3.6 Determinants

We said in Section 3.3 that a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if its *determinant*, $ad - bc$, is nonzero, and we saw the determinant used in the formula for the inverse of a 2×2 matrix. In this section we see how to compute the determinant of $n \times n$ matrices for arbitrary n and also, in the case of 2×2 and 3×3 matrices, how to interpret it geometrically. In the next section we will see one of its important applications: we can use it to write down explicit formulas for solutions of systems of linear equations.

Determinant of an $n \times n$ Matrix

Although it is possible to write down a formula for the determinant of an $n \times n$ matrix in terms of its entries, this formula is rarely actually used to *calculate* determinants. From the point of view of calculation, it is better to specify the determinant of an $n \times n$ matrix *recursively*—we state how to find the determinant of a larger matrix in terms of the determinants of smaller matrices.

If A is a square matrix, we will write its determinant as $\det(A)$. We already know from Section 3.3 how to calculate the determinant of a 2×2 matrix: If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$\det(A) = ad - bc.$$

The determinant of a 1×1 matrix is even simpler: If $A = [a]$, then

$$\det(A) = a.$$

Before generalizing to $n \times n$ matrices, we introduce a new term:

The Minor Matrix and Minor of an Entry

If A is an $n \times n$ matrix with $n \geq 2$ and a_{ij} is one of its entries, the associated **minor matrix** m_{ij} is the $(n-1)\times(n-1)$ matrix obtained by deleting both the row and the column passing through a_{ij}. The *determinant* of the minor matrix m_{ij} is called the **minor** M_{ij}**.** Thus,

m_{ij} = Minor matrix (delete row and column through a_{ij})
M_{ij} = Minor = $\det(m_{ij})$

Quick Examples

1. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$, then

$m_{11} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} = [-2]$ Delete the row and column through a_{11}

$M_{11} = \det(m_{11}) = \det([-2]) = -2$ The determinant of a 1×1 matrix is its only entry.

$m_{12} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} = [2]$ Delete the row and column through a_{12}

$M_{12} = \det(m_{12}) = \det([2]) = 2$

$m_{21} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} = [-1]$ Delete the row and column through a_{21}

$M_{21} = \det(m_{21}) = \det([-1]) = -1$

$m_{22} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} = [1]$ Delete the row and column through a_{22}

$M_{22} = \det(m_{22}) = \det([1]) = 1$

2. If $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 7 & 5 & 1 \end{bmatrix}$, then

$m_{12} = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 7 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 7 & 1 \end{bmatrix}$ Delete the row and column through a_{12}

$M_{12} = \det(m_{12}) = \det\begin{bmatrix} 2 & 0 \\ 7 & 1 \end{bmatrix} = (2)(1) - (0)(7) = 2$

$m_{31} = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 7 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix}$ Delete the row and column through a_{31}

$M_{31} = \det(m_{31}) = \det\begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix} = (1)(0) - (-1)(-2) = -2$

We now give the promised recursive definition of the determinant of a matrix. Since we know how to compute the determinants of 1×1 and 2×2 matrices, we start with the determinants of 3×3 matrices.

Computing the Determinant of a Square Matrix

The **determinant** of the $n \times n$ matrix A**,** written $\det(A)$ or sometimes $|A|$, is an associated real number computed for 1×1 and 2×2 matrices as above, and for larger matrices as follows:

3×3 Matrix

The determinant of the 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is given by

$$\det(A) = a_{11} \times M_{11} - a_{12} \times M_{12} + a_{13} \times M_{13}$$

(The formula involves computing 2×2 minors.)

Quick Example

Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 7 & 5 & 1 \end{bmatrix}$. Then

$$\begin{aligned} \det(A) &= a_{11} \times M_{11} - a_{12} \times M_{12} + a_{13} \times M_{13} \\ &= 3\times\det\begin{bmatrix} -2 & 0 \\ 5 & 1 \end{bmatrix} - 1\times\det\begin{bmatrix} 2 & 0 \\ 7 & 1 \end{bmatrix} + (-1)\times\det\begin{bmatrix} 2 & -2 \\ 7 & 5 \end{bmatrix} \\ &= 3\times(-2) - 1\times2 + (-1)\times24 = -32 \end{aligned}$$

4×4 Matrix

The determinant of the 4×4 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$ is given by

$$\det(A) = a_{11} \times M_{11} - a_{12} \times M_{12} + a_{13} \times M_{13} - a_{14} \times M_{14}$$

Notice the alternating pattern in the signs.

(The formula involves computing 3×3 minors.)

Quick Example

Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ -1 & 2 & 2 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$. Then

$$\det(A) = a_{11} \times M_{11} - a_{12} \times M_{12} + a_{13} \times M_{13} - a_{14} \times M_{14}$$

$$= 1 \times \det\begin{bmatrix} 4 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 3 & 4 \end{bmatrix} - 0 \times \det\begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 3 & 4 \end{bmatrix}$$

$$+ 0 \times \det\begin{bmatrix} 2 & 4 & 0 \\ -1 & 2 & 0 \\ 1 & 2 & 4 \end{bmatrix} - 0 \times \det\begin{bmatrix} 2 & 4 & 0 \\ -1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= 1\times4\times2\times4 = 32$$

Notice that the determinant of a lower triangular matrix like this (no entries above the main diagonal) is just the product of the entries on the main diagonal.

$n\times n$ Matrix

In general, the determinant of an $n\times n$ matrix is given by the following formula with alternating signs:

$$\det(A) = a_{11} \times M_{11} - a_{12} \times M_{12} + a_{13} \times M_{13} - \cdots \pm a_{1n} \times M_{1n}$$

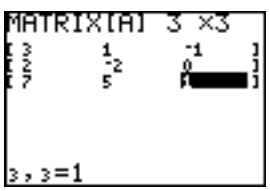
The formula involves computing $(n-1)\times(n-1)$ minors.

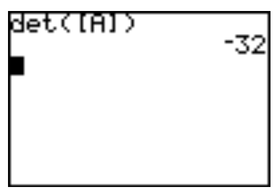
Technology:

Computing Determinants with the TI-83/84 Plus

On a TI-83/84, you can find the inverse of the square matrix [A] by entering

`det([A])` ENTER   det is found in the MATRX MATH menu





Computing Determinants with Excel

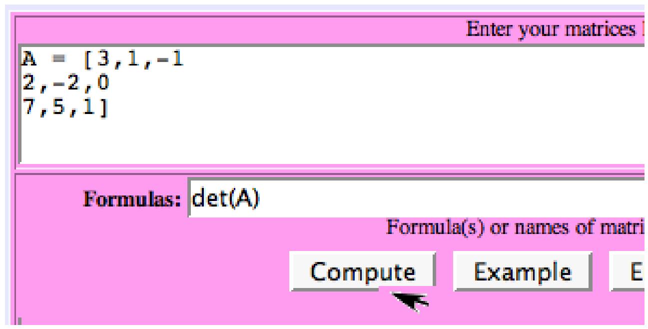
The formula `MDETERM` can be used to compute the determinant of any square matrix.

In the following worksheet, the determinant of $\begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 7 & 5 & 1 \end{bmatrix}$ is computed in cell E3 by entering the formula shown:

	A	B	C	D	E	F	G
1	3	1	-1				
2	2	-2	0				
3	7	5	1		=MDETERM(A1:C3)		
4							

Computing Determinants with the Online Matrix Algebra Tool

On the Web site, follow On-Line Utilities → Matrix Algebra Tool. There, enter your matrix A and the formula `det(A)` as shown, and press "Compute". The figure shows how one would compute the determinant of $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 7 & 5 & 1 \end{bmatrix}$:



Question Now we know how to compute the determinant, but what is it good for?

