$ satisfies $\nabla^2\phi = 0$— see homework. (This is Laplace's equation—see the discussion of the steady state heat equation as an example.)

Meaning of the Divergence—an Example

Motion of a compressible fluid (This covers gases and "vapors.") We also assume no presence of sources and sinks. Now the motion of the fluid is described by its velocity vector field $\mathbf{v} = [v_1, v_2, v_3]$.

Consider a little parallelepiped W in the fluid with dimensions Δx, Δy and Δz, with one corner situated at the point (x,y,z). We calculate the total mass leaving W per unit time. (This is called the flux across the boundary of W.) The way to calculate it is to add up the mass/unit time leaving each face of W. For example, the mass/unit time leaving the faces $\Delta y\Delta z$ has a contribution only by v_1, since the other components are parallel to it. Since W is assumed small, we assume that v_1 is constant over each of these faces. For the left face, we evaluate v_1 at (x,y,z) and calculate

$$\begin{aligned}\text{mass/unit time entering } W &= \text{density}\times\text{volume/unit time}\\ &= \rho\Delta y\Delta z \times (x\text{-distance})/\text{unit time}\\ &= \rho v_1(x,y,z)\ \Delta y\Delta z.\end{aligned}$$

On the opposite face, the mass/unit time leaving W is similarly seen to be $\rho v_1(x+\Delta x,y,z)\Delta y\Delta z$. Thus the net mass leaving per unit time is

$$\rho v_1(x+\Delta x,y,z)\Delta y\Delta z - \rho v_1(x,y,z)\ \Delta y\Delta z$$

$$= [\ \rho\ v_1(x+\Delta x,y,z)\ -\ \rho v_1(x,y,z)\ ]\ \Delta y\Delta z$$
$$= \Delta(\rho v_1)\ \Delta y\Delta z$$
$$= \frac{\Delta(\rho v_1)}{\Delta x}\ \Delta V.$$

Summing up over the other faces, the total flux (loss of mass) becomes

$$\left(\frac{\Delta(\rho v_1)}{\Delta x} + \frac{\Delta(\rho v_2)}{\Delta y} + \frac{\Delta(\rho v_3)}{\Delta z}\right) \Delta V$$

On the other hand, the loss of mass/unit time can also be viewed as resulting from the change of density ρ inside W, and an easy related rate consideration shows that

$$-\ \frac{dM}{dt}\ =\ -\Delta V\frac{\partial\rho}{\partial t}$$

Equating them, and letting $\Delta x,\ \Delta y$ and $\Delta z \longrightarrow 0$ gives

$$\text{div}(\rho \boldsymbol{v})\ =\ -\partial\rho/\partial t,$$

the **continuity equation for compressible fluid flow**. Other things:

1. For steady flow (time-independent), RHS = 0, so div $(\rho \boldsymbol{v})\ =\ 0$.

2. For incompressibility, ρ = const., so also div $\boldsymbol{v} = 0$.

Note div $(\rho \boldsymbol{v})$ is the total mass/unit time/unit volume leaving the point where it is evaluated.

Examples Involving the Curl

A. Find $\boldsymbol{\nabla}\times\boldsymbol{v}$ if $\boldsymbol{v} = [yz,\ 3zx,\ z]$.

B. Rotation of a rigid body: The velocity field is given by $\boldsymbol{v}\ =\ \boldsymbol{\omega}\times\boldsymbol{r},$ where $r\ =\ [x,y,z]$. In the simple case when $\boldsymbol{\omega}\ =\ \omega\boldsymbol{k},$ then $v\ =\ \omega\times\boldsymbol{r}\ =\ [-\omega y,\ \omega x,\ 0]$, and $\boldsymbol{curl}\ \boldsymbol{v}\ =\ 2\boldsymbol{\omega}$. Thus curl is describing the rotational motion.

3. By direct computation, **curl**(grad ϕ) = 0 for every scalar field ˙.

4. curl v = 0 if v is a gravitational field. (Why?)

