Math 171 Notes

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These notes are based, in part, on the following texts:
Mathematical Analysis by T. M. Flett (out of print; formerly published by McGraw Hill in 1966)
An Introduction to Analysis (2nd. Ed.) by W. R. Wade (Prentice Hall, 2000)

1. Sets

The primitive notions we assume given (and undefined) are the notions of a "set" (a collection of objects) and the predicate "is a member of," (as in "Jason is a member of the set of all people whose first names end in the letter *n*.") The sets we consider in this course are the following.

Examples of Sets 1.1
A. $\mathbb{N}$ is the set of all natural numbers $0, 1, 2, 3, ...$
B. $\mathbb{Z}$ is the set of all integers $0, 1, -1, 2, -2, ...$
C. $\mathbb{Q}$ is the set of all rational numbers (ratios of integers), e.g. $1/2, 3/6, 7/3, 1, 0, -6$ etc.
D. $\mathbb{R}$ is the set of all real numbers—to be discussed below.
E. $\mathbb{C}$ is the set of all complex numbers,
$\mathbb{C} = \{a+ib \mid a\in\mathbb{R}, b\in\mathbb{R}\}$.
F. If $\mathbb{A}$ is one of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, let $\mathbb{A}^* = \{a \in \mathbb{A} \mid a \neq 0\}$

Definitions 1.2
A. $x \in A$ means x is a member of the set A.
B. $x \notin A$ means x is not a member of A.
C. $\varnothing$ denotes the empty set, (containing no elements).
D. A **nonempty** set is one that contains at least one element. The two sets A and B are **equal** if and only if (written "iff") they have the same elements. That is,

$$A = B \text{ iff } x \in A \Leftrightarrow x \in B$$

E. If A and B are sets, and $x \in A \Rightarrow x \in B$, then we write $A \subset B$, or equivalently, $B \supset A$, and say that ***A* is a subset of *B***.

$$A \subset B \text{ iff } x \in A \Rightarrow x \in B$$

(Note that, if $A = B$, then $A \subset B$. If $A \subset B$ but if B contains at least one element not in A, we say that the inclusion $A \subset B$ is **strict**.)

Here is an example of how we can use these definitions to prove things.

Proposition 1.3 $A = B$ if and only if $A \subset B$ and $B \subset A$

Proof $A = B$

$\Leftrightarrow [x \in A \Leftrightarrow x \in B]$ by definition of equality

$\Leftrightarrow [x \in A \Rightarrow x \in B]$ and $[x \in B \Rightarrow x \in A]$

$\Leftrightarrow A \subset B$ and $B \subset A$ by definition of $\subset$.

❐

Example 1.4 One has

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

Further, all of these inclusions are strict.

Scholium 1.5 $\sqrt{2}$ is irrational.

Proof in class

Exercise Set 1

1. How would you justify the claim in Example 1.4?

2. Prove that, if $A \subset B$ and $B \subset C$, then $A \subset C$.

3. Prove that, if $A \subset B$, $B \subset C$ and $C \subset A$, then all three sets are equal.

4. The **Cartesian Product**, $A \times B$, of two sets A and B is defined to be the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$.

(a) Describe the set $\{a, b, c\} \times \{1, 2, 3\}$.

(b) Describe $\mathbb{R} \times \mathbb{R}$ (also called $\mathbb{R}^2$) and $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ (also called $\mathbb{R}^3$) in familiar terms.

5. Prove that every set is a proper subset of another set. [Hint: You can assume that no set can be a member of itself.]

6. Prove that $\sqrt{3}$ is irrational.

2. The Natural Numbers and Induction

We begin by looking briefly at $\mathbb{N}$, the set of natural numbers, and deriving a few of their formal properties. When we're done with this section, we shall go back to assuming all the properties that we like. All the familiar properties of the natural numbers can be derived from the following list of axioms:

Axioms of the Natural Numbers 2.1
We assume we are given:
(a) a set $\mathbb{N}$, whose elements are called **natural numbers**,
(b) an element $0 \in \mathbb{N}$, called **zero**
(c) an element $1 \in \mathbb{N}$, called **one**, and different from zero
(d) a rule that assigns to each ordered pair (m, n) of natural numbers a unique (that is, a single) natural number called **the sum of *m* and *n***, and denoted by $m+n$
(e) a rule that assigns to each ordered pair (m, n) of natural numbers a unique natural number called **the product of *m* and *n***, and denoted by mn
(f) Further, the following properties hold:

(1) For all $m, n, p \in \mathbb{N}$,	$(m+n)+p = m+(n+p)$	**Associative Property for +**
(2) For all $m, n \in \mathbb{N}$,	$n+m = m+n$	**Commutative Property for +**
(3) For all $n \in \mathbb{N}$,	$n+0 = 0+n = n$	**Additive Identity Law**
(4) For all $m, n, p \in \mathbb{N}$,	$(mn)p = m(np)$	**Associative Property for $\times$**
(5) For all $m, n \in \mathbb{N}$,	$mn = nm$	**Commutative Property for $\times$**
(6) For all $n \in \mathbb{N}$,	$n \cdot 1 = 1 \cdot n = n$	**Multiplicative Identity Law**
(7) For all $m, n, p \in \mathbb{N}$	$m(n+p) = mn + mp$	**Distributive Law**

(8) For each pair of elements $m, n \in \mathbb{N}$, exactly one of the following three properties holds: If $m, n \in \mathbb{N}$, then *exactly one* of the following hold:

(i) $m = n$
(ii) there exists $p \neq 0$ in $\mathbb{N}$ with $m+p = n$, and we write $m < n$
(iii) there exists $p \neq 0$ in $\mathbb{N}$ with $n+p = m$, so that $n < m$, by (ii).

Trichotomy Property

(9) If $\mathcal{N}$ is any set of natural numbers such that:

(a) $0 \in \mathcal{N}$
(b) If $n \in \mathcal{N}$, then $n+1 \in \mathcal{N}$,
then $\mathcal{N} = \mathbb{N}$. **The Axiom of Induction**

Remarks 2.1 These axioms lead to all the properties of the natural numbers you are familiar with (and we won't spend time deriving them); for example:
1. For every $n \in \mathbb{N}$, $n \neq 0$ implies $n > 0$.
Proof: $0 + n = n$, whence $0 < n$ by 8(ii)
2. 0 is the smallest element in $\mathbb{N}$ (this is just a restatement of (1).)
3. $\mathbb{N} = \{0, 1, 1+1, 1+1+1, 1+1+1+1, \dots\}$ (We commonly write 1+1 as 2, 1+1+1 as 3, etc. Thus, $\mathbb{N} = \{0, 1, 2, 3, \dots\}$.)
This follows directly from the induction axiom.
4. **(Total Ordering of $\mathbb{N}$)** The set $\mathbb{N}$ is totally ordered. That is:

(a) If m and n are any elements of $\mathbb{N}$, then exactly one of the following statements is true: $m = n, m < n, n < m$.
(b) If $m < n$ and $n < r$ then $m < r$.

5. (Properties of the Order Relation) If $m, n,$ an $r \in \mathbb{N}$, then:
(a) $m < n \Rightarrow m+r < n+r$
(b) $m < n$ and $r \neq 0 \Rightarrow mr < nr$.

Theorem 2.4 (Principle of Mathematical Induction)
Let there be associated with every natural number a proposition P(n) which is either true or false.* Then, if:
(a) P(0) is true;
(b) If P(n) is true, then so is P(n+1),
then P(n) is true for every $n \in \mathbb{N}$.

Proof in class.

Examples of a proof by induction

In class, we prove inductively that $1 + \ldots + n = \frac{n(n+1)}{2}$ for all $n \geq 1$.

On Line Discussion How come there are no "infinitely big" natural numbers? Which axiom prevents this? Some people define all the natural numbers as sets:

$0 = \emptyset$
$1 = \{0\}$
$2 = \{0, 1\}$
$3 = \{0, 1, 2\}$ etc.

We can then define $\omega = \{0, 1, 2, ...\} = \mathbb{N}$,

$\omega+1 = \{\omega\}$
$\omega+2 = \{\omega, 1\}$, etc.

What can you say about these strange new numbers?

Exercise Set 2

1. **(a)** Use Property (8) to prove that, if m and n are natural numbers with $n+m = n$, then $m = 0$.
(b) Now deduce the **additive cancellation law**: if $n+m = n+k$, then $m = k$.

2. Prove the following statements by induction:
(a) $1 + 3 + 5 + \ldots + 2n-1 = n^2$
(b) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{(n-1)n} = 1 - \frac{1}{n}$
(c) For every collection $\{A_i\}$ of sets, one has $\bigcap_{i=0}^{n} A_i \subset \bigcup_{i=0}^{n} A_i$.
(d) For every real number x and $n \in \mathbb{N}$ with $n \geq 1$, one has

$$x^n - 1 = (x-1)(x^{n-1} + \ldots + x + 1).$$

* For example, P(n) is the proposition "n is divisible by 2," "n < n+1," or "there are *n* shopping days 'till Halloween."

3. Prove that the Principle of Mathematical Induction implies the Axiom of Induction. (In other words, use induction to prove the Axiom of Induction!)

4. Prove the following variant of the principle of induction:

Let there be associated with every natural number a proposition P(n) which is either true or false, and let $k \in \mathbb{N}$. Then, if:

(a) P(k) is true;

(b) If P(n) is true, then so is P(n+1),

then P(n) is true for every $n \in \mathbb{N}$ with $n \geq k$.

5. Prove inductively that, for any real numbers $r \neq 1$ and a, one has

$$a + ar + ar^2 + \ldots + ar^n = \frac{a(1-r^{n+1})}{1-r}$$

6. Well-Ordering Principle A totally ordered set S is **well ordered** if every non-empty subset A of S has a least element—that is, an element $x \in A$ such that $x \leq a$ for every $a \in A$. Prove that $\mathbb{N}$ is well-ordered. [Hint: If S is any non-empty subset of $\mathbb{N}$, let

$$\mathcal{N} = \{n \in \mathbb{N} : n < s \text{ for every } s \in S\}$$

($\mathcal{N}$ is the set of "(strict) lower bounds" of S.) Assume that S has no least element, and conclude that $\mathcal{N} = \mathbb{N}$.]

Extra Credit (Due exactly one month from assignment of this exercise set)

Prove the following generalization of the Principle of Mathematical Induction: Let Ω be a well-ordered set with least element 0, and let there be associated with every element $\omega \in \Omega$ a proposition P(ω) which is either true or false. Then, if

(a) P(0) is true

(b) If P(ω) is true for every $\omega < \alpha \Rightarrow$ P(α) is true,

then P(ω) is true for every $\omega \in \Omega$.

3. The Real Numbers

All the properties of the real numbers can be derived from the following axioms, as well as a mystery extra axiom.