Answer Determinants give us a method to compute volumes, to determine whether a square matrix is singular, and to compute the inverse of a nonsingular matrix. In the next section we will see how they give us explicit solutions for systems of linear equations. There are also numerous theoretical applications that go beyond the scope of this book.

Computing Areas and Volumes

Consider the parallelogram shown on the left in Figure 1. Notice that its shape and size are completely determined by the coordinates of the two points (a, b) and (c, d)—once we know these points, we can draw in the rest of the parallelogram, as shown on the right.

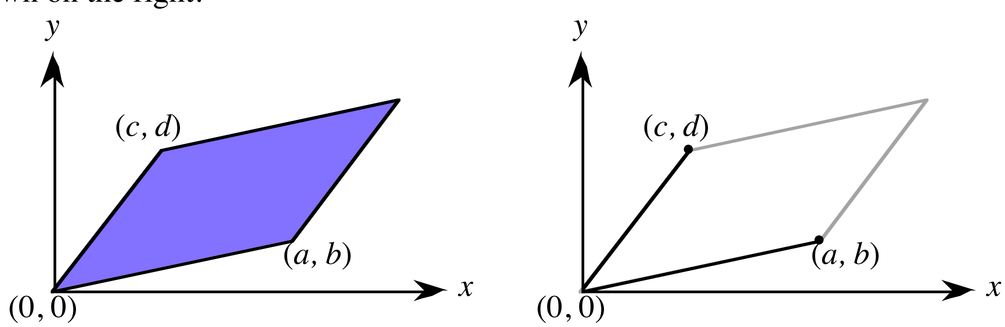


Figure 1

It follows that the *area* of this parallelogram is also determined by the four numbers $a, b, c,$ and d, and, in fact, the area is the absolute value of the following determinant:

$$\text{Area of parallelogram} = \left|\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| = |ad - bc|$$

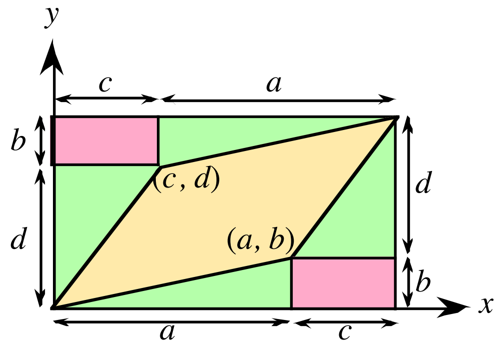


Figure 2

Question Why?

Answer Figure 2 shows the parallelogram inside a rectangle. The area of the rectangle is

$$(a+c)(b+d) = ab + cb + ad + cd$$

To obtain the area of the parallelogram we subtract the combined area of the four (green) triangles and two (pink) rectangles, which is

$$2(\tfrac{1}{2}ab) + 2(\tfrac{1}{2}cd) + 2bc = ab + cd + 2bc$$

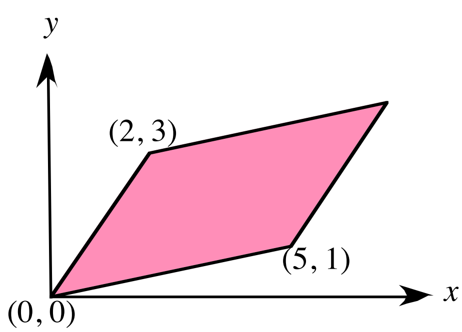
So, the area of the parallelogram is

$$ab + cb + ad + cd - (ab + cd + 2bc)$$
$$= ad - bc.$$

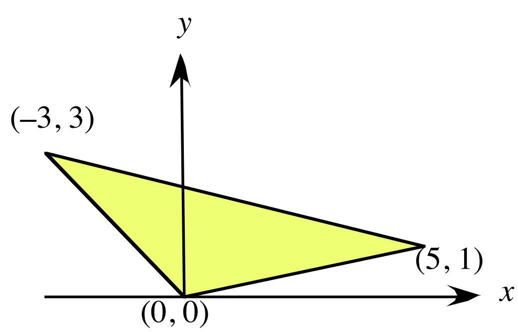
Example 1 Computing Areas

Use determinants to compute the areas of the following regions:

a.



b.



Solution

a. We are given a parallelogram with $(a,b) = (5, 1)$ and $(c, d) = (2, 3)$. (You could also reverse the choice by taking $(a, b) = (2, 3)$ and $(c, d) = (5, 1)$—see *Before we go on* below). Therefore,

$$\text{Area} = \left|\det\begin{bmatrix} 5 & 1 \\ 2 & 3 \end{bmatrix}\right| = |(5)(3) - (1)(2)| = |13| = 13 \text{ square units}$$

b. Although the figure is not a parallelogram, it can be thought of as *half* a parallelogram (Figure 3).

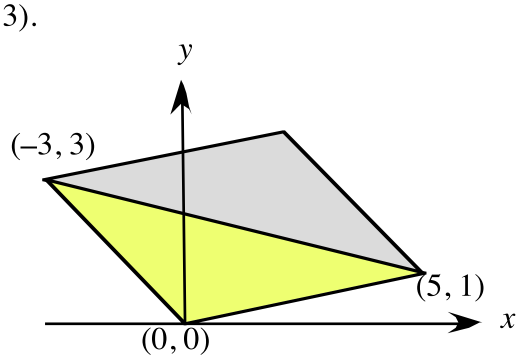


Figure 3

The area of the complete parallelogram is

$$\text{Area of parallelogram} = \left| \det \begin{bmatrix} 5 & 1 \\ -3 & 3 \end{bmatrix} \right| = |18| = 18 \text{ square units.}$$

Therefore, the area of the original triangle is half of that:

$$\textit{Area of triangle} = \frac{1}{2}18 = 9 \text{ square units.}$$

Before we go on...

Question In Example 1, which point do I take as (a, b) and which point do I take as (c, d)?

Answer It makes no difference: If we reverse our choice, we get

$$\text{Area} = \left| \det \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \right| = |(2)(1) - (3)(5)| = |-13| = 13 \text{ square units}$$

This is always true: Changing the order of the rows does not affect the absolute value of the determinant, only its sign.

Parallelepipeds are three-dimensional versions of parallelograms: You can form one by taking two identical parallelograms that are parallel to each other, and then joining corresponding corners (Figure 4).

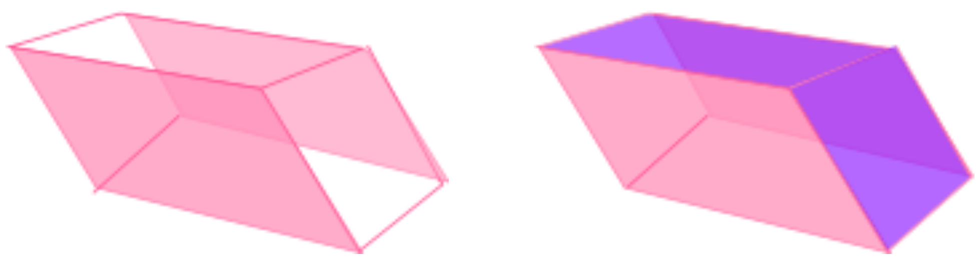


Figure 4

To specify a parallelepiped, one of whose corners is at the origin, we use three points as shown in Figure 5.