Exercise Set 14

p. 409 every second odd number 1-35,
p. 413 #1, 3, 5, 7, 11, 13(a), 13(b), 15, 17, 19
p. 416 #3, 5, 7, 9, 13, 15, 17

Reading Assignment: **Review of Double Integrals**

Review double integrals of the form

$$\iint_R f(x,y)\ dx\ dy,$$

where R is a region in the x/y-plane. A brief review is given in the book pp. 478–482

Hand In (Value = 5)

Verify that the gravitational potential $\phi(\boldsymbol{r})\ =\ \dfrac{MG}{|\boldsymbol{r}-\boldsymbol{r}_0|}$ satisfies $\boldsymbol{\nabla}^2\phi\ =\ 0$.

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15 Path Integrals

Paramaterized Paths

Definitions 15.1 A **curve** C may be represented by a vector function $\boldsymbol{r}: \mathbb{R} \longrightarrow \mathbb{R}^3$, so that $\boldsymbol{r}(t) = \langle x(t),\ y(t),\ z(t)\rangle$. This is called a **parametric representation of C** with **parameter t.** The curve is **smooth** if $x(t)$, $y(t)$ and $z(t)$ are differentiable functions of t.

Sometimes, we restrict the domain in order to get an arc of a curve, and take, for example, $\mathbf{r}: [a,b] \longrightarrow \mathbb{R}^3$.

Examples 15.2
(a) Straight line through the points $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{r}(t) = t\boldsymbol{b} + (1-t)\boldsymbol{a}.$$

(b) Straight line segment from $\boldsymbol{a}$ to $\boldsymbol{b}$. Here, $\boldsymbol{r}$: $[0,1] \longrightarrow \mathbb{R}^3$ with the same formula as above.
(c) Ellipse, circle: $\boldsymbol{r}$: $[0,2\pi] \longrightarrow \mathbb{R}^3$ given by $\boldsymbol{r}(t) = (a \cos \omega t,\ b \sin \omega t,\ 0)$
(d) Elliptical helix: $\boldsymbol{r}$: $[0,\mp) \longrightarrow \mathbb{R}^3$ given by $\boldsymbol{r}(t) = (a \cos \omega t,\ b \sin \omega t,\ vt)$ − (right-handed).
(r) Spiral: $\boldsymbol{r}$: $[0,+\infty) \longrightarrow \mathbb{R}^3$ given by $\boldsymbol{r}(t) = (\frac{1}{t}\cos t,\ \frac{1}{t}\sin\ t,\ vt)$.
(c) Parametrize the parabola $y = x^2$ in two different ways.

We have seen that the same curve may be given different parametrizations. Eg., we can replace t by $s = 2t$ in any of the above paramaterizations and come up with the same curve. This is saying that the parametric equations contain more information than just the shape of the curve - they also tell you how fast and in what direction the position vector is moving as the parameter changes.

Definition 15.3 Given a curve C and a vector field $\boldsymbol{F} = \langle F_1,\ F_2,\ F_3\rangle$, we define the **path integral of $\boldsymbol{F}$ along C** by

$$\int_C \boldsymbol{F}.d\boldsymbol{r} = \int_C F_1(x,y,z)\ dx + \ F_2(x,y,z)\ dy + F_3(x,y,z)dz$$

$$= \int_a^b [F_1(x(t),y(t),z(t))\cdot x'(t) + F_2(x(t),y(t),z(t))\cdot\partial y'(t) + F_3(x(t),y(t),z(t))\cdot z'(t)]\ dt$$

Similarly, we define other variants of this as follows:

$$\int_C f(x,\ y)\ dx = \int_a^b f(x(t),y(t))\cdot x'(t)\ dt\ ,$$

$$\int_C f(x,\ y)\ dy = \int_a^b f(x(t),y(t))\cdot y'(t)\ dt\ , \text{etc.}$$

Notes

1. If $\boldsymbol{F}$ is a force field acting on a particle, then $\int_C \boldsymbol{F}.d\boldsymbol{r}$ is the **work done by the force $\boldsymbol{F}$** in moving the particle along C.

2. One can show by using a change of variables that this integral is independent of the choice of paramaterization, (see book, p. 506), provided the parameterizations give the same direction along the curve. Reversing the direction of the curve reverses the sign of the path integral.