Axioms of the Real Numbers 3.1
We suppose that we are given
(a) a set $\mathbb{R}$ whose elements are called **real numbers**
(b) an element 0 of $\mathbb{R}$, called **zero**
(c) an element 1 of $\mathbb{R}$, different from 0, called **one**
(d) a rule that assigns to each ordered pair (x, y) of real numbers a unique real number called **the sum of x and y**, and denoted by $x+y$
(e) a rule that assigns to each ordered pair (x, y) of real numbers a unique real number called **the product of x and y**, and denoted by xy
(f) a relation $x < y$, also written as $y > x$, holding between certain pairs of elements of $\mathbb{R}$.
We suppose also that the following groups of properties hold:

Field Axioms The real numbers form a **field**; that is
(1) For all $x, y, z \in \mathbb{R}$, $(x+y)+z = x+(y+z)$ **Associative Property for +**
(2) For all $x, y \in \mathbb{R}$, $x+y = y+x$ **Commutative Property for +**
(3) For all $x \in \mathbb{R}$, $x+0 = 0+x = x$ **Additive Identity Law**
(4) For each $x \in \mathbb{R}$, there exists an element $-x \in \mathbb{R}$ such that
$x+(-x) = (-x)+x = 0$ **Existence of Additive Inverses**
(5) For all $x, y, z \in \mathbb{R}$, $x(yz) = (xy)z$ **Associative Property for ×**
(6) For all $x, y \in \mathbb{R}$, $xy = yx$ **Commutative Property for ×**
(7) For all $x \in \mathbb{R}$, $x \cdot 1 = 1 \cdot x = x$ **Multiplicative Identity Law**
(8) For each non-zero $x \in \mathbb{R}$, there exists an element $x^{-1} \in \mathbb{R}$ such that
$xx^{-1} = x^{-1}x = 1$ **Existence of Multiplicative Inverses**
(9) For all $x, y, z \in \mathbb{R}$, $(x+y)z = xz + yz$ **Distributive Law**

Order Axioms The field of real numbers is an ordered field:
(10) For each pair of elements x, y in $\mathbb{R}$, exactly one of the following three properties holds: $x < y$, $x > y$, $x = y$. **Trichotomy Property**
(11) If $x < y$ and $y < z$, then $x < z$. **Transitive Property**
(12) If $x < y$, then $x+w < y+w$ for all $w \in \mathbb{R}$ **Additive Property**
(13) If $x < y$ and $w > 0$, then $xw < yw$. **Multiplicative Property**

Notes
(a) All the usual rules of arithmetic follow from the field axioms. For instance, if we define x^n as $x \cdot x \cdot \cdot x$ (n times), then we get the usual rules of exponents. (See any book on precalculus). We are more interested in proving things about order.
(b) Define a subset $\mathcal{N}$ of $\mathbb{R}$ inductively as follows: $0 \in \mathcal{N}$, and, if $x \in \mathcal{N}$, then $x+1 \in \mathcal{N}$. Then $\mathcal{N} = \{0, 1, 1+1, 1+1+1, \ldots\}$, and satisfies all the axioms of the natural numbers. Thus we henceforth identify it with the set of natural numbers. In other words:

We once and for all regard the set of natural numbers as a subset of $\mathbb{R}$.

Lemma 3.1
(a) If $x \le y$ and $y \le x$, then $x = y$.
(b) Axioms (11) through (13) continue to hold with "<" replaced by "≤".
(c) If $x < y$, then $-x > -y$.
(d) If $x < y$ and $w < 0$, then $xw > yw$.
(e) If $x \neq 0$, then $x^2 > 0$. In particular, $1 > 0$.
(f) If $x > 0$, then $\frac{1}{x} > 0$.
(g) if $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$.

Proof: (a) is immediate from Axiom 10, and you are required to prove (b) in Exercise Set 3. (c) follows from two invocations of (12) and (d) follows from (c). (e) (13) and (d), and (f) and (g) are exercises for you.

Following is our first little theorem on analysis.

Theorem 3.2 (See Theorem 1.9 in Wade)
If a and b are real numbers such that $a \le b+\varepsilon$ for every $\varepsilon>0$, then $a \le b$.

Proof If, on the contrary, $a > b$, then take $\varepsilon = \frac{1}{2}(a-b) > 0$. The hypothesis $a \le b+\varepsilon$ then gives, with this value of ε and a little fiddling with inequalities, $a \le b$, a contradiction. ✣

Corollary 3.3 (Existence of "In-Between Numbers")
Between every pair of real numbers lies a third. That is, if a and b are real numbers with $a < b$, then there exists an $\varepsilon>0$ with $a < a+\varepsilon < b$.

Proof S'pose not. Then, for all $\varepsilon>0$, one has $a+\varepsilon \ge b$, that is, $b \le a+\varepsilon$. But then Theorem 3.2 implies that $b \le a$, contradicting $b > a$. ♣

Definition The **absolute value** $|x|$ of the real number x is defined as

$$|x| = \begin{cases} x & \text{if } x \ge 0, \\ -x & \text{if } x < 0. \end{cases}$$

In other words, $|x|$ is either x or $-x$, whichever is positive (or zero).

Lemma 3.4 One has, for all $x, y \in \mathbb{R}$,
(a) $|x| = |-x|$
(b) $|xy| = |x|\cdot|y|$

Proof is in Exercise Set 3

Proposition 3.5 If $a > 0$, then the following statements are equivalent.
(a) $|x| < a$
(b) $-a < x < a$
(c) $x^2 < a^2$
A similar result holds if "<" is replaced throughout by "≤".

Proof in Brief:
(a)⇒(b): If $x \geq 0$, then (a) says that $0 \leq x < a$, whence $-a < x < a$. If $x < 0$, then (a) says that $0 \leq -x < a$, whence (turning this around) $-a < x < a$ again.
(b)⇒(c): If $x > 0$, then (b) implies that $0 < x < a$, so $x^2 < ax < a^2$ by Axiom (13). If $x < 0$, then (b) implies that $0 < -x < a$ and so $x^2 = (-x)^2 < (-x)a < a^2$ by Axiom (13). If $x = 0$, the result is immediate, since $a > 0$.
(c)⇒(a): Assume (c) is true, and suppose, on the contrary, that (a) was false, so that $|x| \geq a$. Then $x^2 = |x|^2 \geq |x|a \geq a^2$, contradicting (c). ☠

Proposition 3.6 (Absolute Value of a Sum)
For any pair of real numbers x and y,

$$|x+y| \leq |x| + |y|.$$

Proof: By Proposition 3.5, it suffices to compare the squares of both sides, and this boils down to saying that $xy \leq |xy|$ (by Lemma 3.4). But that is true. ❍

On Line Discussion We still haven't proved the following very "obvious" fact: for every real number x (no matter how large) there exists an integer n with $n > x$. What difficulties does this present to you? A related question is this: Is the set of real numbers a well-ordered set? If it were, what would this imply?

Exercise Set 3
1. Prove Lemma 3.1(b), (f) and (g).
2. Prove:

(a) If $x > 0$, then $0 < \frac{x}{2} < x$

(b) If $0 < x < y$, then $0 < \sqrt{x} < \sqrt{y}$ (assuming the square roots exist—see later)
.

(c) If $\delta > 0$, there exists an $\varepsilon > 0$ with $\varepsilon < \delta$.

3. Prove Lemma 3.4.
4. (Generalization of Proposition 3.6) Prove that, for all real numbers x and y, one has

$$\big||x| - |y|\big| \leq |x \pm y| \leq |x| + |y|$$

5. Justify the claim in Note (b) after the axioms.
6. Prove the following "in-between" result:

If $x^2 < b$, then there exists an $\varepsilon>0$ with $(x+\varepsilon)^2 < b$.

Note: You cannot assume the existence of square roots, since we haven't shown that they exist yet! [Hint: By considering the cases $x<0$ and $x\geq 0$ separately, show that there exists an $\varepsilon>0$ with $2x\varepsilon + \varepsilon^2 < \delta$. (You might also find Exercise 2(c) above helpful.)]
7. Prove the further "in-between" result:

If $x^2 > b$, then there exists an $\varepsilon>0$ with $(x-\varepsilon)^2 > b$.

[Hint: the proof here is similar; first use 3.3 to show that is a $\delta>0$ with $x^2-\delta>b$. Then, by considering the cases $x<0$ and $x\geq 0$ separately, show that there exists an $\varepsilon>0$ with $2x\varepsilon - \varepsilon^2 < \delta$.]

4. Maximum, Minimum, Supremum and Infimum

Definition 4.1 Let A be a non-empty set of real numbers. Then an element $d \in A$ is called the **greatest element of A** if $d \geq a$ for every $a \in A$. We write

$$d = \max A.$$

Similarly, $\min A$, the **least element of A**, is an element e such that $e \leq a$ for every $a \in A$.

Notes

1. $\max A$ and $\min A$ need not exist; e.g. look at the interval $(0, 1)$ (and Exercise Set 3 # 2).
2. $\max A$, if it exists, is unique by Lemma 3.1(a).

Examples 4.1

A. $A = \{-4, -8, 0, 100\}$ **B.** $A = [0, 1]$

C. $A = (0, 1]$ **D.** $A = \{\frac{1}{n} \,:\, n \in \mathbb{N}, n \geq 1\}$

E. $A = \{\frac{1}{n} \,:\, n \in \mathbb{N}, n \geq 1\} \cup (0)$ **F.** $A = \mathbb{R}$

We do have the following, however.

Lemma 4.2 If A is finite and non-empty, then $\max A$ and $\min A$ exist.

Proof: We do induction on the number of elements n in the set. If $n = 1$, then its only element satisfies the definition of min and max, so we are done. If $|A| = n+1$, then choose any element $a \in A$, and let $B = A - \{a\}$, so that $|B| = n$, and so B has a max. Then take

$$d = \begin{cases} \max B & \text{if } \max B > a \\ a & \text{if } a > \max B. \end{cases}$$

Then $d \geq$ every element in A, and thus is its max. Similarly for $\min A$. ✺

Definition 4.3 Let A be any set. Then an **upper bound of A** is an element $M \in \mathbb{R}$ such that $M \geq a$ for every $a \in A$. M is also called a **majorant of A**. Similarly, a **lower bound of A** is an element $m \in \mathbb{R}$ such that $m \leq a$ for every $a \in A$. m is also called a **minorant of A**. A set A is **bounded** if it is bounded below and bounded above.

Examples 4.4 Which of the following sets is bounded above/bounded below/bounded? (Note how heavily we use the existence of "in-between numbers:" Corollary 3.3.

A. $A = \emptyset$ **B.** $A = [-1, 1)$

C. $A = (0, +\infty)$ **D.** $A = \left\{ \frac{n^2-1}{n} : n \in \mathrm{N},\, n \geq 1 \right\}$

E. Find the set of majorants of $\{a \in \mathbb{R} \;:\; a < x\}$.

Definition 4.5 If A is any set of real numbers, then the **least upper bound**, or **supremum** of A is the least of all upper bounds, if it exists. We write this as $\sup A$ (or sometimes $\text{lub}A$).

Notes

1. $\sup A$ is the minimum of the set of all upper bounds, if it exists.

2. Since the minimum of a set is unique, it follows by Note (2) above that the supremum is also unique: a set can have at most one supremum.
3. Since an upper bound of A may be a member of A, sup$A$ may also be a member of A.
4. To prove that a real number s is the supremum of a set, we must show two things:
(a) that it is an upper bound of A
(b) that it is ≤ any other upper bound of A.

In other words:

Proving That s is the Supremum of A
(a) Prove that, for all $a \in A$, one has $s \geq a$ (upper bound)
(b) Prove that, for all upper bounds m of A, one has $s \leq m$ (least upper bound)

Examples 4.6

A. $\sup(0, 1) = 1$

B. $\sup(0, 1] = 1$

C. $A = \emptyset$ is bounded but has no supremum

D. $A = \mathbb{R}$

E. $A = \left\{ \frac{1}{n} : n \in \mathrm{N}, n \geq 1 \right\}$

F. $A = \left\{ \frac{n^2-1}{n} : n \in \mathrm{N}, n \geq 1 \right\}$

G. $A = \{x \in \mathbb{R} : x^2 < 2\}$

H. If max$A$ exists, then sup$A$ exists and equals max$A$ (proved in the exercises).

We can now add the last remaining axiom of the real numbers:

Completeness Axiom of the Real Numbers
Every non-empty subset of $\mathbb{R}$ that is bounded above has a supremum.

Note In other words, if A is nonempty and bounded, then sup$A$ exists. This makes the supremum useful, because the supremum always exists, whereas the maximum element may not exist.

Definition 4.7 If A is any set of real numbers, then the **greatest lower bound**, or **infimum** of A is the greatest of all lower bounds, if it exists. We write this as inf$A$ (or sometimes glb$A$).

Examples 4.8 Find the infima of the sets in Examples 4.6, where they exist.

We now have the following consequence of the completeness axiom:

Equivalent Form of Completeness Axiom of the Real Numbers
Every non-empty subset of $\mathbb{R}$ that is bounded below has an infimum.