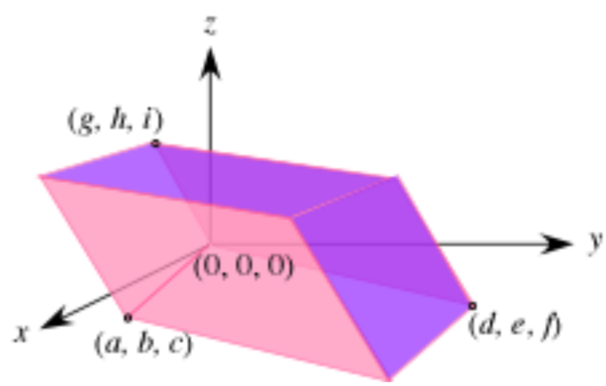


Figure 5

Notice that the three labeled points are on the ends of the three edges that contain the origin (0, 0, 0).

Question Why are we labeling points with 3 coordinates now?
Answer We need 3 coordinates to specify a point in 3-dimensional space. Look at the point (a, b, c). We get to this point from the origin (0, 0, 0) as follows: Move a units in the x-direction (toward you if a is positive, away from you if a is negative), then move b units in the y-direction (to the right if a is positive, to the left if a is negative), and finally move c units in the z-direction (straight up if a is positive, down if a is negative).

Just as the area of a parallelogram is given by the determinant of a 2×2 matrix, so the volume of a parallelepiped is given by the determinant of a 3×3 matrix:

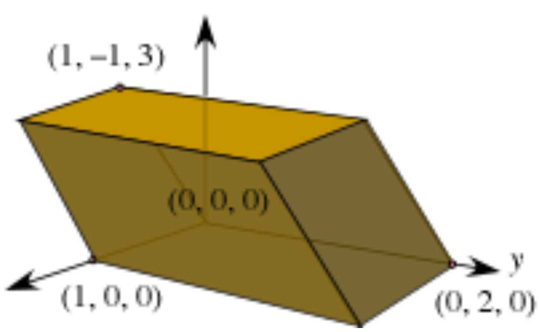
$$\text{Volume of parallelepiped} = \left| \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \right|$$

Question Does it matter in what order we write the rows?
Answer No. Changing the order of the rows in a matrix affects only the *sign* of its determinant, not the absolute value.

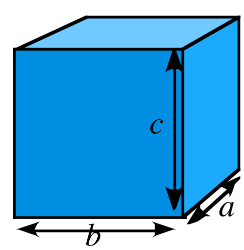
Example 2 Computing Volumes

Use determinants to compute the volumes of the following solids:

a.



b.



Rectangular Solid

Solution

a. Since we are given the coordinates of the three points on the ends of the three edges containing the origin, we can use the formula directly:

$$\text{Volume of parallelepiped} = \left| \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix} \right| = (1)(2)(3) = 6.$$

Notice that we arranged the three points in such a way as to obtain an upper triangular matrix, so that the determinant is just the product of the diagonal entries. As we said above, the determinant of a matrix does not change in absolute value if we rearrange the rows.

b. Since the figure is a rectangular solid, we know that its volume is

$$\text{depth} \times \text{width} \times \text{height} = abc$$

However, we were asked to compute it using determinants. To do this we first place one corner at the origin and find the coordinates of the three adjacent points. Figure 6 shows a way of doing that.

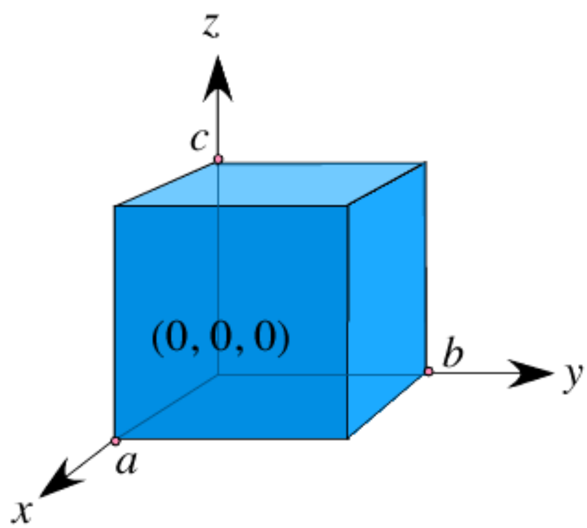


Figure 6

We have placed the far corner at the origin so that the point a has coordinates $(a, 0, 0)$, the point b has coordinates $(0, b, 0)$, and the point c has coordinates $(0, 0, c)$.

Question Why?
Answer To get to the point labeled a, just move a units in the x-direction, and no units in any of the other directions. Therefore, its coordinates are $(a, 0, 0)$. The coordinates of the other points are computed in a similar way.

We now have

$$\text{Volume of parallelepiped} = \left|\det\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\right| = abc,$$

as expected.

Some Shortcuts

There are quicker ways of calculating the determinants of matrices of certain types. The justifications of the following shortcuts are beyond the scope of this book, but can be found in standard linear algebra texts.

Shortcuts and Special Cases:

- The determinant of a triangular matrix (one in which either all the entries above the main diagonal are zero or all the entries below it are) is the product of the entries on the diagonal.

Quick Example: $\det\begin{bmatrix} 5 & 198 & 44 \\ 0 & 2 & -101 \\ 0 & 0 & 1 \end{bmatrix} = (5)(2)(1) = 10$ Check this by calculating minors.

- The determinant of a matrix is the same as the determinant of its transpose: $\det(A) = \det(A^T)$

Quick Example: $\det\begin{bmatrix} 0 & 30 & -1 \\ 1 & 200 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \det\begin{bmatrix} 0 & 1 & 0 \\ 30 & 200 & 0 \\ -1 & 4 & 1 \end{bmatrix} = -30$

- Switching two rows changes the sign of the determinant, but leaves its magnitude unchanged. (The same is true if we switch two columns.)

Quick Example: $\det\begin{bmatrix} -99 & 13 & 4 \\ 6 & 1 & 0 \\ 3 & 0 & 0 \end{bmatrix} = -\det\begin{bmatrix} 3 & 0 & 0 \\ 6 & 1 & 0 \\ -99 & 13 & 4 \end{bmatrix} = -12$ $R_1 \leftrightarrow R_3$

- If a matrix has a row or column of zeros, or if one row or column is a multiple of another, then its determinant is zero.