Examples 15.4

A. Evaluate $\int_C y^2 dx + x \; dy$, where C is the line segment from $(-5, -3)$ to $(0, 2)$.

B. Evaluate the same integral where C is the arc of the parabola $x = 4-y^2$ from $(-5, -3)$ to $(0, 2)$.

C. Evaluate $\int_C ydx + z \; dy + x \; dz$ where C consists of the line segments from (2, 0, 0) to (6, 6, 6) followed by the vertical segment from (6, 6, 6) to (6, 6, 0).

D. Evaluate $\int_C \boldsymbol{F}.d\boldsymbol{r}$, where $\boldsymbol{F} = \langle xy, yz, zx\rangle$ and C is the curve $\langle t, t^2, t^3\rangle$; $0 \le t \le 1$.

E. Ditto for $\boldsymbol{F} = -\dfrac{GmM\boldsymbol{r}}{|\boldsymbol{r}|^3}$ and C the unit semicircle in the positive xy−plane center the origin.

Line Integrals Independent of the Path We say that the integral $\int_C \boldsymbol{F}.d\boldsymbol{r}$ is **independent of the path** if $\int_C \boldsymbol{F}.d\boldsymbol{r} = \int_D \boldsymbol{F}.d\boldsymbol{r}$, for any two piecewise smooth paths C and D with the same starting point and ending point. Just when the heck does this happen? Well, we have the following great big theorem:

Theorem 15.5 ***(Great Big)***
Let $\boldsymbol{F}$ be a vector field defined on the region R. Then the following are equivalent:

- **(a)** $\int_C \boldsymbol{F}.d\boldsymbol{r}$ is independent of the path;
- **(b)** There is a "potential" scalar field ϕ *with* $\boldsymbol{F} = \nabla\phi$;
- **(c)** If C is any *closed* path in R, then $\oint_C \boldsymbol{F}.d\boldsymbol{r} = 0$;

In the event that any (and hence all) of these conditions hold, we have:

$$\int_C \boldsymbol{F}.d\boldsymbol{r} = \phi(B) - \phi(A),$$

where A and B are respectively the beginning point and end point of C. Further, we refer to $\boldsymbol{F}$ *as* a **conservative vector field**. ❖

Examples 15.6

A. Gravitational fields; find $\int_C \boldsymbol{F}.d\boldsymbol{r}$ where C is any curve from (2, 2, 2) to (3, 4, 5) not passing through $\boldsymbol{r}_0$.

B. Show that the vector field $\langle 2xy,\ xy^3\rangle$ is conservative.

C. Show that the vector field $\langle y^2,\ 2xy + e^{3z},\ 3ye^{3z}\rangle$ is conservative.

Tests for Conservatism

These tests only work for fields defined on a *simply connected* region R:
1. Two-Dimensional Vector Fields: The field $\langle P(x,y),\ Q(x,y)\rangle$ is conservative if $P_y = Q_x$.
2. Three-Dimensional Vector Fields. The field $\boldsymbol{F}$ is conservative if $\nabla\times\boldsymbol{F} = \boldsymbol{0}$.

Examples Test the fields in Examples B and C above.

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Review of Double Integrals

Unfortunately, the student will now be required to review double integrals of the form

$$\iint_R f(x,y)\ dx\ dy,$$

where R is a region in the x/y-plane. A brief review is given in the book pp. 515-522. One thing we may discuss however, is the formula for change-of coordinates, (analogous with the change-of coordinates for ordinary integrals)

$$\iint_R f(x,y)\ dx\ dy = \iint_R f(x(u,\ v),y(u,\ v))\ \left|\frac{\partial(x,\ y)}{\partial(u,\ v)}\right|\ du\ dv,$$

Examples

(a) Polar coords; $x = r\cos\theta$, $y = r\sin\theta$.

(b) Also see Example 3 on p. 520.

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Theorem 15.7 (Green's Theorem)

Let R be any region in $\mathbb{R}^2$ whose boundary C consists of finitely many closed curves, If P and Q are continuously differentiable real-valued functions defined on R, then

$$\int_C P dx + Q\ dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA.$$

Notes:

1. Here, the path integral is taken in the direction leaving R always on the left.
2. In vector form, let $\boldsymbol{F} = \langle P, Q, 0\rangle$. then the theorem says that

$$\int_C \boldsymbol{F}.\, d\boldsymbol{r} = \iint_R (\nabla\times\boldsymbol{F}).\boldsymbol{n}\, dA,$$

where $\boldsymbol{n} = \boldsymbol{k}$ is a unit normal to the region R.