Thus—and here is yet another form of the completeness axiom—every non-empty bounded set has both an infimum and a supremum.

Lemma 4.9 Let A be any non-empty subset of $\mathbb{R}$. Then $s = \sup A$ iff s has the following properties:

(1) $s \geq a$ for every $a \in A$

(2) If $t < s$, then there exists an $a \in A$ such that $a > t$.

Exercise Set 4

1. (a) Prove that the set of majorants of $A \subset \mathbb{R}$ is either empty or infinite.

(b) Give two examples of sets A with majorants that are also minorants.

2. Prove that a set S of real numbers is bounded iff there exists a real number M such that $|s| \leq M$ for every $s \in S$.

3. Prove each of the following:

(a) $\sup\left\{ 1 - \frac{1}{n} : n \geq 1,\ n \in \mathrm{N} \right\} = 1$

(b) $\inf \bigcup_{n=1}^{+\infty} \left(1 - \frac{n}{n+1},\ 1 \right) = 0$

4. Prove the claim in example 4.6 (H): If $\max A$ exists, then $\sup A$ exists and equals $\max A$.

5. Show that the set $\mathbb{Q}$ of rational numbers is not complete.

6. Let A be any non-empty set that is bounded above, and let $B = \{-a : a \in A\}$ be the set of negatives of the elements in A. Show that B is bounded below, and that $\inf B = -\sup A$. Now deduce that the two forms of the axiom of completeness stated above are equivalent.

7. Let A and B be non-empty and bounded above with $\sup A = s$ and $\sup B = t$. Which of the following claims are true? (Prove or give a counterexample in each case.)

(a) $\sup(A \cup B)$ exists and equals $\max\{s,\ t\}$

(b) $\sup(A \cap B)$ exists and equals $\min\{s,\ t\}$

(c) If $A \subset B$, then $\sup A \leq \sup B$

8. State and prove an analog of Lemma 4.9 for infima.

On Line Discussion What does the axiom of completeness get you (look ahead to Section 5 for some examples)? Could you live without it? Are there any instances from geometry where one needs the square roots of rational numbers to exist? (Think of situations where you have had to take square roots.)

5. Further Properties of the Real Numbers

We can now use the completeness axiom to get more interesting results about $\mathbb{R}$.

Theorem 5.1 Every positive real number has a square root in $\mathbb{R}$.

Proof Let $a \in \mathbb{R}$, $a > 0$, and let $A = \{x \in \mathbb{R} \mid x^2 \le a\}$.Then A is non-empty ($0 \in A$) and bounded above by max{a,1}, and hence has a supremum, s. Note that s, being an upper bound, must be positive (there certainly is a positive ε with $\varepsilon^2 \le a$, namely $\varepsilon = \min\{1, a\}$). Claim: $s^2 = a$ (showing the result). Indeed, if $s^2 < a$, then by Exercise Set 3, #6, there exists an $\varepsilon > 0$ with $(s+\varepsilon)^2 < a$, whence $s+\varepsilon \in A$ by definition of A. This contradicts the fact that s is an upper bound (since the element $s+\varepsilon$ of A is greater than s). Further, if $s^2 > a$, then by Exercise Set 3 #7, there exists an $\varepsilon > 0$ with $s-\varepsilon > 0$ and $(s-\varepsilon)^2 > a$.† But then $s-\varepsilon$ is an upper bound of A (since if $x \in A$, then $x^2 \le a < (s-\varepsilon)^2$, so that $x < (s-\varepsilon)$ by Exercise Set 3 #2(b)). This contradicts the fact that s is the *least* upper bound of A. Thus the only remaining possibility is that $s^2 = a$. ✶

Theorem 5.2 (Archimedean Property of the Reals)
If a and b are any positive real numbers, there exists $n \in \mathbb{N}$ such that $na > b$.

Note This says that, given any real number, no matter how large, there is a larger integer.

Proof of Theorem 5.2 Suppose the conclusion were false. Then the non-empty set $A = \{na \mid n \in \mathbb{N}\}$ would be bounded above (by b) and hence have a supremum s. By Lemma 4.9, since $s-a < s$, is it also < some element of A. In other words, $s-a < na$ for some $n \in \mathbb{N}$. Thus $s < (n+1)a \in A$, contradicting the fact that s is an upper bound of A. ❒

Corollary 5.3 (Alternative Form of the Archimedean Property)
The Archimedean property is equivalent to the following:
If a is any positive real number (no matter how small) there exists an $n \in \mathbb{N}$ with $\frac{1}{n} < a$.

Proof Assume the Archimedean property, and let a be any positive integer. By the Archimedean property, we can find an $n \in \mathbb{N}$ with $na > 1$. The desired inequality follows.
Conversely, assume the Alternative form, and let a and b be positive numbers. Choose $n \in \mathbb{N}$ so that $\frac{1}{n} < \frac{a}{b}$. Then $na > b$ by manipulation of inequalities. ①

Recall that, if $a \le b$, the **open interval $(a,\ b)$** is defined by $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$. (Note that $(a, a) = \emptyset$ if $a \in \mathbb{R}$.)

---

† The quotes exercise alleged only that there is an ε_1 $(s-\varepsilon_1)^2 > a$, but we know from above that $s > 0$, so that there is an ε_2 with $s-\varepsilon_2 > 0$. Choosing ε t be $\min\{\varepsilon_1, \varepsilon_2\}$ gives ther result.

Definition 5.4 A subset $C \subset \mathbb{R}$ is a **dense subset** of $\mathbb{R}$ if every non-empty open interval contains at least one element of C.

Note If C is dense in $\mathbb{R}$, then in fact every non-empty open interval contains *infinitely many* element of C. (Proof in the exercises.)

Theorem 5.5 (Density Theorems)
(a) $\mathbb{Q}$ is dense in $\mathbb{R}$.
(b) $\mathbb{R}-\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof is left an exercise

Exercise Set 5

1. Prove the claim after Definition 5.4; namely: If C is dense in $\mathbb{R}$, then every non-empty open interval contains infinitely many element of C.

2. (a) Prove that, if a and b are positive real numbers, there exists $n \in \mathbb{N}$ such that

$$(n-1)a \leq b < na$$

(b) Prove that, it x is a positive real number then there exists $n \in \mathbb{N}$ such that

$$x < n \leq x+1$$

3. Prove Theorem 5.5. (You might find part (b) of the previous exercise useful.)

4. Use your method of proving Theorem 5.5 to produce five irrational numbers between 0.14 and 0.15.

On Line Discussion What does the fact that the rational are dense in the reals suggest to you about the relative "magnitudes" of $\mathbb{R}$ and $\mathbb{Q}$?

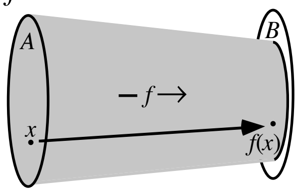
6. Functions and Countability

We need one more primitive notion: that of a *function*.

Definition 6.1 A **function or map f from A to B** is a triple (f, A, B) where A and B are sets, and f is a rule of correspondence that associates with each element $x \in A$ a unique element $y \in B$. We refer to this element y as $f(x)$, and write $f: A \to B$ to express the fact that f is a function from A to B. Also, A is called the **domain** (or **source**) of f and B is called the **codomain** (or **target**) of f.

Notes

1. We think of f a rule which assigns to every element of A a unique element $f(a)$ of B, and we can picture a function $f: A \to B$ as follows:



2. The codomain of f is *not* the "range" of f; that is, not every element of B need be of the form $f(x)$.

3. The sets A and B are part of the information of f; specifying f by saying, for instance, $f(x) = 2x-1$ is not sufficient. We should instead say something like this: "Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x-1$."

4. The "x" in $f(x) = . . $ is called a dummy variable; the functions $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x-1$ and $g(a) = 2a-1$ are the *same thing*.

Examples 6.2

Some in class, plus:

A. The **identity map** $1_A: A \to A$; $1_A(a) = a$ for every $a \in A$, for any set A

B. If $B \subset A$, then we have the **inclusion map** $\iota: B \to A$; $\iota(b) = b$ for all $b \in B$

C. The **empty map** $\emptyset: \emptyset \to A$ for any set A.

Definition 6.3 Let $f: A \to B$ be a map. Then f is **injective** (or **one-to-one**) if

$$f(x) = f(y) \Rightarrow x = y$$

In other words, *if* $x \neq y$, then $f(x)$ cannot equal $f(y)$.

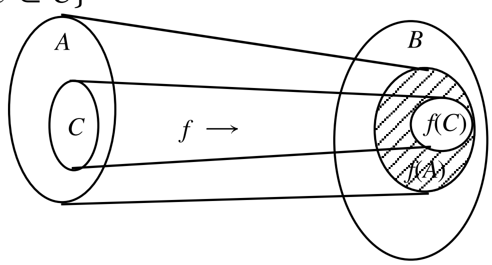
Note This definition gives us the following procedure.

Proving that a Function is Injective
To prove that f: $X \to Y$ is injective, assume $f(x) = f(y)$, and prove that $x = y$.
To prove that f is *not* injective, produce two elements x_1 and x_2 *of* X with $x_1 \neq x_2$ but $f(x_1) = f(x_2)$.

Examples 6.4
A. f: $\mathbb{R} \to \mathbb{R}$; $f(x) = 2x-1$ is injective
B. f: $\mathbb{R} \to \mathbb{R}$; $f(x) = x^2 + 1$ is not
C. identity 1_A: $A \to A$ is always injective
D. inclusion ι: $B \to A$ is always injective

Definitions 6.5 Let f: $A \to B$ be a map, and let $C \subset A$. Then the **image of *C* under *f*** is the subset

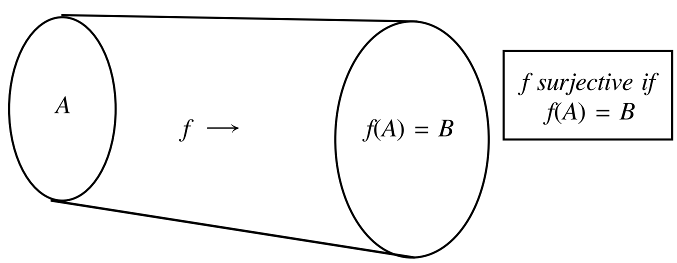
$$f(C) = \{f(c) \mid c \in C\}$$



f is **surjective** (or **onto**) if $f(A) = B$. In other words, given $b \in B$, there exists an $a \in A$ such that $f(a) = b$. That is

$$b \in B \Rightarrow \exists\ a \in A \text{ such that } f(a) = b$$

Thus, f "hits" every element in B.



Notes
1. If f: $A \to B$, then $f(A)$ is sometimes called the **range of *f*,** or the **image of *f*,** and denoted by Imf.
2. The definition gives us the following procedure.

Proving that f: $A \to B$ is Surjective
Choose a general element $b \in B$, and then prove that b is of the form $f(a)$ for some $a \in A$.

Examples 6.6
A. f: $\mathbb{R} \to \mathbb{R}$; $f(x) = x^2 + 1$. Find $f(\mathbb{R})$ and $f[0,+\infty)$

B. Identity maps are always surjective.
C. The inclusion ι: $C \to B$ is surjective iff $C = B$.
D. The canonical projections of a (possibly infinite) product.
E. Let S be any set and let $\approx$ be an equivalence relation on S. Denote the set of equivalence classes in S by $S/\approx$. Then there is a natural surjection ν: $S \to S/\approx$..

Lemma 6.7 Let f: $A \to B$. Then:
(a) $f^{-1}(f(C)) \supset C$ for all $C \subset A$, with equality iff f is injective.
(b) $f(f^{-1}(D)) \subset D$ for all $D \subset B$, with equality iff f is surjective.

Proof in class

Definition 6.8 f: $A \to B$ is **bijective** if it is both injective and surjective.