Quick Examples: $\det\begin{bmatrix} -99 & 0 & 4 \\ 0 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} = 0$ Second column is zero

$\det\begin{bmatrix} 3 & 0 & 0 \\ 6 & 1 & 0 \\ 12 & 2 & 0 \end{bmatrix} = 0$ R_3 is twice R_2

Example 3 Shortcuts and Special Cases

Compute the determinant of each of the following matrices.

a. $A = \begin{bmatrix} 3 & 0 & 0 \\ 6 & -1 & 0 \\ -99 & 0 & 4 \end{bmatrix}$ **b.** $B = \begin{bmatrix} 1 & 2 & -3 \\ -2 & -4 & 6 \\ 99 & 40 & 1 \end{bmatrix}$ **c.** $C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix}$

Solution

a. The matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 6 & -1 & 0 \\ -99 & 0 & 4 \end{bmatrix}$ is lower triangular (it has only zeros above the main diagonal). Therefore, its determinant is the product of the diagonal entries:

$$\det(A) = (3)(-1)(4) = -12.$$

b. Notice that in the matrix $B = \begin{bmatrix} 1 & 2 & -3 \\ -2 & -4 & 6 \\ 99 & 40 & 1 \end{bmatrix}$, Row 2 is (-2) times Row 1. Therefore its determinant is zero:

$$\det(B) = 0$$

c. We notice that the second row of $C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix}$ is almost all zero. It would therefore be easier to compute the determinant if we first switched Rows 1 and 2:

$$\det(C) = \det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix} = -\det\begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix} \quad \text{Rows 1 and 2 switched}$$

$$= -[a_{11} \times M_{11} - a_{12} \times M_{12} + a_{13} \times M_{13} - a_{14} \times M_{14}]$$

$$= -[(-1) \times M_{13}] \quad \text{All the other terms are zero.}$$

$$= \det\begin{bmatrix} 1 & 2 & 4 \\ 1 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= -\det\begin{bmatrix} 0 & 0 & 4 \\ 1 & 0 & 3 \\ 1 & 2 & 4 \end{bmatrix} \quad \text{Rows 1 and 3 switched}$$

$$= -4 \times \det\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = -4 \times 2 = -8$$

3.6 Exercises

Let $A = \begin{bmatrix} -1 & 0 & 5 \\ 3 & -1 & 5 \\ 2 & 0 & 1 \end{bmatrix}$. In Exercises 1–6, write the associated minor matrix and then compute the indicated minor.

1. M_{23} **2.** M_{32} **3.** M_{22}

4. M_{11} **5.** M_{12} **6.** M_{21}

In Exercises 7–16 compute the determinant of the given matrix directly (no shortcuts).

7. $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & -1 \\ 5 & 0 & 2 \end{bmatrix}$ **8.** $\begin{bmatrix} 2 & 1 & 0 \\ -2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

9. $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 3 \\ 2 & 0 & 0 \end{bmatrix}$ **10.** $\begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & 0 \\ 3 & 3 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 1 \\ 3 & 4 & 5 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 4 \\ 3 & -4 & 5 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix}$

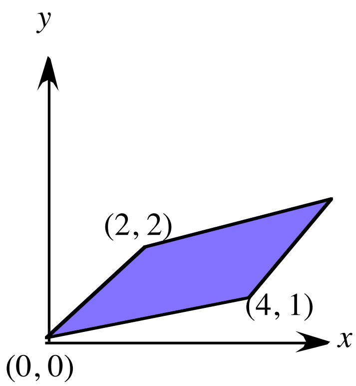
14. $\begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & 0 & -4 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ -2 & 3 & 1 & 0 \\ 1 & 1 & -3 & -1 \end{bmatrix}$

In Exercises 17–22, use a determinant to compute the area of the given region.

17.



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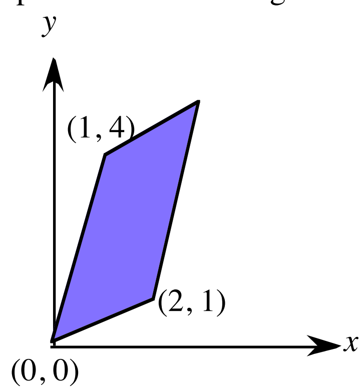

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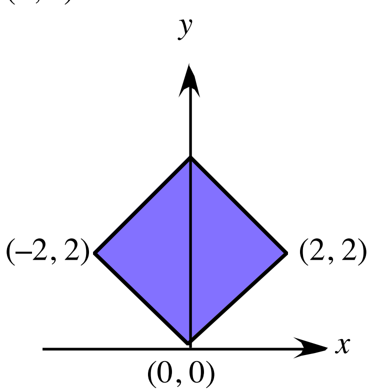

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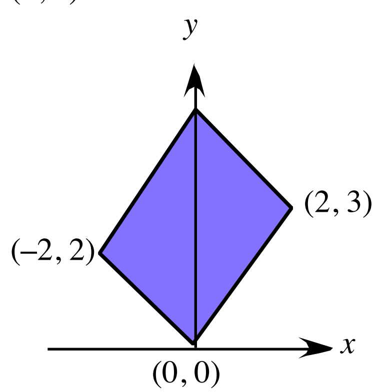

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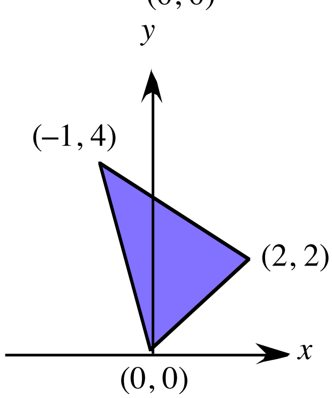

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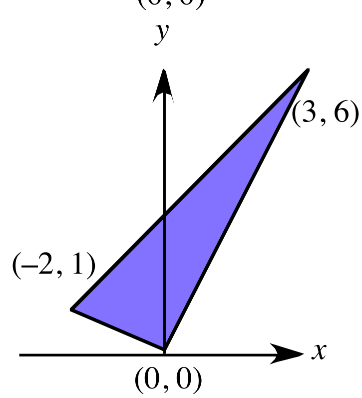
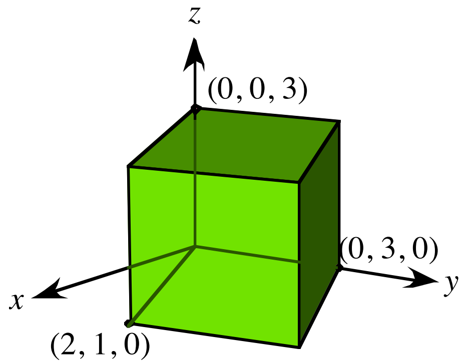


In Exercises 23–28, use a determinant to compute the volume of the given solid.

23.



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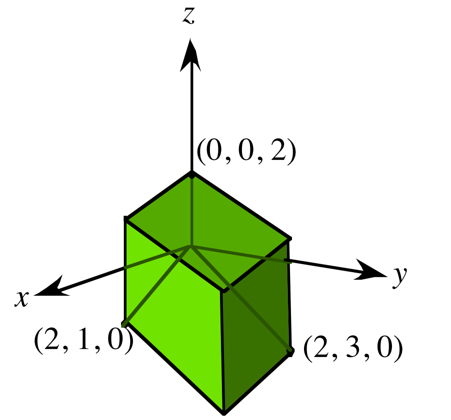

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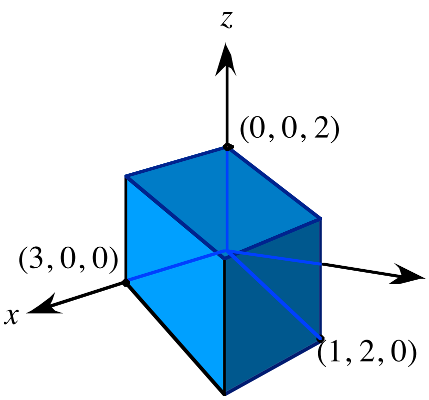
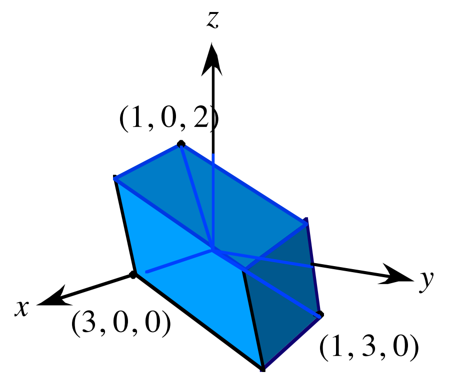
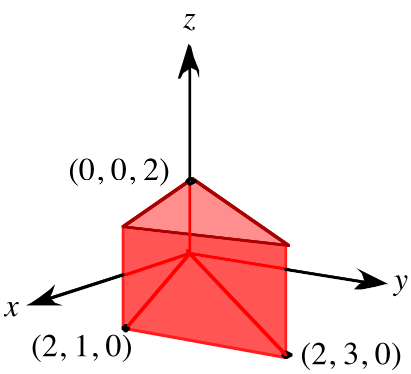