Sketch of Proof of Green's Theorem
First assume that R is a Type 1 region, and directly evaluate the integral

$$\iint_R \frac{\partial P}{\partial y}\, dA,$$

getting

$$\iint_R \frac{\partial P}{\partial y}\, dA = \int_C P dx$$

Then assume it is a Type 2 region to get the other half. For more general regions, piece together nice regions.

Examples 15.8

A. Evaluate $\int_C x^4 dx + xy\, dy$ where C is the triangle with vertices (0, 0), (1, 0) and (0, 1).
B. Show that the area of a region R is given by

$$A = \int_C x\ dy = -\int_C y\, dx = \frac{1}{2}\int_C x\, dy - y\, dx.$$

C. Find the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$.
D. **Polar Coords** Under the change of coordinates $x = r\cos\theta$, $y = r\sin\theta$, the formula

$$A = \frac{1}{2}\int_C x\, dy - y\, dx$$

becomes

$$A = \frac{1}{2}\int_C r^2\, d\theta$$

(which you may recall from Calc II)

E. Let $\boldsymbol{F}(x,y) = \langle -y, x\rangle/(x^2 + y^2)$. Show that

$$n(C) = \frac{1}{2\pi}\int_{\mathrm{C}} \boldsymbol{F}.d\boldsymbol{r}$$

counts the number of times the closed path C goes around the origin. (We call this the **winding number** of the path.)
F. Verification of the first test for independence of the path.

G. Evaluate the integral $\int_C \boldsymbol{F}.d\boldsymbol{r}$ where $\boldsymbol{F} = \langle y^2e^x, 2ye^x\rangle$ and C is the curve $(\sin t^2, \; t^2+1)$; $0 \le t \le r\pi$.

Exercise Set 15
p. 398 #1-29 every second odd number (eso)
p. 425 #1–9 odd
p. 432 #1-19 eso
p. 444 #1–19 eso

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16 Surfaces, Surface Integrals, and Stokes' Theorem

Recall that the parametric equations of a curve are $\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$. For a *surface*, we have:

$\boldsymbol{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$.

We get a grid by holding $u = const$ and then $v = const$. Filling in the grid gives a surface. Sometimes, we restrict (u, v) to lie in some region R of the uv-plane.

Examples

A.

$x = a\sin u \cos v$; $y = a\sin u \sin v$; $a\cos u$; $0 \le u \le \pi$; $0 \le v \le 2\pi$.

Sphere Radius a

B. $x = u+v$; $y = u-v$; $z = v$.

C. Paramaterize the graph of $f(x,y) = x^2 - y^2$.

D. Parameterize the graph of an arbitrary function of two variables.

E. Cylinder

F. Torus $\boldsymbol{r} = a\langle\cos u, \sin u, 0\rangle + b\langle\cos u, \sin u, 0\rangle\cos v + b\sin v\langle 0, 0, 1\rangle\sin v$, so that

$x = (a+b\cos v)\cos u$; $y = (a+b\cos v)\sin u$; $z = b\sin v$; $0 \le u, v \le 2\pi$

Torus Radii $a > b$

G. A general plane parallel to the vectors $\langle a, b, c\rangle$ and $\langle a', b', c'\rangle$ and passing through the point (x_0, y_0, z_0).

Summary of the Tree Types of Surfaces:

1. Explicit Surfaces. These are the graphs of functions of two variable and take the form

$S = \{(x, y, z) : z = f(x,y)\}$.

Example: The elliptic paraboloid $z = x^2 + 3y^2$ is the graph of $f(x,y) = x^2 + 3y^2$.

2. Implicit Surfaces, or Varieties These are the level surfaces of functions of three variables, and take the form

$S = \{(x, y, z) : F(x,y,z) = k\}$

Example: The ellipsoid $x^2 + 3y^2 + 2z^2 = 5$ is a level surface of $F(x,y,z) = x^2 + 3y^2 + 2z^2$.