Examples 6.9
A. Exponential map $\mathbb{R} \to \mathbb{R}^+$ **B.** f: $[0, +\infty) \to [0, +\infty)$; $f(x) = \sqrt{x}$.
C. f: $[0, +\infty) \to [0, +\infty)$; $f(x) = x^2$ **D.** f: $\mathbb{R} \to [0, +\infty)$; $f(x) = x^2$ is not.
E. Restricted trig functions, e.g. g: $[-\pi, \pi] \to [-1, 1]$; $g(x) = \sin x$.
F. Inverse Trig functions **G.** The identity map on any set
H. Any linear map f: $\mathbb{R} \to \mathbb{R}$; $f(x) = mx+b$ with $m \neq 0$.

Definition 6.10 The **graph** of the function f: $A \to B$ is the subset of $A \times B$ given by

$$\text{graph}(f) = \{(x, y) \in A \times B \mid f(x) = y\}$$

Examples in class

Definition 6.11 If f: $A \to B$ and g: $B \to C$, then the **composite** $g \circ f$: $A \to C$ is the function $g \circ f(a) = g(f(a))$.

Examples in class

Lemma 6.12 Let f: $A \to B$ and g: $B \to C$,. Then:
(a) If f and g are injective, then so is $g \circ f$.
(b) If f and g are surjective, then so is $g \circ f$.
(c) If $g \circ f$ is injective, then so is f.
(d) If $g \circ f$ is surjective, then so is g.

Proof in Exercise Set 6

Definition 6.13 f: $A \to B$ and g: $B \to A$ are called **inverse maps** if $g \circ f = 1_A$ and $f \circ g = 1_B$. In this event, we write $g = f^{-1}$ (and say that g is the inverse of f) and $f = g^{-1}$. If f has an inverse, we say that f *is* **invertible**.

Form the definition, we have:

Showing that f and g are Inverses
(1) Check that the domain of f = codomain of g and codomain of f = domain of g.
(2) Check that $g(f(x)) = x$ for every x in the domain of f.
(3) Check that $f(g(y)) = y$ for every y in the domain of g.

Examples 6.14
A. Exp and log to any base
B. Inverse of a linear function
C. Inverse trig functions
C. Radicals, reciprocals
D. Look through Examples 6.9 to find any non-invertible ones.

Theorem 6.15
(a) f: $A \to B$ is invertible iff f is bijective
(b) The inverse of an invertible map is unique.

Proof in class.

Definition 6.16 The non-empty set A is:
(a) finite if there exists a bijection ϕ: $\{1, 2, \ldots, n\} \to A$. (In this case, we say that the **cardinality** of A is n, and write $|A| = n$.)
(b) infinite if it is not finite
(c) countably infinite (or aleph-0, $\aleph_0$) if there is a bijection ϕ: $\{1, 2, 3, \ldots\} \to A$ (or, equivalently, $\mathbb{N} \to A$)
(d) uncountably infinite ($\aleph_n$; $n \geq 1$) if it is infinite, but not countably infinite.

Examples 6.17
A. Finite sets
B. $\mathbb{N}$, $\mathbb{Z}$, $2\mathbb{Z}$, etc. are countably infinite.
C. $\mathbb{Q}$ is countably infinite
D. The set of all books with ≤ 500,000 words is finite.
E. The set of all books is countably infinite.
F. $\mathbb{R}$ is not uncountably infinite.

Exercise Set 6
1. Show that f: $\mathbb{R} \to \mathbb{R}$, $f(x) = ax^2 + bx + c$ is injective iff $a = 0$ and $b \neq 0$.
2. Give examples of functions f and sets C, D with $f^{-1}(f(C)) \neq C$ and $f(f^{-1}(D)) \neq D$.
3. Prove Lemma 6.12.
4. (a) Give an example of maps f: $A \to B$ and g: $B \to C$ with $g \circ f$ injective but g not injective.
(b) Give an example of maps f: $A \to B$ and g: $B \to C$ with $g \circ f$ surjective but f not surjective.
5. Let f: $\mathbb{R} \to \mathbb{R}$ be given by interchanging the first two decimal places. That is,

$$f(n.d_1 d_2 d_3 \ldots d_n \ldots) = n.d_2 d_1 d_3 \ldots d_n \ldots .$$

Show that f is bijective, and sketch the graph of f for $1 \leq x \leq 1.5$.
6. Prove that composition of functions is associative: $(f \circ g) \circ h = f \circ (g \circ h)$ and unital: $f \circ 1_A = 1_B \circ f = f$ for all f, g, h for which the expressions make sense.

7. (a) Show that if f: $A \to B$ and g: $B \to C$ are bijective, then so is $g \circ f$.

(b) Give an example of maps f: $A \to B$ and g: $B \to C$ with $g \circ f$ bijective, but with neither f nor g bijective.

8. Prove each of the following:

(a) If each A_i is countably infinite, ($i = 1, \ldots, n$) then $\overset{n}{\underset{i=1}{\times}} A_i$ is countably infinite.

(b) the union of finitely many countably infinite sets is countably infinite.

(b) the infinite product $\{0, 1\} \times \{0, 1\} \times \{0. 1\} \times \ldots$ is uncountably infinite.

(c) the set of irrational numbers is uncountably infinite.

On Line Discussion Why is the following profound?: The rationals are dense in the reals.

7. Real Valued Functions

Definition 7.1 A **real-valued function** is a function f: $A \to \mathbb{R}$ for some set A. A **real-valued function of a single real variable** is a real-valued function whose domain is a subset of $\mathbb{R}$.

Notes Real-valued functions of a single real variable are the functions we shall be looking at for the remainder of this course, and we shall simply refer to them as "real-valued functions." Denote the domain of the function f by $\mathcal{D}(f)$.

Definition 7.2 If f and g are real-valued functions, then define functions $f+g$, $f-g$, fg, $\max\{f, g\}$, $\min\{f, g\}$, all with domain $\mathcal{D}(f) \cap \mathcal{D}(g)$, and f/g with domain $\mathcal{D}(f) \cap g^{-1}(\mathbb{R}^*)$ (recall from p. 1 that $\mathbb{R}^*$ is the set of non-zero real numbers) as follows.

$$(f \pm g)(x) = f(x) \pm g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$\max\{f, g\}(x) = \max\{f(x), g(x)\}$$
$$\min\{f, g\}(x) = \min\{f(x), g(x)\}$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Further, if λ is any real number, define the function λf to have domain $\mathcal{D}(f)$ and to be given by

$$(\lambda f)(x) = \lambda f(x).$$

Examples 7.3

A. $f(x) = \sin x$, $g(x) = \cos x$. We look at various things you can do with these.

B. $\tan = \frac{\sin}{\cos}$, etc.

Remarks 7.4

It is easy to see that these operations obey all the expected rules; for instance, addition and multiplication are associative, commutative and distributive:

$(f + g) + h = f + (g + h)$.................(1)
$f + g = g + f$....................................(2)
$f(gh) = (fg)h$ ....(3)
$fg = gf$ ....(4)
$f(g + h) = fg + fh$.............................(5)
$(\lambda\mu)f = \lambda(\mu f)$ ....(6)
$1f = f$...(7)
$\lambda(f + g) = \lambda f + \lambda f$............................(8)
$(\lambda + \mu)f = \lambda f + \mu f$ ....(9)

We'll prove one or two in class, and you'll prove one or two, as well as some additional ones, at home

Definition 7.5 If f is a real-valued function, also define $|f|$, f^+, f^-, all with domain $\mathcal{D}(f)$, by

$f^+ = \max\{f, 0\}$ (here, 0 denotes the zero function 0: $\mathbb{R} \to \mathbb{R}$)

$f^- = \max\{-f, 0\}$
$|f|(x) = |f(x)|$

Definition 7.6 The real-valued function f is **bounded above, bounded below,** or **bounded** if its image Imf is, respectively, bounded above, bounded below or bounded. If $A \subset \mathcal{D}(f)$, then f is **bounded above, bounded below,** or **bounded on A** if $f(A)$ is, respectively, bounded above, bounded below or bounded.

Thus, for instance, the function f is bounded* iff there exists $M > 0$ such that
$|f(x)| \leq M$ for every $x \in \mathcal{D}(f)$
and bounded on A iff there exists $M > 0$ such that
$|f(x)| \leq M$ for every $x \in A$.

Examples 7.7
A. f: $\mathbb{R}\to\mathbb{R}$; $f(x) = x^2$ is unbounded, but bounded on every finite interval.
B. f: $\mathbb{R}^*\to\mathbb{R}$; $f(x) = 1/x$ is unbounded on $(0, +\infty)$, but bounded on $(a, +\infty)$ for any $a > 0$.

Exercise Set 7
1. Prove the claims in Remarks 7.4 (3), (5) and (6).
2. (a) Prove that $\max\{f, \max\{g, h\}\} = \max\{\max\{f, g\}, h\}$.
(b) Prove or give a counterexample: $\max\{f + g, h\} = \max\{f, h\} + \max\{g, h\}$.
3. Prove that, for every real-valued function f,
(a) $f = f^+ - f^-$
(b) $|f| = f^+ + f^-$.
4. Say which of the following functions is bounded, justifying your claim.
(a) $f(x) = \dfrac{\sin x}{x}$, on $\mathbb{R}^*$ (b) $f(x) = \tan x$, on $(0, \pi/2)$
(c) $f(x) = \dfrac{1-\cos x}{x^2}$, on $(0, +\infty)$ [Hint: consider $(0, 1]$ and $[1, +\infty)$ separately.]
5. The rational number p/q is in **lowest terms** if $q > 0$, and p and q have no common divisor except ±1. Define f: $\mathbb{R}\to\mathbb{R}$ by

$$f(x) = \begin{cases} q & \text{if } x = p/q \text{ in lowest terms} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Prove that f is unbounded.

On Line Discussion
- Can you formulate a definition of $\sup\{f\}$ and $\inf\{f\}$ for a collection $\{f\}$ of functions?
- Also, what might the graph of the function in Exercise #5 look like? What if we used $f(x) = 1/q$ instead of q in the definition?

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* See Exercise Set 4 #2.

8. Sequences

(Here we will follow the book closely.) Mathematical Analysis begins with the study of limits, and the theory of limits is best introduced in terms of sequences of real numbers.

Definition 8.1 A **sequence** of real numbers is just a function

$$f\colon \{1, 2, 3, \ldots\} \to \mathbb{R}$$

If $n \geq 1$, we call $f(n)$ the ***n*th term** of the sequence.

Notes

1. We usually write $f(n)$ as a_n, and write the sequence whose nth term is a_n as $\{a_n\}$.

2. Sometimes, it is more convenient to use a sequence indexed on $\{0, 1, 2, 3, \ldots\}$ or perhaps $\{2, 4, 6, \ldots\}$ or some other subset of $\mathbb{N}$, and we shall do so. However, by renumbering if necessary, we can regard such a sequence as having domain $\{1, 2, 3, \ldots\}$. (Some authors define a sequence as an function whose domain is an arbitrary infinite subset of $\mathbb{N}$, but then they get into a pickle defining subsequences, and their result is so messy that they avoid subsequences altogether!)

Examples 8.2

A. $a_n = 1/n \ (n \geq 1)$

B. $\left\{\dfrac{n}{n+1}\right\}_{n\geq 0}$

C. $b_n = 1$

D. $\{(-1)^{n+1}\}$

Following is the most important definition in the course so far. Think about it for a long time.

Definition 8.3 The sequence $\{a_n\}$ **converges to the real number *a*** if, given any real number $\varepsilon > 0$ (no matter how small) there exists an integer $N > 0$ such that

$$|a_n - a| < \varepsilon \text{ for all } n \geq N.$$

When this occurs, we write

$$a_n \to a \text{ as } n \to +\infty,$$

or

$$\lim_{n\to+\infty} a_n = a.$$

If no such number a exists, we say that the sequence $\{a_n\}$ is **divergent.**

Notes

1. For a convergent sequence, the inequalities say that a_n is sandwiched between $a - \varepsilon$ and $a + \varepsilon$ whenever $n \geq N$. (Picture in class) We have the following equivalent definition of a limit.