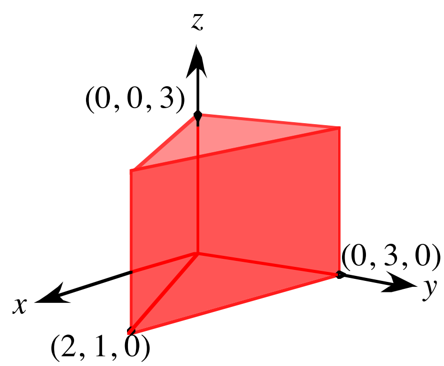
26.



27.



28.



In Exercises 29–44, use shortcuts to find the determinant of the given matrix.

29. $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

30. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

31. $\begin{bmatrix} -1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 0 & 4 \end{bmatrix}$

32. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

33. $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 3 \\ -2 & 4 & -6 \end{bmatrix}$

34. $\begin{bmatrix} 4 & -2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$

35. $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

36. $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & -2 & 0 \end{bmatrix}$

37. $\begin{bmatrix} 0 & -3 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix}$

38. $\begin{bmatrix} -3 & 3 & 2 \\ 1 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$

39. $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & 0 & -4 \end{bmatrix}$

40. $\begin{bmatrix} 4 & -1 & 2 & 3 \\ 3 & 4 & 0 & 0 \\ -2 & 3 & 1 & 0 \\ -8 & 2 & -4 & -6 \end{bmatrix}$

41. $\begin{bmatrix} 1 & 3 & -2 & 2 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 4 & -4 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

42. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 \end{bmatrix}$

43. $\begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

44. $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 5 & -3 & -1 \end{bmatrix}$

Communication and Reasoning Exercises

45. Multiple choice: If the $n \times n$ matrix B is obtained from A by switching two rows, then:

(A) $\det(B) = -\det(A)$
(B) $\det(B) = \det(A)$
(C) $\det(B) = 1/\det(A)$
(D) $\det(B) = 2\det(A)$

46. Multiple choice: If A is an $n \times n$ matrix all of whose entries are 1s, then:

(A) $\det(A) = 1$
(B) $\det(A) = 0$
(C) $\det(A) = n^2$
(D) $\det(A) = n$

47. Thinking of 3×3 matrices as volumes, explain why the determinant of a matrix is zero if two rows are identical.

48. Thinking of 3×3 matrices as volumes, decide what effect doubling all the entries in one row has on the magnitude of the determinant.

3.7 Using Determinants to Solve Systems: Cramer's Rule

As we claimed, determinants can be used to write down formulas for solutions of systems of linear equations. To see how, let us first take a look at a general system of two linear equations in two unknowns:

$$a_{11}x + a_{12}y = b_1$$

$$a_{21}x + a_{22}y = b_2$$

Matrix form $\overset{A}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}} \overset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \overset{B}{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}$

We can solve the system by the elimination method described in Section 2.1: To eliminate y, multiply the first equation by a_{22} and the second by a_{12} and subtract:

$$\begin{array}{r} a_{22}a_{11}x + a_{22}a_{12}y = a_{22}b_1 \\ a_{12}a_{21}x + a_{12}a_{22}y = a_{12}b_2 \\ \hline (a_{22}a_{11} - a_{12}a_{21})x = a_{22}b_1 - a_{12}b_2 \end{array}$$

so $$x = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$$ Assuming that $a_{11}a_{22} - a_{12}a_{21} \neq 0$

If we instead eliminate x by multiplying the first equation by a_{21} and the second by a_{11} and subtracting, we similarly obtain

$$y = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$ Again assuming that $a_{11}a_{22} - a_{12}a_{21} \neq 0$

The denominator in both cases, $a_{11}a_{22} - a_{12}a_{21}$, you might recognize as the determinant of the coefficient matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. The numerators are also determinants:

$$a_{22}b_1 - a_{12}b_2 = \det \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$$
Numerator of solution for x

$$a_{11}b_2 - a_{21}b_1 = \det \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$$
Numerator of solution for y

In the first, we have replaced the first column of the coefficient matrix A by the column B of right-hand sides, and in the second, we have replaced the second column of A by B. We can now write the solutions as follows:

Cramer's Rule for Solution of a System of 2 Linear Equations in 2 Unknowns

The system of two linear equations in two unknowns

$$\begin{aligned} a_{11}x + a_{12}y &= b_1 \\ a_{21}x + a_{22}y &= b_2 \end{aligned}$$

Matrix form
$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{B}{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}$$

has a unique solution if and only if $\det(A) = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21} \neq 0$, in which case the solution is given by

$$x = \frac{\det \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}}{\det(A)} \qquad y = \frac{\det \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}}{\det(A)}$$

Quick Examples

1. The system

$$\begin{aligned} x + 2y &= 3 \\ 3x + 4y &= 5 \end{aligned}$$

has $\det(A) = \det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = (1)(4) - (2)(3) = -2 \neq 0$. Therefore the system has the unique solution

$$x = \frac{\det \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}}{\det(A)} = \frac{\det \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}}{-2} = \frac{(3)(4)-(2)(5)}{-2} = \frac{2}{-2} = -1$$

$$y = \frac{\det \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}}{\det(A)} = \frac{\det \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}}{-2} = \frac{(1)(5)-(3)(3)}{-2} = \frac{-4}{-2} = 2$$

2. The system

$$\begin{aligned} x - y &= 3 \\ 2x - 2y &= 5 \end{aligned}$$

has $\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \det \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} = (1)(-2) - (-1)(2) = 0$

As the coefficient matrix is singular, Cramer's Rule does not apply (in fact the given system is inconsistent), and so we would need to analyze the system using the methods of Chapter 2.

Question What happens when the determinant of the coefficient matrix is zero?

Answer Notice first that in this case the Cramer's Rule formulas have zero in their denominators and hence make no sense. In general, the coefficient matrix of a system of n linear equations in n unknowns has determinant zero if and only if the system is inconsistent (there is no solution) or underdetermined (there are infinitely many solutions). In either case, we would need to analyze the system using a method like row-reduction discussed in Section 2.2.

One advantage of Cramer's Rule over row reduction is that the explicit formulas it gives allow us to write down the solution of a linear system even when the coefficients are parameters (algebraic variables) instead of numbers. (In such cases, attempting to solve the system by row-reduction might be extremely messy.) The next example illustrates the use of Cramer's Rule for solving such a system.