3. Parametric Surfaces These have the form

$S = \{(x, y, z) : x = x(u,v);\ y = y(u,v);\ z = z(u,v);\ (u,v)\ ´\ R\}$.

Note: Every explicit surface can be described as an implicit one; if $f(x, y)$ is a function of two variables, let $F(x,y,z) = z - f(x,y)$. Then the graph of $f(x,y)$ consists of all (x, y, z) such that $z = f(x,y)$, or $F(x,y,z) = 0$. Thus it is a level surface of $F(x,y,z)$. Thus we shall preferably describe all surfaces either implicitly or parametrically.

Normal and Unit Normal
In class, we see that a normal vector is given by

$$\boldsymbol{N} = \boldsymbol{r_u} \times \boldsymbol{r_v}\ .$$

The magnitude of $(\boldsymbol{r_u} \times \boldsymbol{r_v}\ du\ dv)$ is the area of the obtained by increasing u by du and v by dv. Thus,

$$\boldsymbol{r}_u \times \boldsymbol{r}_v\ \ dudv = \boldsymbol{n}\ dS = d\boldsymbol{S},$$

where $\boldsymbol{n}$ is a unit normal vector to the surface, and dS is the area of the small increment of surface corresponding to the increment dA in R.

Example Surface form for a sphere is $r^2\sin\phi\ d\theta\ d\phi$

Surface Integrals
Definition 16.1 If S is a surface (represented parametrically) and $\boldsymbol{F}$ is a vector field, then the **surface integral of $\boldsymbol{F}$ over S** is given by

Surface Integral

$$\boxed{\iint_S \boldsymbol{F}.\ d\boldsymbol{S} = \iint_S \boldsymbol{F}.\ \boldsymbol{n}\ dS = \iint_R \boldsymbol{F}.(\boldsymbol{r}_u \times \boldsymbol{r}_v)\ dA.}$$

Notes
1. The surface integral measure the total **flux** of $\boldsymbol{F}$ across S. For example, if $\boldsymbol{F}$ = velocity of a fluid, then $\iint_S \boldsymbol{F}.\ d\boldsymbol{S}$ is the total amount of fluid crossing the surface S per unit time.
2. The convention is to make sure that $\boldsymbol{n}$ is *the outward normal.* This we must check for the paramaterization we are using. Some surfaces have no sense of "outward," eg. the Moebius Band, and application of the above formula to this will not give the flow across it, as there will be a discontinuity in the integrand. The surfaces we use will possess an outward sense and are called **oriented**.

Examples 16.2
A. Find the flux of the vector field $\langle z, y, x\rangle$ over the unit sphere center (0, 0, 0).
B. Same for $\boldsymbol{F} = \langle x^2, 0, 3y^2\rangle$; S = that portion of the plane $x+y+z = 1$ in the first octant.
C. Same for $\boldsymbol{F} = \langle y, x, z\rangle$ and S = the boundary of the solid region enclosed by $z = 1 - (x^2 + y^2)$ and $z = 0$.
D. The **surface area of a surface S** is given by

$$A = \iint_S dS = \iint_R |\boldsymbol{r}_u \times \boldsymbol{r}_v|\ dA\ .$$

E. Area of a sphere.
F. Area of a torus
G. A Special Case: Explicitly Defined Surface: S'pose that S is given explicitly as $z = f(x,y)$, where $(x,y) \in R$. Here, we take $x = u$, $y = v$, $z = f(u,v)$, so

$$\boldsymbol{r}_u = \langle 1, 0, f_u\rangle, \ \ \boldsymbol{r}_v = \langle 0, 1, f_v\rangle, \ \ \boldsymbol{r}_u \times \boldsymbol{r}_v = \langle -f_u, -f_v, 1\rangle,$$

whence

$$A = \iint_R |\boldsymbol{r}_u \times \boldsymbol{r}_v|\, dA = \iint_R \sqrt{f_u^2 + f_v^2 + 1}\ du\, dv$$

$$= \iint_R \sqrt{f_x^2 + f_y^2 + 1}\ dx\, dy.$$

Area of an Explicit Surface $z = f(x, y)$ **With** (x, y) **Constrained to Lie in the Region** R

$A = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\ dA$ where A is the given region of the $xy-$plane.
Note: $dA = dx\, dy$ here

H. Find the surface area of that portion of the paraboloid $z = x^2 + y^2$ above the unit circle.

I.. Find the moment of inertia of a spherical shell radius a about any axis through its center.