Rephrasing of Definition of Limit
The sequence $\{a_n\}$ **converges to the real number *a*** if, given any real number $\varepsilon > 0$ (no matter how small) there exists an integer $N > 0$ such that

$$n \geq N \Rightarrow a - \varepsilon < a_n < a + \varepsilon,$$

2. The definition also gives us the following method of proof

Proving that $\{a_n\}$ Converges to *a*
1. Assume that $\varepsilon > 0$ be given. (You have no say as to how big it is.)
2. Find an integer N so that, if $n \geq N$, $|a_n - a| < \varepsilon$.
3. Do the rough work. To do this, it often helps to work backwards: You want $|a_n - a|$, to be $< \varepsilon$, so calculate and simplify $|a_n - a|$, then set the result $< \varepsilon$, and try to find an n which makes this work (by "solving" the inequality or some other method, such *Stef's Sure fire method:* as finding something bigger that *can* be solved for n. This is then the N.
4. Write up your proof. Here is the skeleton of a proof:

Let $\varepsilon > 0$, and let $N = \ldots$ (or $N > \ldots$) .

Then, if $n \geq N$, one has

$$\begin{aligned} |a_n - a| &= \ldots \quad \text{(substitute formula)} \\ &= \ldots \quad \text{(simplify)} \\ &\ldots \\ &\leq \ldots \quad \text{(something with } N\text{'s)} \\ &< \varepsilon \ , \end{aligned}$$

since N was chosen $> \ldots$. QED
(Boxed part gives the definition. In other words, a proof is just the definition with some justification.)

Examples 8.4

A. $1/n \to 0$ as $n \to +\infty$

B. $\left\{\frac{n}{n+2}\right\}$ Use the above method to do it.

C. Use Set's Sure Fire Method to prove that $\frac{n^2 - 1}{2n^3 - n} \to 0$ as $n \to +\infty$.

Definition 8.5 (Negation of Defn. 8.3)
(a) The sequence $\{a_n\}$ does **not converge to *a*** if there exists some $\varepsilon > 0$ such that, for every integer $N > 0$, there is an $n \geq N$ with $|a_n - a| \geq \varepsilon$.
Equivalently:
The sequence $\{a_n\}$ does **not converge to *a*** if there exists some $\varepsilon > 0$ such that infinitely many of the terms of the sequence are a distance of ε or more from a.

(b) The sequence $\{a_n\}$ does **not converge** if it does not converge to any real number a. In other words, for all $a > 0$, there exists some $\varepsilon > 0$ such that, for every integer $N > 0$, there is an $n \geq N$ with $|a_n - a| \geq \varepsilon$.

Proving that $\{a_n\}$ Does Not Converge to a
There are two ways to do this:
1. Use the above definition: draw a picture to find an $\varepsilon > 0$ such that $|a_n - a| \geq \varepsilon$ for infinitely many n.
2. Assume it *does* converge to a, and then obtain a contradiction.

Examples 8.6

A. The sequence $\{\sqrt{n}\}$ does not converge. We prove this by method 1.

Proof. Let a be any real number, and choose $\varepsilon = 1$. (Actually, any positive number will do.) Now choose n to be any number $\geq (a+1)^2$. Then $\sqrt{n} \geq a+1$, and so

$$|a_n - a| = |\sqrt{n} - a| = \sqrt{n} - a \text{ (since } \sqrt{n} > a)$$
$$\geq 1 = \varepsilon.$$

In other words, $|a_n - a| \geq \varepsilon$ for infinitely many n. so we are done. ✣

B. $(-1)^n$ does not converge to any real number. We prove this by contradiction (method 2).

Proof

(Given)

S'pose the sequence $\{(-1)^n\}$ did converge to a real number a.

(What does this mean?)

Then, given any $\varepsilon > 0$ there would be a $N > 0$ such that

$$|(-1)^n - a| < \varepsilon \text{ whenever } n \geq N.$$

(Obtain a contradiction: Looking at a picture of the sequence, we notice that each term is a distance 1 from the next, thus the terms are at least 2 apart. so we choose $\varepsilon = 1$ to cause trouble.)

Now choose $\varepsilon = 1$. Then we are told there is an $N > 0$ such that $|(-1)^n - a| < \varepsilon$ whenever $n \geq N$. For n even, this means that $|1 - a| < 1$, so

$$0 < a < 2,$$

But then, for n odd (and larger than N)

$$|(-1)^n - a| = |-1-a| > 1 = \varepsilon,$$

a contradiction. ❢

Theorem 8.7 (Uniqueness of the Limit)
A sequence can converge to only one limit.

Proof in class

Definition The sequence $\{a_n\}$ is **bounded** if it is bounded as a set.† Similarly, the sequence $\{a_n\}$ is **bounded above** or **bounded below** if it has this property as a set.

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† That is, there exists a positive constant M such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. (See Exercise Set 4 #2.)

Examples 8.8

A. $(-1)^n$ **B.** $\left\{\frac{1}{n}\right\}$ **C.** Not $\{n\}$

Proposition 8.9
Every convergent sequence is bounded.

Proof Since $\{a_n\}$ converges, it has a limit a, and so, choosing $\varepsilon = 1$, there exists an integer N such that $|a_n - a| < 1$ for $n \geq N$. But then

$$|a_n| = |a_n - a + a| \leq |a_n - a| + |a| < 1 + |a|$$

for $n \geq N$. Hence, if

$$M = \max\{|a|+1, |a_1|, |a_2|, \ldots, |a_{n-1}|\},$$

one has $|a_n| \leq M$ for every n. ■

Definition 8.10 We say that $\{a_n\}$ **diverges to infinity,** and write $a_n \to +\infty$ as $n \to +\infty$ if, given any $M > 0$ (no matter how large),there exists an integer $N > 0$ such that $a_n > M$ whenever $n \geq N$. Similarly, $\{a_n\}$ **diverges to negative infinity,** and write $a_n \to -\infty$ as $n \to +\infty$ if, given any $M > 0$ (no matter how large),there exists an integer $N > 0$ such that $a_n < -M$ whenever $n \geq N$.

Examples 8.11

A. $\{n\}$ **B.** $\left\{\dfrac{3n^2 - n}{n+100}\right\}$

Theorem 8.12 ("$\frac{1}{0^+} = +\infty$ and $\frac{1}{+\infty} = 0$")

If $\{a_n\}$ is a sequence with positive terms, then $a_n \to +\infty$ as $n \to +\infty$ iff $\dfrac{1}{a_n} \to 0$ as $n \to +\infty$.

Proof in Exercise Set 8

Lemma 8.13 (Zero Limit)
The sequence $\{a_n\}$ converges to zero iff $\{|a_n|\}$ converges to zero.

Proof The result follows formally from the fact that

$$||a_n|-0| = |a_n| = |a_n-0|.$$

Exercise Set 8

1. Prove that, if $a_n = a$ for every n, then $a_n \to a$ as $n \to +\infty$.

2. Wade, p.37 # 1 a, b, c, d.

3. Prove that, if A is bounded above and non-empty, and $s = \sup A$, then there exists a sequence $\{a_n\}$ of points in A converging to s.

4. **(a)** Prove that, if $a_n \to a$ as $n \to +\infty$, then $|a_n| \to |a|$ as $n \to +\infty$. (You might want to consult Exercise Set 3 #4 for this. You might not.)
(b) is the converse of part (a) true? Prove of give a counterexample.

5. Prove that, if $|a| < 1$, then $a^n \to 0$ as $n \to \infty$ as follows.† First note that it suffices to show that $|a|^n \to 0$ as $n \to \infty$ by Lemma 8.13.

(a) The case $a = 0$ is trivial, so assume that $a \neq 0$, $|a| < 1$. Show that we can write $|a| = 1/(1+b)$ for some $b > 0$.

(b) Show that $(1+b)^n > 1+nb$, and hence that $|a|^n \leq 1/(1+nb)$ for $n \geq 1$.

(c) Deduce that $|a|^n \to 0$ as $n \to \infty$.

6. Prove that, if $a_n = b_n$ for all but a finite number of terms, then $\{a_n\}$ converges iff $\{b_n\}$ converges.

7. (Do not hand in)

(a) Prove: If $\{a_n\}$ is a convergent sequence and a is a real number such that $a_n = a$ for infinitely many n, then a_n converges to a.

(b) Does the result still hold if the word "convergent" is omitted? Prove, or give a counterexample.

8. Prove Theorem 8.12.

9. Prove:

Closed Intervals Are Sequentially Closed
If $\{x_n\}$ is any convergent sequence contained in the closed interval $[a, b]$ then its limit is also contained in $[a, b]$.

† We will see an easier proof using completelness in §10.

9. Rules for Limits

Theorem 9.1 (Rules for Limits) If $a_n \to a$ and $b_n \to b$ as $n \to +\infty$, then:

(a) $\lim_{n\to+\infty} (a_n \pm b_n) = a \pm b$

(b) $\lim_{n\to+\infty} (a_n b_n) = ab$

(c) $\lim_{n\to+\infty} \left(\frac{a_n}{b_n}\right) = \frac{a}{b}$, provided $b_n \neq 0$ for all n, and $b \neq 0$

(d) $\lim_{n\to+\infty} (a_n{}^p) = a^p$ for $p \in \mathbb{N}$, $p \geq 1$.

(e) $\lim_{n\to+\infty} \sqrt[k]{a_n} = \sqrt[k]{a}$, for $k > 0$ in $\mathbb{N}$, provided $a_n > 0$ for all n, and $a > 0$.

(f) (Preservation of order) If there exists $k \in \mathbb{N}$ such that $a_n \leq b_n$ for all $n \geq k$, then $a \leq b$.

Some of the Proof

(a) In class

(b) (First read the explanation in the book (p. 68).)

By Proposition 8.9, $\{b_n\}$ is a bounded sequence, whence there exists $M > 0$ with

$|b_n| < M$ ....(1)

for all n. Now let $\varepsilon > 0$ be given. Then, since $a_n \to a$ as $n \to +\infty$, there exists N_1 such that

$|a_n - a| < \frac{\varepsilon}{2M}$ ....(2)

whenever $n \geq N_1$. Also, since $b_n \to b$ as $n \to +\infty$, there exists N_2 such that

$|b_n - b| < \frac{\varepsilon}{2|a|+1}$ ....(3)

whenever $n \geq N_2$. (The reason for the +1 is in case $a = 0$.) Then, choosing $N = \max\{N_1, N_2\}$, we have

$$\begin{aligned}
|a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\
&\leq |a_n b_n - ab_n| + |ab_n - ab| \\
&= |b_n|\,|a_n - a| + |a|\,|b_n - b| \\
&< M|a_n - a| + |a|\,|b_n - b| && \text{(by (1))} \\
&< M\frac{\varepsilon}{2M} + |a|\frac{\varepsilon}{2|a|+1} && \text{(by (2) and (3))} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

(c) This will follow from (b) if we can prove that, if $b_n \neq 0$ and $b \neq 0$, then $1/b_n \to 1/b$ as $n \to +\infty$. Since $b_n \to b \neq 0$, there exists N_1 such that $n \geq N_1$ implies

$|b_n - b| < \frac{|b|}{2}$, so that, by Exercise Set 3 #4,

$\big||b_n| - |b|\big| \leq |b_n - b| < \frac{|b|}{2}$ also, giving |

$$|b_n| > b - \frac{|b|}{2} = \frac{|b|}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

for $n \ge N_1$. There also exists N_2 with

$$|b_n - b| < \frac{|b|^2\varepsilon}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

for $n \ge N_2$. Thus, choosing $N = \max\{N_1, N_2\}$, we have

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b_n - b|}{|bb_n|} = \frac{1}{|b|}\frac{1}{|b_n|}\,|b_n - b|$$

$$< \frac{1}{|b|} \cdot \frac{2}{|b|} \cdot \frac{|b|^2\varepsilon}{2} = \varepsilon,$$

and we are done. ☎

(d) Homework

(e) Proved later in the course, using functions

(f) In class

Theorem 9.2 (Sandwich Rule)
If $a_n \to L$ and $c_n \to L$ as $n \to +\infty$, and if there exists $k \in \mathbb{N}$ such that for all $n \ge k$, $a_n \le b_n \le c_n$, then $b_n \to L$ as $n \to +\infty$.