Example 1 Using Cramer's Rule with Parameters: Regression

We shall see in Chapter 15 <NOTE this is the chapter number for the combined book.> that the equations for the slope m and intercept b of the regression line associated with a set of data points are given by solving the system

$$\begin{aligned} m\Sigma(x^2) + b\Sigma x &= \Sigma xy \\ m\Sigma x + nb &= \Sigma y \end{aligned}$$

for m and b. Here, n is the number of data points, Σx is the sum of their x-coordinates, Σxy is the sum of the products xy, and $\Sigma(x^2)$ is the sum of the squares of the x-coordinates. What are m and b, and what condition is necessary to ensure a unique solution?

Solution

The determinant of the coefficient matrix is

$$\det(A) = \det\begin{bmatrix} \Sigma(x^2) & \Sigma x \\ \Sigma x & n \end{bmatrix} = n\Sigma(x^2) - (\Sigma x)^2$$

For a unique solution, we require that $n\Sigma(x^2) - (\Sigma x)^2 \neq 0$. (It can be shown that this condition holds whenever there is more than a single x-coordinate.) When this condition is satisfied, the unique solution is given by

$$m = \frac{\det\begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}}{\det(A)} = \frac{\det\begin{bmatrix} \Sigma xy & \Sigma x \\ \Sigma y & n \end{bmatrix}}{\det(A)} = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{n\Sigma(x^2) - (\Sigma x)^2}$$

$$b = \frac{\det\begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}}{\det(A)} = \frac{\det\begin{bmatrix} \Sigma(x^2) & \Sigma xy \\ \Sigma x & \Sigma y \end{bmatrix}}{\det(A)} = \frac{\Sigma(x^2)\Sigma y - (\Sigma xy)(\Sigma x)}{n\Sigma(x^2) - (\Sigma x)^2}$$

The method described above can be extended to systems of n linear equations in n unknowns. To see how to extend it, is useful to look first a general system of three equations in three unknowns:

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

As in the case of two equations in two unknowns, it is possible to solve this system by elimination: First eliminate z from the first two equations by multiplying the first by a_{23} and the second by a_{13} and subtracting. Then eliminate z from the second and third equations in a similar way (multiply the second by a_{33} and the third by a_{23} and subtract). This will leave us with two equations in x and y:

$$(a_{11}a_{23} - a_{21}a_{13})x + (a_{12}a_{23} - a_{13}a_{22})y = a_{23}\, b_1 - a_{13}\, b_2$$
$$(a_{21}a_{33} - a_{31}a_{23})x + (a_{22}a_{33} - a_{23}a_{32})y = a_{33}\, b_2 - a_{23}\, b_3$$

At this point we can calculate x and y as we did earlier: Eliminate y by multiplying each equation by the coefficient of y in the other and subtracting to obtain x, and similarly we can obtain y by eliminating x. To obtain z with this method, we would start all over again by first eliminating x and then y. If we actually went through these remaining steps we would find that the results can again be expressed in terms of determinants:

Cramer's Rule for Solution of System of 3 Linear Equations in 3 Unknowns

The system of 3 linear equations in 3 unknowns

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned} \qquad \text{Matrix form} \quad \overset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} \overset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} \overset{=}{=} \overset{B}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$$

has a unique solution if and only if $\det(A) = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \neq 0$, in which case the solution is given by

$$x = \frac{\det \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}}{\det(A)}, \quad y = \frac{\det \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}}{\det(A)}, \quad z = \frac{\det \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}}{\det(A)}$$

Notice again that the matrices in the numerators are obtained from the coefficient matrix A by replacing each column in turn by the column B of right-hand sides.

Cramer's Rule for Solution of a System of *n* Linear Equations in *n* Unknowns

If A is an $n \times n$ matrix, then the system of linear equations $AX = B$ has a unique solution if and only if $\det(A) \neq 0$, in which case the unique solution is given by

$$x_1 = \frac{\det \begin{bmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \dots & \dots & & \dots \\ b_n & a_{n2} & \dots & a_{nn} \end{bmatrix}}{\det(A)}$$

$$x_2 = \frac{\det \begin{bmatrix} a_{11} & b_1 & a_{13} & \dots & a_{1n} \\ a_{21} & b_2 & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & & \dots \\ a_{n1} & b_n & a_{n3} & \dots & a_{nn} \end{bmatrix}}{\det(A)}$$

$\dots$

$$x_n = \frac{\det \begin{bmatrix} a_{11} & \dots & a_{1(n-1)} & b_1 \\ a_{21} & \dots & a_{2(n-1)} & b_2 \\ \dots & & \dots & \dots \\ a_{n1} & \dots & a_{n(n-1)} & b_n \end{bmatrix}}{\det(A)}$$

Example 2 Using Cramer's Rule: 3 Equations in 3 Unknowns

Use Cramer's Rule to solve the system

$$\begin{aligned} 2x \quad\;\; + z &= 1 \\ 2x + y - z &= 1 \\ 3x + y - z &= 1 \end{aligned}$$

Solution

We first compute the determinant of the coefficient matrix:

$$\begin{aligned} \det(A) &= \det \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & -1 \end{bmatrix} \\ &= (2)\det \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} - (0)\det \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} + (1)\det \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \\ &= (2)(0) - (0)(1) + (1)(-1) = -1 \end{aligned}$$

Since the determinant is nonzero, the system has a unique solution. The unknowns are

$$x = \frac{\det \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}}{\det(A)}$$

$$= \frac{(1)\det\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} - (0)\det\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} + (1)\det\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}{-1}$$

$$= \frac{(1)(0) - (0)(0) + (1)(0)}{-1} = \frac{0}{-1} = 0$$

$$y = \frac{\det\begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & -1 \end{bmatrix}}{\det(A)}$$

$$= \frac{(2)\det\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} - (1)\det\begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} + (1)\det\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}}{-1}$$

$$= \frac{(2)(0) - (1)(1) + (1)(-1)}{-1} = \frac{-2}{-1} = 2$$

$$z = \frac{\det\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}}{\det(A)}$$

$$= \frac{(2)\det\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - (0)\det\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} + (1)\det\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}}{-1}$$

$$= \frac{(2)(0) - (0)(-1) + (1)(-1)}{-1} = \frac{-1}{-1} = 1$$

Thus, the unique solution is $(x, y, z) = (0, 2, 1)$.

In the next example we solve a system of four linear equations in four unknowns with the aid of a spreadsheet. From the general case for $n\times n$ systems, we can write down the solution of the 4×4 system

$$\begin{aligned}
a_{11}x + a_{12}y + a_{13}z + a_{14}t &= b_1 \\
a_{21}x + a_{22}y + a_{23}z + a_{24}t &= b_2 \\
a_{31}x + a_{32}y + a_{33}z + a_{34}t &= b_3 \\
a_{41}x + a_{42}y + a_{43}z + a_{44}t &= b_4
\end{aligned}$$

as

$$x = \frac{\det\begin{bmatrix} b_1 & a_{12} & a_{13} & a_{14} \\ b_2 & a_{22} & a_{23} & a_{24} \\ b_3 & a_{32} & a_{33} & a_{34} \\ b_4 & a_{42} & a_{43} & a_{44} \end{bmatrix}}{\det(A)}, \qquad y = \frac{\det\begin{bmatrix} a_{11} & b_1 & a_{13} & a_{14} \\ a_{21} & b_2 & a_{23} & a_{24} \\ a_{31} & b_3 & a_{33} & a_{34} \\ a_{41} & b_4 & a_{43} & a_{44} \end{bmatrix}}{\det(A)}$$

$$z = \frac{\det\begin{bmatrix} a_{11} & a_{12} & b_1 & a_{14} \\ a_{21} & a_{22} & b_2 & a_{24} \\ a_{31} & a_{32} & b_3 & a_{34} \\ a_{41} & a_{42} & b_4 & a_{44} \end{bmatrix}}{\det(A)} \qquad z = \frac{\det\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \\ a_{41} & a_{42} & a_{43} & b_4 \end{bmatrix}}{\det(A)}$$

$$\text{provided } \det(A) = \det\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \neq 0,$$