Stokes' Theorem

Recall the vector form of Green's theorem, which said that

$$\int_C \boldsymbol{F}.\, d\boldsymbol{r} = \iint_R (\nabla\times\boldsymbol{F}).\boldsymbol{n}\, dA$$

for a region R in the xy-plane. Stokes' theorem asserts that if S is any oriented piecewise smooth surface bounded by the piecewise smooth curve C, then

$$\int_C \boldsymbol{F}.\, d\boldsymbol{r} = \iint_S (\nabla\times\boldsymbol{F}).\boldsymbol{n}\, dS,$$

where the right-hand rule is applied to the direction of C. To prove Stokes' theorem. we first show it true for a small flat triangular region. If the region is parallel to the $x/y-$plane, this follows from Green's Theorem. If not, we can rotate the whole region and the vector field $\boldsymbol{F}$ to make it so, since dot products are preserved by rotations. Finally, to get the whole theorem, we use a subdivision argument and take a limit.

Examples

A. Use Stokes' theorem to evaluate $\int_C \boldsymbol{F}.\, d\boldsymbol{r}$ where $\boldsymbol{F}(x,y,z) = \langle -y^2, x, z^2\rangle$ and C is the curve of intersection of the plane $y+z = 2$ and the cylinder $x^2 + y^2 = 1$.

B. Use Stokes to evaluate $\iint_S (\nabla\times\boldsymbol{F}).d\boldsymbol{S}$ where $\boldsymbol{F} = \langle yz, xz, xy\rangle$ and S is that part of the sphere $x^2 + y^2 + z^2 = 4$ inside the cylinder $x^2 + y^2 = 1$ and above the $x/y-$plane.

Note If C is a tiny little circle centered at $\boldsymbol{r}_0$ and normal to $\boldsymbol{n}$, then we treat $\nabla\times\boldsymbol{F}$ as constant, equal to $\nabla\times\boldsymbol{F}(\boldsymbol{r}_0)$, so that Stokes' Theorem says that $\int_C \boldsymbol{F}.\, d\boldsymbol{r} \approx \nabla\times\boldsymbol{F}(\boldsymbol{r}_0).\boldsymbol{n}A$, where A is the area of the disc. Thus, $\nabla\times\boldsymbol{F}(\boldsymbol{r}_0).\boldsymbol{n} \approx \frac{1}{A}\int_C \boldsymbol{F}.d\boldsymbol{r}$, which is saying that $\nabla\times\boldsymbol{F}$ is measuring the circulation per unit area at each point. (If we choose $\boldsymbol{n}$ in the direction of $\nabla\times\boldsymbol{F}$ at that point, then the LHS is just $|\nabla\times\boldsymbol{F}|$ at that point.)

The Divergence Theorem

This says that

$$\iiint_G \nabla.\boldsymbol{F}\ \ dV = \iint_S \boldsymbol{F}\ .\ d\boldsymbol{S},$$

where S is an oriented piecewise smooth surface enclosing G. $\nabla.\boldsymbol{F}$ is called the **divergence** of $\boldsymbol{F}$, and sometimes written as div$\boldsymbol{F}$. We prove the theorem in class.

Examples

A. Find the flux of the vector field $\langle z, y, x\rangle$ out of the unit sphere center (0, 0, 0) radius 1.

B. Evaluate $\iint_S \boldsymbol{F}\ .\ d\boldsymbol{S}$, where $\boldsymbol{F}(x, y, z) = \langle xy,\ y^2+e^{xz^2},\ sin(xy)\rangle$ where S is the surface bounded by $z = 1-x^2$ and the planes $z=0$, $y=0$ and $y+z = 2$.

Deriving the Heat Equation (p. 551-552)

Exercise Set 16

p. 448 #1-25 eso
p. 456 #1-19 eso
p. 463 #1-19 eso
p. 473 #1–13 eso