Proof By the hypotheses, there exist N_1 and N_2 such that

$n \ge N_1 \Rightarrow L - \varepsilon < a_n < L + \varepsilon$, and

$n \ge N_2 \Rightarrow L - \varepsilon < c_n < L + \varepsilon$.

Putting these together with the hypotheses gives

$n \ge \max\{N_1, N_2, k\} \Rightarrow L - \varepsilon < a_n \le b_n \le c_n < L + \varepsilon$, whence

$L - \varepsilon < b_n < L + \varepsilon$,

as required. ✙

Examples 9.3

A. $\lim_{n\to+\infty} 4/n + 3(2 - 1/n)^2$ **B.** $\lim_{n\to+\infty} \dfrac{2n^3 + 4n - 2}{\sqrt{3n^6 - n}}$

C. $\lim_{n\to+\infty} \dfrac{1}{n} \sin n$

Definition 9.4 The sequence $\{a_n\}$ **diverges to +∞** if, given any real number M, there exists $N \in \mathbb{N}$ such that $a_n > M$ whenever $n \ge N$. Similarly, the sequence $\{a_n\}$ **diverges to −∞** if, given any real number M, there exists $N \in \mathbb{N}$ such that $a_n < M$ whenever $n \ge N$.

Examples in class

Lemma 9.5
(a) $a_n \to +\infty$ as $n \to +\infty$ iff $-a_n \to -\infty$ as $n \to +\infty$
(b) (Comparison) If $a_n \to +\infty$ as $n \to +\infty$, and $b_n \ge a_n$ for $n \ge k \in \mathbb{N}$, then $b_n \to +\infty$ also.

Exercise Set 9

1. Prove that, if $c \in \mathbb{R}$, and $a_n \to a$ as $n \to +\infty$, then $ca_n \to ca$ as $n \to +\infty$ in two ways:

(a) directly, without quoting any part of Theorem 9.1;

(b) by quoting a certain part of 9.1 and a certain exercise from the past.

2. Wade, p. 42, #1a, 1b, 1d, 2a, 2d

3. (a) Prove the following:

> **Theorem 9.X2 (Bounded Sequence $\times$ Sequence Going To Zero)**
> If $\{a_n\}$ is a bounded sequence, and $b_n \to 0$ as $n \to +\infty$, then $a_n b_n \to 0$ as $n \to +\infty$.

(b) Give an example to show necessity of the assumption that $\{a_n\}$ is a bounded sequence.

4. Use the preceding exercise to redo Example C without using the sandwich rule.

5. Prove Theorem 9.1(d).

6. Let $a_n = \frac{1}{2} + \frac{1}{6} + \ldots + \frac{1}{n(n+1)}$. Determine whether $\{a_n\}$ converges, and if so, find its limit. (No hints. You're on your own.)

7. (Do not hand in) Prove:

> **Theorem 9.X7 (More $1/\infty = 0$)**
> S'pose that $a_n > 0$ for all n. Then: $a_n \to +\infty$ as $n \to +\infty$ iff $1/a_n \to 0$ as $n \to +\infty$.

10 Consequences of Completeness: Monotone Sequences and Cauchy Sequences

Here we see that the completeness axioms tell us that certain limits have to exist, even if we can't evaluate them explicitly.

Definition 10.1 The sequence $\{a_n\}$ is **(monotone) increasing** if $a_n \leq a_{n+1}$ for every n. It is **strictly (monotone) increasing** if $a_n < a_{n+1}$ for every n. It is **eventually (strictly) increasing** if there exists $k \in \mathbb{N}$ such that $a_n \leq a_{n+1}$ (resp. $a_n < a_{n+1}$) for every $n \geq k$.

Examples 10.2

A. $\{1, 1, 2, 2, 3, 3, \ldots\}$

B. $\{3, 1, 6, 5, 8, 4, 4, 4, 3, 3, 3, 3, 2, 2, 2, \ldots\}$

C. $\left\{1-\frac{1}{n}\right\}$

D. $\frac{n!}{2^n}$

E. $\left(1+\frac{1}{n}\right)^n$ is strictly increasing with n—we'll prove this much later, but think about it in the meantime.

From the definition, we have:

Showing that $\{a_n\}$ is Monotone Increasing

Show that $a_n \leq a_{n+1}$ for all n,

OR: Show that $a_{n+1} - a_n \geq 0$ for all n,

OR: Show that $\dfrac{a_{n+1}}{a_n} \geq 1$ for all n,

(OR: Take the derivative with respect to n and show it's positive.)

Recall from **§8** that the sequence $\{a_n\}$ is **bounded** (**bounded above; bounded below**) if it has this property as a *set*.

Proposition 10.3 (**Limits of Monotone Sequences**)
Let $\{a_n\}$ be a monotone increasing sequence. Then either:

(a) It is bounded above by M, in which case it converges to a limit $a \leq M$;

(b) It is not bounded above, in which case it diverges to $+\infty$.

Proof in class

Note The above proposition holds if "increasing" is replaced by "decreasing," "≥" by "≤," "above" by "below," and "+∞" by "−∞."

An Application

Prove that , if $|a| < 1$, then $a^n \to 0$ as $n \to \infty$.

(We already proved it the hard way in the Exercise Set 8. Here is an easier way.)

Proof By Lemma 8.13, it suffices to prove that $|a|^n \to 0$ as $n \to \infty$. Since

$$|a|^{n+1} = |a| \cdot |a|^n < |a|^n$$

we see that $\{|a|^n\}$ is a decreasing sequence, and bounded above by 0. Hence it converges to a limit $L \geq 0$. Applying Theorem 9.1 (limit of a product) to the equation

$$|a|^{n+1} = |a|\cdot|a|^n$$

now yields

$$L = |a|\cdot L,$$

showing that, either $|a| = 1$ (which it isn't) of $L = 0$.❇

Definition 10.4 The sequence $\{a_n\}$ is a **Cauchy sequence** if, given any $\varepsilon > 0$, there exists an integer $N > 0$ such that $|a_n - a_m| < \varepsilon$ whenever $n, m \geq N$.

Proposition 10.5
Every Cauchy sequence is bounded.

Proof Exercise Set 10

Definition 10.6 If $\{a_n\}$ is any sequence indexed on $\{1, 2, 3, \ldots\}$, and if $\{n_1 < n_2 < \ldots < n_r < \ldots\}$ is any subset $\{1, 2, 3, \ldots\}$, then the sequence $\{a_{n_1}, a_{n_2}, \ldots, a_{n_r}, \ldots\}$ is called a **subsequence** of $\{a_n\}$, and is written as $\{a_{n_r}\}$.

Notes
1. A subsequence is just a subset of the sequence, where the terms are written in the same order as in the original sequence.
2. By definition, one has $n_r \geq r$ for $r = 1, 2, 3, \ldots$

Examples 10.7
A. $\{2, 4, 6, 8, 10, \ldots, 2n, \ldots\}$ is a subsequence of $\{1, 2, 3, \ldots, n, \ldots\}$, with $a_{n_r} = 2r$.
B. $\{1, 1, 1, \ldots\}$ is a subsequence of $\{1, 2, 1, 2, 3, 1, 2, 3, 4, \ldots\}$
C. Every sequence is a subsequence of itself.

Lemma 10.8
(a) If $\{a_n\}$ converges to a, then so does every subsequence.
(b) Every subsequence of a monotone sequence is monotone.
(c) Every subsequence of a bounded sequence is bounded.

Proof of (a) Let $\varepsilon > 0$. Since $a_n \to a$ as $n \to +\infty$, there is an N such that $r \geq N$ implies $|a_r - a| < \varepsilon$. But then, since $n_r \geq r$, one has, for $r \geq N$, $|a_{n_r} - a| < \varepsilon$ for $r \geq N$ also. ◆

Lemma 10.9
If a subsequence of a Cauchy sequence converges, then so does the original sequence.

Proof in Exercise Set 10

Theorem 10.10
Every sequence has a monotone subsequence

Proof S'pose $\{a_n\}$ has no decreasing subsequences. Then, as a set, it must have a least element, which we call a_{n_1}, say. (Otherwise, we can define a decreasing subsequence inductively. . .) Similarly, $\{a_k \mid k > n_1\}$ also has a least element, which we call a_{n_2} . Note that $a_{n_2} \geq a_{n_1}$ by definition. Continuing in this way gives the required increasing subsequence. ✣

Putting these all together gives:

Theorem 10.11 (Completeness of ℝ)
Every Cauchy sequence converges.

Proof If $\{a_n\}$ is Cauchy, then it is bounded by Proposition 10.5, and has a monotone subsequence, by Theorem 10.10. Since subsets of bounded sets are bounded, this monotone subsequence is bounded as well, whence it converges by Proposition 10.3. Finally, since now the subsequence of the Cauchy sequence $\{a_n\}$ converges, so does the sequence $\{a_n\}$ itself, by Lemma 10.9 ❖

Exercise Set 10

1. Give examples of:

(a) a sequence that diverges to $+\infty$ but is not increasing
(b) a sequence that is bounded but does not converge
(c) a sequence that converges, but is not monotone

2. Show that the following are convergent:

(a) $\frac{n}{2^n}$ (b) $\frac{n+1}{2n+1}$ (c) $\frac{n!}{1\cdot 2\cdot 3 \cdots (2n-1)}$.

3. Let $a_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2}$. Show that $\{a_n\}$ converges to a limit a with $1 \leq a \leq 2$. Is $\frac{5}{4} \leq a$?

4. Prove Proposition 10.5: Every Cauchy sequence is bounded. (You might want to use 8.9 as a guide, but there are other ways of proving it.)

5. Prove Lemma 10.9: If a subsequence of a Cauchy sequence converges, then so does the original sequence.

6. Prove:

Bolzano-Weierstrass Theorem for Sequences
If $\{x_n\}$ is any sequence in the closed interval $[a, b]$, then it has a subsequence converging to a limit in $[a, b]$.

(You might want to glance back to Exercise set 8 for a part of this.)

11. Limit of a Real-Valued Function

(§3.1 in Wade) The rigorous definition of a limit and continuous function is one of the greatest accomplishments of 17th century mathematics, perfected by Cauchy, Bolzano and others. (The exact definition still varies from author to author—see below.) Before we define a limit, we need to take about *limit points* of a set. Intuitively, these are points at which the elements of the set "approach arbitrarily closely."

Definition 11.1 Let D be any set of real numbers. A **limit** or **accumulation point** of D is a point $a \in \mathbb{R}$, not necessarily in D, such that every open interval containing a contains points of D other than a.

In other words—and this is how you prove that a point a is a limit point:

Proving that a is a Limit Point of D
Let $\varepsilon>0$ be arbitrary. Prove that $(a-\varepsilon, a+\varepsilon)\cap D$ contains points other than a.

Examples 11.2

A. We find all the limit points of $[0, 1\}$.

B. $D = \{1, 2, 3\}$

B. $D = \mathbb{Q}$

C. $D = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N}, n \geq 1\right\}$

D. $D = \left\{ \frac{n}{|n|+1} \;\middle|\; n \in \mathbb{Z}\right\}$

Note a is a limit point of D iff there is a sequence $\{d_n\}$ of points in D with limit a. (Proved in homework)

In class, we arrive at the following definition after some discussion.

Definition 11.3 Let f: $D \to \mathbb{R}$, and let a be a limit point of D, and let L be a real number. Then we say that f **has limit L as x approaches a** if, given any $\varepsilon>0$ there exists $\delta>0$ such that if $x \in D$, then

$$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

We also say $f(x)\to L$ as $x\to a$, or $\lim_{x\to a} f(x) = L$. Also, if f has no limit as x approaches a, we say that f **has no finite limit as x approaches a**.