T *Example 3 Four Equations in Four Unknowns with Excel*

Use Cramer’s Rule to solve the system

$$\begin{aligned} x \quad\quad + z - t &= 1 \\ 2x - y - z \quad\quad &= 1 \\ x + y + z - t &= 2 \\ x + y + z + t &= 1 \end{aligned}$$

Solution

Doing this calculation by hand would be tedious. We show how to use Excel to help. First enter the coefficients and the right-hand sides in your spreadsheet:

	A	B	C	D	E
1	1	0	1	-1	1
2	2	-1	-1	0	1
3	1	1	1	-1	2
4	1	1	1	1	1

The formulas for the solution shown above require the determinants of four more matrices, each obtained from the original coefficient matrix (A1:D4) by changing a single column. We therefore make four copies of (A1:D4) (this takes seconds using copy-and-paste) and then paste the column (E1:E4) in the appropriate place of each (again using copy-and-paste):

	A	B	C	D	E
1	1	0	1	-1	1
2	2	-1	-1	0	1
3	1	1	1	-1	2
4	1	1	1	1	1
5					
6	1	0	1	-1	
7	1	-1	-1	0	
8	2	1	1	-1	
9	1	1	1	1	
10					
11	1	1	1	-1	
12	2	1	-1	0	
13	1	2	1	-1	
14	1	1	1	1	
15					
16	1	0	1	-1	
17	2	-1	1	0	
18	1	1	2	-1	
19	1	1	1	1	
20					
21	1	0	1	1	
22	2	-1	-1	1	
23	1	1	1	2	
24	1	1	1	1	

Next, we compute the determinant of the coefficient matrix in Cell A5 using the MDETERM function we saw on p. 4:

	A	B	C	D	E
1	1	0	1	-1	1
2	2	-1	-1	0	1
3	1	1	1	-1	2
4	1	1	1	1	1
5	=MDETERM(A1:D4)				
6					

↓ Control+Shift+Enter

	A	B	C	D	E
1	1	0	1	-1	1
2	2	-1	-1	0	1
3	1	1	1	-1	2
4	1	1	1	1	1
5	6				

Since the determinant is nonzero, the system has a unique solution. We next obtain the numerators of the solutions for x, y, z, and t by copying and pasting the formula of Cell A5 into cells A10, A15, A20, and A25:

	A	B	C	D	E
1	1	0	1	-1	1
2	2	-1	-1	0	1
3	1	1	1	-1	2
4	1	1	1	1	1
5	6				
6	1	0	1	-1	
7	1	-1	-1	0	
8	2	1	1	-1	
9	1	1	1	1	
10	5				
11	1	1	1	-1	
12	2	1	-1	0	
13	1	2	1	-1	
14	1	1	1	1	
15	6				
16	1	0	1	-1	
17	2	-1	1	0	
18	1	1	2	-1	
19	1	1	1	1	
20	-2				
21	1	0	1	1	
22	2	-1	-1	1	
23	1	1	1	2	
24	1	1	1	1	
25	-3				

Finally, the solution for x is computed in Cell B10 an then pasted into B15, B20, and B25. Note the use of the absolute reference to cell \$A\$5:

	A	B	C
1	1	0	1
2	2	-1	-1
3	1	1	1
4	1	1	1
5	6		
6	1	0	1
7	1	-1	-1
8	2	1	1
9	1	1	1
10	5	=A10/$A$5	
11	1	1	1
12	2	1	-1
13	1	2	1
14	1	1	1
15	6	=A15/$A$5	
16	1	0	1
17	2	-1	1
18	1	1	2
19	1	1	1
20	-2	=A20/$A$5	
21	1	0	1
22	2	-1	-1
23	1	1	1
24	1	1	1
25	-3	=A25/$A$5	

→

	A	B	C
1	1	0	1
2	2	-1	-1
3	1	1	1
4	1	1	1
5	6		
6	1	0	1
7	1	-1	-1
8	2	1	1
9	1	1	1
10	5	0.83333333	
11	1	1	1
12	2	1	-1
13	1	2	1
14	1	1	1
15	6	1	
16	1	0	1
17	2	-1	1
18	1	1	2
19	1	1	1
20	-2	-0.3333333	
21	1	0	1
22	2	-1	-1
23	1	1	1
24	1	1	1
25	-3	-0.5	

so we conclude that $(x, y, z, t) = (\frac{5}{6}, \frac{6}{6}, \frac{-2}{6}, \frac{-3}{6}) \approx (0.8333, 1, -0.3333, -0.5)$

3.7 Exercises

In Exercises 1–12 solve the given system of linear equations using Cramer's Rule.

1. $x + y = 4$
$x - y = 1$

2. $2x + y = 2$
$2x - 3y = 2$

3. $0.1x - 0.2y = 0$
$0.4x + 0.2y = 1.2$

4. $0.5x - 0.1y = 0.3$
$2.5x + 0.3y = 2.2$

5. $\frac{x}{3} + \frac{y}{2} = 0$
$\frac{x}{2} + y = -1$

6. $\frac{2x}{3} - \frac{y}{2} = \frac{1}{6}$
$\frac{x}{2} - \frac{y}{2} = -1$

7. $-x + 2y + z = 0$
$-x - y + 2z = 0$
$2x - z = 7$

8. $x + 2y = 4$
$y - z = 0$
$x + 3y - 2z = 5$

9. $-x - 4y + 2z = 4$
$x + 2y - z = 3$
$x + y - z = 8$

10. $-x - 4y + 2z = 8$
$x - z = 3$
$x + y - z = 2$

11. $-0.1x + 0.2z = 4$
$0.2y - 1.1z = 2$
$x + z = 2$

12. $0.1x - 0.2z = 6$
$y - z = 6$
$0.1x - 1.1y = 3$

T In Exercises 13–18 use Cramer's Rule with technology to solve the given system of linear equations in the event that is has a unique solution. If there is no unique solution, indicate why. [Hint: See Example 3]

T 13. $x + y + 5z = 1$
$y + 2z + w = 1$
$x + 3y + 7z + 2w = 2$
$x + y + 5z + w = 1$

T 14. $x + y + 4w = 1$
$2x - 2y - 3z + 2w = -1$
$4y + 6z + w = 4$
$2x + 4y + 9z = 6$

T 15. $x + y + 5z = 1$
$y + 2z + w = 1$
$x + y + 5z + w = 1$
$x + 2y + 7z + 2w = 2$

T 16. $x + y + 4w = 1$
$2x - 2y - 3z + 2w = -1$
$4y + 6z + w = 4$
$3x + 3y + 3z + 6w = 4$

T 17. $x + y + 5z = 1$
$y + 2z + w = 1$
$x + y + 5z + w = 1$
$2x + 2y + 7z + 2w = 2$

T 18. $x + y + 4w = 1$
$2x - 2y - 3z + 2w = -1$
$4y + 6z + w = 4$
$3x + 3y + 3z + 7w = 4$

In Exercises 19–22, write down an equation the parameters must satisfy for there to be a unique solution, and then solve for the indicated variables assuming that condition is met. [Hint: See Example 1]

19. $(a+b)p + cq = a-b$

$cp - (a-b)q = b-a$

Solve for p and q.