Notes

1. Saying that $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$ is equivalent to saying that

$$x \neq a,\ a-\delta < x < a+\delta \Rightarrow L-\varepsilon < f(x) < L+\varepsilon.$$

2. If a is not a limit point of the domain of f, then we have not defined what it means for f to have limit L as $x\to a$.

3. Important We shall *not* say that "$\lim_{x\to a}f(x)$ does not exist" if f has no finite limit as $x\to a$ (and this is where we differ from the book) as this need not be true: we will consider *infinite limits* later on, and write $\lim_{x\to a}f(x) = +\infty$. The book is confusing on this point, and says that, eg. $\lim_{x\to 0}(1/x^2)$ does not exist, and then later says that it equals $+\infty$. This is very ugly. When *we* say that the limit does not exist, we shall really mean it! In the meantime, don't say it at all. Just say "f has no finite limit as $x\to a$" or "f does not converge as $x\to a$."

4. Some authors define limits as follows—and this is the way it was taught to me at Liverpool: if a is any *cluster* point of the domain of f,† f has limit L as x approaches a if, given any $\varepsilon>0$ there exists $\delta>0$ such that $|x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$. The difference is that x is allowed to be a. This has the advantage that it is more natural and makes discrete domains easier to deal with, but it is not the same as the way you were taught in Calc I, when you were told that the limit of $f(x)$ *as* $x\to a$ has nothing to do with $f(a)$ itself.

5. Wade requires more than we do: he says that the function f must be defined on some deleted interval of a in order to talk about the limit as $x\to a$. Our use of limit points permits talking about limits in more general cases; for example, let $f(x) = x$ if x is irrational, so that the domain of f is $\mathbb{R}-\mathbb{Q}$. Then we can still say that $f(x)\to 0$ as $x\to 0$, but Wade cannot. Also, we can say that, for example, $\lim\limits_{x\to 0} \sqrt{x} = 0$, whereas Wade needs to use two-sided limits to talk about this.

Our definition gives us the following method for proving that $\lim\limits_{x\to a} f(x) = L$..

Proving that $f(x)\to L$ as $x\to a$
(1) Let $\varepsilon>0$ be arbitrary.
(2) Find a suitable $\delta>0$ such that $\lvert f(x) - L\rvert < \varepsilon$ whenever $0 < \lvert x-a\rvert < \delta$.

Examples 11.4

A. $\lim\limits_{x\to 3} 2x-1 = 5$

B. $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2} = 4$

C. $\lim\limits_{x\to 2} x^2+1 = 5$

D. $\lim\limits_{x\to a} \dfrac{1}{x} = \dfrac{1}{a}$ if $a \neq 0$

E. $\lim\limits_{x\to a} \sqrt{x} = \sqrt{a}$

G. $\lim\limits_{x\to a} \sin x = \sin a$

Remarks 11.5

In logical syntax, the statement that $f(x)\to L$ as $x\to a$ has the following form:

Logical Form of $f(x)\to L$ as $x\to a$
$\forall\varepsilon>o\ \exists\delta>0$ such that $\forall x,\ 0 < \lvert x-a\rvert < \delta \Rightarrow \lvert f(x)-L\rvert < \varepsilon$

This gives us the following form of its negation:

Logical Form of $f(x)\not\to L$ as $x\to a$
$\exists\varepsilon>o$ such that $\forall\delta>0$, $\exists x$ with $0 < \lvert x-a\rvert < \delta$ but $\lvert f(x)-L\rvert \geq \varepsilon$

† that is, either a limit point or a point in the domain itself

Thus, to prove that $f(x)$ does not approach the limit L as $x \to a$, you must do the following.

> **Proving that $f(x) \not\to L$ as $x \to a$**
> (1) Find a suitable $\varepsilon > 0$ such that
> (2) Given any $\delta > 0$, there are elements x with $0 < |x-a| < \delta$ but with $|f(x)-L| \geq \varepsilon$.

Examples 11.6

A. $\lim_{x \to 3} \frac{1}{x-1} \neq 1$

Proof Let $\varepsilon = 2/3$. Then, given any $\delta > 0$, choose $x \in (3-\delta, 3+\delta) \cap (2, 4)$. Then two interesting things are true: **(a)** $|x-3| < \delta$ (which we need for the contradiction), and **(b)** $2 < x < 4$, which is useful for inequalities.

Now, for such x, we have $|x-3| < \delta$ and yet

$$|f(x)-1| = \frac{|x|}{|x-1|} = \frac{x}{x-1} \quad \text{since everything in sight is positive}$$
$$> \frac{2}{3}$$
$$= \varepsilon \quad \text{by some strange coincidence}$$

(Highlighted parts give proof that $f(x) \not\to 1$ *as* $x \to 3$.) ❇

B. $\frac{1}{x^2}$ has no (real) limit as $x \to 0$

Proof postponed until after the following theorem

> **Theorem 11.7 (Sequential Criterion for Existence of Limits)**
> Let a be a limit point of $\mathcal{D}(f)$. Then the following are equivalent;
> **(a)** $f(x) \to L$ as $x \to a$.
> **(b)** For every sequence $\{a_n\} \subset \mathcal{D}(f)$ with $a_n \neq a$ for each n and $a_n \to a$ as $n \to \infty$, the sequence $\{f(a_n)\}$ converges to L.

Note In the above proof, we used the following fact:

> **Fact** If a is a limit point of D, then there exists a sequence $\{d_n\}$ of points in D converging to a (proved in the exercise set below).

Theorem 11.7 gives us a new recipe box for proving *negative* results:

Proving that $f(x) \not\to L$ as $x \to a$ Using a Sequence
Come up with a sequence of points $\{a_n\}$ in $\mathcal{D}(f)$ with $a \neq a_n \to a$, but $f(a_n) \not\to L$.

Proving that $f(x)$ Has No Finite Limit Using a Sequence
Come up with a sequence of points $\{a_n\}$ in $\mathcal{D}(f)$ with $a \neq a_n \to a$, but with $\{f(a_n)\}$ having no finite limit.

Example Prove that $\frac{1}{x^2}$ has no (real) limit as $x \to 0$.

Proof (using recipe box) Let $a_n = \frac{1}{n}$. Then $a_n \to 0$ as $n \to +\infty$, but

$$f(a_n) = f\left(\frac{1}{n}\right) = n^2,$$

and we know from the rules of limits that $n^2 \to +\infty$ as $n \to +\infty$, so that $\{f(a_n)\}$ has no real limit as $n \to +\infty$. ❤

Infinite Limits

Definition 11.8 Let a be a limit point of $\mathcal{D}(f)$. We say that **$f(x) \to +\infty$ as $x \to a$** if, given any $N \in \mathbb{R}$, there exists $\delta > 0$ such that

$$0 < |x-a| < \delta \Rightarrow f(x) > N.$$

Similarly, we say that **$f(x) \to -\infty$ as $x \to a$** if, given any $N \in \mathbb{R}$, there exists $\delta > 0$ such that

$$0 < |x-a| < \delta \Rightarrow f(x) < N.$$

Note This gives us the following procedure box:

To prove $f(x) \to +\infty$ as $x \to a$
(1) Let $N > 0$ be arbitrary.
(2) Find a suitable $\delta > 0$ such that $f(x) > N$ whenever $0 < |x-a| < \delta$.
To prove $f(x) \not\to +\infty$ as $x \to a$
(1) Choose a suitably large N such that
(2) Given any $\delta > 0$ there exist x with $0 < |x-a| < \delta$, but $f(x) \leq N$.
OR
Come up with a sequence of points $\{a_n\}$ in $\mathcal{D}(f)$ with $a_n \to a$, but $f(a_n) \not\to +\infty$.

Note The last step in the above procedure box is justified by the "observation" that Theorem 11.7 continues to hold if L is replaced by $+\infty$. (Think about it after you have proved the theorem.)

Examples 11.9 The following limits do not exist, either as finite numbers, or as infinite limits. We'll just say that **the limits do not exist**. (Careful: some authors say that a limit doesn't exist even if it does exist but is infinite, e.g. Wade.)

A. $\lim_{x\to 0} \frac{1}{x^2}$ **B.** $\lim_{x\to 0} \frac{1}{x}$ does not exist

C. $\lim_{x\to 0} \frac{\cos x}{x^2}$

Last Example Let us investigate $\lim_{x\to a} f(x)$, where f: $\mathbb{R}\to\mathbb{R}$ is given by

$$f(x) = \begin{cases} x & \text{if } x \text{ is irrational;} \\ 0 & \text{if } x \text{ is rational.} \end{cases}$$

Exercise Set 11

1. Find all limit points of the each of the following. Justify all claims.

(a) $A = \mathbb{N}$ (b) $A = \mathbb{R}-\mathbb{Q}$

(c) $A = \left\{ \frac{n}{n+1} \;\middle|\; n \in \mathrm{N} \right\}$ (d) $A = \left\{ \frac{p^2}{q^2} \;\middle|\; p, q \in \mathrm{Z}, q \neq 0 \right\}$

2. Prove each of the following.

(a) $\lim_{x\to 4} 3-2x = -5$ (b) $\lim_{x\to a} x^2 = a^2$

(c) $\lim_{x\to 3} \frac{x-2}{x-3}$ does not exist (d) $\lim_{x\to 3} \frac{x-2}{(x-3)^2} = +\infty$

(e) $\lim_{x\to a} |x| = |a|$

3. Prove that a is a limit point of D iff there is a sequence $\{d_n\}$ of points in D with limit a.

4. Prove Theorem 11.7.

5. A **neighborhood** of $a \in \mathbb{R}$ is a set $U \subset \mathbb{R}$ such that there exists $\delta>0$ with $(a-\delta, a+\delta) \subset U$.

(a) Show that the union of an arbitrary collection of neighborhoods of a is again a neighborhood of a.
(b) Show that the intersection of two neighborhoods of a is a neighborhood of a.
(c) Use part (b) to show that by induction the intersection of finitely many neighborhoods of a is a neighborhood of a.
(d) Show that the intersection of an arbitrary collection of neighborhoods of a need not be a neighborhood of a.

6. A **deleted neighborhood** of $a \in \mathbb{R}$ is a set $N \subset \mathbb{R}$ of the form $N = U - \{a\}$, where U is a neighborhood of a. In other words, N is any set that:

(1) contains $(a-\delta, a+\delta)-\{a\}$ (called a *deleted interval*) for some $\delta>0$, but

(2) does not contain the point a.

Prove:

(a) a is a limit point of D iff every deleted neighborhood of a contains points of D.

(b) The intersection of two deleted neighborhoods of a is a deleted neighborhood of a.

(c) If a is a limit point of $\mathcal{D}(f)$, then:

Definition of Limit in Terms of Neighborhood

$f(x)\to L$ (with L finite) as $x\to a$ iff, given any $\varepsilon>0$, there exists a deleted neighborhood N of a such that $|f(x) - L| < \varepsilon$ whenever $x \in N$.

7. Dirichlet's Function Let f: $(0, 1)\to\mathbb{R}$ be given by

$$f(x) = \begin{cases} \dfrac{1}{q} & \text{if } x = \dfrac{p}{q} \text{ in lowest terms with } p, q \in \mathbb{N}; \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that $f(x)\to 0$ as $x\to a$ for every a in the domain of f!

Thoughts Did you ever feel that mathematics should include "infinitesimal numbers?" What kinds of properties should these infinitesimal numbers have? How might you define what is meant by a limit using infinitesimal numbers? How does our definition of a limit effect your view of infinitesimals?

12. Properties of Limits

We can finally justify many of the "rules for limits" we learned in Calc I. We shall see that their proofs follow from the sequential criterion (11.7).

Theorem 12.1 (Uniqueness of the Limit)
If $f(x) \to L_1$ as $x \to a$ and $f(x) \to L_2$ as $x \to a$, then $L_1 = L_2$.