20. $a^2p - (r+s)q = a^3$

$(r+s)q - \frac{q}{a^2} = r$

Solve for p and q.

21.

$$\begin{aligned} ax_1 \quad\quad\; + qx_3 &= q \\ rx_1 + x_2 \quad\quad\quad &= 0 \\ r^2x_1 - x_2 + ax_3 &= a \end{aligned}$$

Solve for x_1, x_2, and x_3.

22.

$$\begin{aligned} bx_1 + ax_2 + qx_3 &= 2a \\ a^2x_2 + qx_3 &= 2a^2 \\ bx_1 \quad\quad\quad + qx_3 &= 0 \end{aligned}$$

Solve for x_1, x_2, and x_3.

Applications

Some of the following exercises are similar or identical to exercises and examples in Section 3.3. All should be solved using Cramer's rule.

23. ***Resource Allocation*** You manage an ice cream factory that makes three flavors: Creamy Vanilla, Continental Mocha, and Succulent Strawberry. Into each batch of Creamy Vanilla go two eggs, one cup of milk, and two cups of cream. Into each batch of Continental Mocha go one egg, one cup of milk, and two cups of cream. Into each batch of Succulent Strawberry go one egg, two cups of milk, and one cup of cream. Your stocks of eggs, milk, and cream vary from day to day. How many batches of each flavor should you make in order to use up all of your ingredients if you have the following amounts in stock?
(a) 350 eggs, 350 cups of milk, and 400 cups of cream
(b) 400 eggs, 500 cups of milk, and 400 cups of cream

24. ***Resource Allocation*** The Arctic Juice Company makes three juice blends: PineOrange, using 2 quarts of pineapple juice and 2 quarts of orange juice per gallon; PineKiwi, using 3 quarts of pineapple juice and 1 quart of kiwi juice per gallon; and OrangeKiwi, using 3 quarts of orange juice and 1 quart of kiwi juice per gallon. The amount of each kind of juice the company has on hand varies from day to day. How many gallons of each blend can it make on a day with the following stocks?
(a) 800 quarts of pineapple juice, 650 quarts of orange juice, 350 quarts of kiwi juice.
(b) 650 quarts of pineapple juice, 800 quarts of orange juice, 350 quarts of kiwi juice.

Investing In Mutual Funds Exercises 25 and 26 are based on the following data on three mutual funds.[1]

	2007 Yield
FHIFX (Fidelity Focused High Income Fund)	6%
FFRHX (Fidelity Floating Rate High Income Fund)	5%
FASIX (Fidelity Asset Manager 20%)	7%

[1] Yields are for the year ending September, 2007 and rounded. Source: money.excite.com, October, 2007.

25. You invested a total of $9,000 in the three funds at the beginning of 2007, including an equal amount in FFRHX and FASIX. Your 2007 yield for the year from the first two funds amounted to $400. How much did you invest in each of the three funds?

26. You invested a total of $6,000 in the three funds at the beginning of 2007, including an equal amount in FHIFX and FFRHX. Your total yields for 2007 amounted to $360. How much did you invest in each of the three funds?

Investing in Stocks Exercises 27 and 28 are based on the following data on three computer related stocks.[2]

	Price per Share	Dividend Yield
MSFT (Microsoft)	$30	1.5%
INTC (Intel)	25	1.8
YHOO (Yahoo)	25	0

27. You invested a total of $5,400 in Microsoft, Intel, and Yahoo shares at the above prices, and expected to earn $45 in annual dividends. If you purchased a total of 200 shares, how many shares of each stock did you purchase?

28. You invested a total of $5,800 in Microsoft, Intel, and Yahoo shares at the above prices, and expected to earn $54 in annual dividends. If you purchased a total of 220 shares, how many shares of each stock did you purchase?

Communication and Reasoning Exercises

29. Name one advantage and one disadvantage of Cramer's Rule versus row-reduction for solving a system of linear equations.

30. What does it mean about a system of n linear equations in n unknowns when the determinant of the coefficient matrix is zero?

31. Multiple choice: If the determinant of the coefficient matrix is zero for a system of linear equations, then:

(A) Cramer's Rule yields the exact solution.
(B) Cramer's Rule fails, but we can obtain the solution by row-reducing the augmented matrix.
(C) Cramer's Rule fails, but we can obtain the solution by using the inverse of the coefficient matrix.
(D) There is only the zero solution.

32. Multiple choice: If the determinant of the coefficient matrix is nonzero for a system of linear equations, but the right-hand sides are zero, then:

(A) There are infinitely many solutions.

[2] Stocks were trading at or near the given prices in September, 2007. Dividends are rounded. Source: http://money.excite.com, October 2007.

(B) Cramer’s Rule fails, but we can obtain the solution by row-reducing the augmented matrix.
(C) Cramer’s Rule fails, but we can obtain the solution by using the inverse of the coefficient matrix.
(D) There is only the zero solution.

Answers to Odd-Numbered Exercises

3.6

1. $m_{23} = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}$; $M_{23} = 0$ **3.** $m_{22} = \begin{bmatrix} -1 & 5 \\ 2 & 1 \end{bmatrix}$; $M_{22} = -11$ **5.** $m_{12} = \begin{bmatrix} 3 & 5 \\ 2 & 1 \end{bmatrix}$; $M_{12} = -7$ **7.** 7 **9.** –18 **11.** –12 **13.** –1 **15.** –24 **17.** 6 **19.** 8 **21.** 5 **23.** 18 **25.** 12 **27.** 4 **29.** 0 **31.** –12 **33.** 0 **35.** 0 **37.** –60 **39.** 0 **41.** 0 **43.** –4 **45.** (A) **47.** The determinant of a 3×3 matrix gives the volume of the solid parallelepiped obtained with corner points the three rows of the matrix. If two are the same, then two of the three edges are on top of each other, and the solid has zero volume.

3.7

1. (2.5, 1.5) **3.** (2.4, 1.2) **5.** (6, –4) **7.** (5, 1, 3) **9.** (10, –5, –3) **11.** (–12, 87, 14) **13.** (−1.5, 0, 0.5, 0) **15.** No unique solution as det(A) = 0 **17.** (0, 1, 0, 0) **19.** $-a^2 + b^2 - c^2 \neq 0$; $p = \dfrac{(a-b)(-a+b+c)}{-a^2 + b^2 - c^2}$, $q = \dfrac{(b-a)(a+b+c)}{-a^2 + b^2 - c^2}$ **21.** $a^2 - rq(1+r) \neq 0$; $(x_1, x_2, x_3) = (0, 0, 1)$ **23. (a)** 100 batches of vanilla, 50 batches of mocha, 100 batches of strawberry **(b)** 100 batches of vanilla, no mocha, 200 batches of strawberry. **25.** \$5000 in FHIFX, \$2000 in FFRHX, \$2000 in FASIX **27.** 80 MSFT, 20 INTC, 100 YHOO **29.** Advantage: Cramer's Rule allows us to write own the solution explicitly, and this is useful when, for instance, the coefficients are parameters. Disadvantage: Cramer's rule applies only to systems in which the number of equations equals the number of unknowns and then only when there is a unique solution. Row reduction can be used to analyze any system of linear equations. **31.** (B)