Proof of finite case (L_1 and L_2 finite) in class, using sequences and 8.7

Moreover, Theorem 9.1 gives us the following:

Theorem 12.2 (Algebraic Properties of Limits)
Assume that a is a limit point of $\mathcal{D}(f) \cap \mathcal{D}(g)$, L and $\bar{L}$ are finite, $\lim_{x\to a} f(x) = L$, and $\lim_{x\to a} g(x) = \bar{L}$. Then:

(a) $\lim_{x\to a} [f(x)+g(x)] = L+\bar{L}$

(b) $\lim_{x\to a} [f(x)g(x)] = L\bar{L}$

(c) $\lim_{x\to a} \left[\frac{1}{g(x)}\right] = \frac{1}{\bar{L}}$, if $\bar{L} \neq 0$.

(d) $\lim_{x\to a} f(x)^p = L^p$ for $p \in \mathbb{N}$.

(e) $\lim_{x\to a} \sqrt[k]{f(x)} = \sqrt[k]{L}$ for $k > 0$ in $\mathbb{N}$, provided $f(x) > 0$ in a neighborhood of a.

(f) Suppose that there exists a deleted neighborhood N of a such that, for every $x \in N \cap \mathcal{D}(f) \cap \mathcal{D}(g)$, one has $f(x) \leq g(x)$. Then $L \leq \bar{L}$.

Further, if L and/or $\bar{L}$ is infinite, the above results continue to hold whenever they make sense, if we define

$L + \infty = \infty$, $(L \neq -\infty)$

$-\infty + L = -\infty$, $(L \neq +\infty)$

$L \times \pm\infty = \pm\infty$, $(L \geq 0)$

$\frac{1}{\pm\infty} = 0$.

Also, 9.2 gives us:

Theorem 12.3 (Sandwich Rule)
Let a be a limit point of $\mathcal{D}(f) \cap \mathcal{D}(g) \cap \mathcal{D}(h)$, with $f(x) \to L$ and $h(x) \to L$ as $x \to a$, and assume there exists a deleted neighborhood of a such that for all $x \in N \cap \mathcal{D}(f) \cap \mathcal{D}(g) \cap \mathcal{D}(h)$, $f(x) \leq g(x) \leq h(x)$, then $g(x) \to L$ as $x \to a$.

Exercise Set 12

1. Complete the proof of Theorem 12.1 (infinite case)

2. Prove: If $f(x) \to L$ as $x \to a$, then:

(a) If $k < L$, there exists $\delta > 0$ such that $f(x) > k$ whenever $0 < |x-a| < \delta$.

(b) If $K > L$, there exists $\delta>0$ such that $f(x) < K$ whenever $0 < |x-a| < \delta$.
(c) f is bounded on some deleted neighborhood of a.
3. Rewrite the following proof from Parzynski & Zipse, especially the last part. (This is a book I was considering earlier, but decided against): "Consider $\lim_{x\to 0}\sin(1/x)$. The function $f(x) = \sin(1/x)$ is defined for all real numbers $x\neq 0$. We observe that $f(x) = 0$ for the values $x = 1/n\pi$ ($n = 1, 2, 3, \ldots$) and that $f(x) = 1$ for the values $x = 2/(4n+1)\pi$ ($n = 1, 2, 3, \ldots$).† Thus in each deleted neighborhood of 0 there are points x with $f(x) = 0$ and there are points $\bar{x}$ with $f(\bar{x}) = 1$. By Theorem [12.1] the limit of f cannot be both 0 and 1. Hence $\lim_{x\to 0}\sin(1/x)$ does not exist." [Hint, it's only the next-to-last sentence that needs a little work.]

4. Give a trigonometric proof that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$.

5. Prove that $\lim\limits_{x\to 0} x\sin(1/x) = 0$.

6. Let $f(x) = \begin{cases} 0 \text{ if } x \text{ is rational} \\ 1 \text{ if } x \text{ is irrational} \end{cases}$. Use a sequential argument to prove that $\lim\limits_{x\to a} f(x)$ exists for no values of x.

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† The book has a misprint here; it writes $x = 2/(4n-1)\pi$, which is wrong.

13 One-Sided Limits & Limits at Infinity

Definition 13.1 Suppose that a is a limit point of $\mathcal{D}(f)$. Then we say that $f(x) \to L$ as $x \to a^+$ if the following two conditions hold:

(a) The point a is a limit point of $\mathcal{D}(f) \cap (a, +\infty)$.

(b) Given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \in \mathcal{D}(f) \text{ and } 0 < x - a < \delta \text{ implies } |f(x) - L| < \varepsilon.$$

We call L the **right-hand limit** of $f(x)$ as x approaches a, and say that $f(x)$ approaches L as x approaches a from the right. Similarly, we say that $f(x) \to L$ as $x \to a^-$ if:

(a) The point a is a limit point of $\mathcal{D}(f) \cap (-\infty, a)$.

(b) Given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \in \mathcal{D}(f) \text{ and } -\delta < x - a < 0 \text{ implies } |f(x) - L| < \varepsilon.$$

Notes 13.2

1. We make similar definitions for $f(x) \to \pm\infty$ as $x \to a^+$ or $x \to a^-$. (See exercise set.)

2. The right limit cannot exist at all if condition (a) is not met—for instance, $\lim\limits_{x \to 0^-} \sqrt{x}$ does not exist, because 0 is not a limit point of $\mathcal{D}(f) \cap (-\infty, 0)$. Indeed, the latter set is empty, and hence has no limit points.

3. From the definition, it is immediate that, if a is a limit point of both $\mathcal{D}(f) \cap (a, +\infty)$ and $\mathcal{D}(f) \cap (-\infty, a)$, then $f(x) \to L$ as $x \to a$ iff $f(x) \to L$ as $x \to a^+$ and $f(x) \to L$ as $x \to a^-$.

4. If $\mathcal{D}(f)$ contains no points to the left of a, then $f(x) \to L$ as $x \to a$ iff $f(x) \to L$ as $x \to a^+$; if $\mathcal{D}(f)$ contains no points to the right of a, then $f(x) \to L$ as $x \to a$ iff $f(x) \to L$ as $x \to a^-$. (Proved in the exercises.) In other words, depending on the domain of f, a limit can exist even if one of the sided limits do not.

5. We also have the following very useful consequence of the definition, proved in the exercise set. (There is a similar one for left limits.)

Sequential Criterion for Right Limit
Let a be a limit point of $\mathcal{D}(f)$. Then $f(x) \to L$ as $x \to a^+$ iff:
(a) There is a sequence $\{a_n\}$ of points in $\mathcal{D}(f)$ with $a_n \to a$ and $a_n > a$ for all n
(b) For every sequence $\{a_n\}$ of points in $\mathcal{D}(f)$ with $a_n \to a$ and $a_n > a$ for all n, one has $f(a_n) \to L$ as $n \to +\infty$.

6. In view of (5), *all the algebraic rules that hold for ordinary limits also hold for one-sided limits*, so we don't need any more δs and εs here.

Examples 13.3

A. Let $f(x) = \dfrac{x-2}{|x-2|}$. Then $\lim\limits_{x \to 2^-} f(x) = -1$ and $\lim\limits_{x \to 2^+} f(x) = 1$. (Reason: use the sequential criterion.)

B. Let $f(x) = \dfrac{x^2-3x-4}{(x-6)^2}$ .Then $\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} f(x) = +\infty$, whence so does $\lim_{x\to -3} f(x)$. (Reason: Use the rules for limits.)

Definition 13.4 We say that $f(x)\to L$ as $x\to+\infty$ if:

(a) For every $M \in \mathbb{R}$, there exists $x \in \mathcal{D}(f)$ with $x > M$.

(b) Given any $\varepsilon > 0$, there exists M such that

$x \in \mathcal{D}(f)$ and $x > M$ implies $|f(x) - L| < \varepsilon$.

Similarly, we say that $f(x)\to L$ as $x\to-\infty$ if:

(a) For every $M \in \mathbb{R}$, there exists $x \in \mathcal{D}(f)$ with $x < M$.

(b) Given any $\varepsilon > 0$, there exists M such that

$x \in \mathcal{D}(f)$ and $x < M$ implies $|f(x) - L| < \varepsilon$.

Remark We have the following sequential equivalent definition which—as usual—is only useful when we want to prove a negative result.

Lemma 13.5 (Sequential Definition for Limits at Infinity)
$f(x)\to L$ as $x\to+\infty$ iff the following two properties hold:
(a) There is a sequence $\{d_n\} \subset \mathcal{D}(f)$ with $d_n\to+\infty$ as $n\to+\infty$.
(b) For every sequence $\{a_n\} \subset \mathcal{D}(f)$ with $a_n\to+\infty$ as $n\to+\infty$, one has $f(a_n)\to L$ as $n\to+\infty$.

Proof in exercises

Examples 13.6

A. $f(x) = \dfrac{x^2 - 2}{3x^2 + x}$, as $x\to-\infty$. **B.** $f(x) = \sin\sqrt{x}$ has no limit as $x\to+\infty$.

C. $f(x) = \dfrac{\sin x}{x} \to 0$ as $x\to+\infty$. **D.** $f(x) = \dfrac{x^3}{\sqrt{x^5 - 3x}}$ as $x\to+\infty$.

Exercise Set 13

1. Prove the sequential criterion for one-sided limits: Let a be a limit point of $\mathcal{D}(f)$. Then $f(x)\to L$ as $x\to a^+$ iff:

(a) There is a sequence $\{a_n\}$ of points in $\mathcal{D}(f)$ with $a_n\to a$ and $a_n > a$ for all n

(b) For every sequence $\{a_n\}$ of points in $\mathcal{D}(f)$ with $a_n\to a$ and $a_n > a$ for all n, one has $f(a_n)\to L$ as $n\to+\infty$.

2. Prove: If a is a limit point of $\mathcal{D}(f)$ and $\mathcal{D}(f)$ contains no points to the left of a, then $f(x)\to L$ as $x\to a$ iff $f(x)\to L$ as $x\to a^+$.

3. Give a definition of what is means for $f(x)\to+\infty$ as $x\to a^+$, and give an example of a function $f(x)$ with $\lim_{x\to -3^+} f(x) = -\infty$ and $\lim_{x\to -3^+} f(x) = +\infty$, justifying your claims.

4. Prove Proposition 13.5.

14 Continuity

The definition of a continuous function is one of the great breakthroughs in 19th century mathematics, and is due to Augustin Louis Cauchy (1789–1857). Cauchy's papers in the 1820s gave the definition of limits and continuity that we use to this day. His work began the great nineteenth century project of introducing logical rigor into mathematics, establishing a precedent that mathematicians follow to this day.

Definitions 14.1 Let $D \subset \mathbb{R}$ and let f: $D \to \mathbb{R}$, with $a \in D$. Then f is **continuous at a** if, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \in D \text{ and } |x - a| < \delta \text{ implies } |f(x) - f(a)| < \varepsilon.$$

f is **continuous on the subset $E \subset D$** if it is continuous at every point in E.

Remarks 14.2

1. Notice that we no longer require a to be a limit point, but we do require a to be in the domain of f.

2. Notice also that we allow x to equal a when taking the limit.

3. It is a consequence of the definition that, if a does happen to be a limit point of f, then f is continuous at a iff $f(x) \to f(a)$ as $x \to a$. (This is the definition you learned in elementary school: that f is continuous at a iff it has a limit as $x \to a$, and this limit equals $f(x)$.) In fact, we have the following theorem.

Theorem 14.3 (Continuity and Limits)
f: $D \to \mathbb{R}$ is continuous at the point $a \in \mathbb{D}$ iff:
(a) The point a is not a limit point of D.
or
(b) The point a is a limit point of D and $\lim_{x \to a} f(x)$ exists and equals $f(a)$.

Proof in class

Note The theorem says that, if a is not a limit point of the domain of f, then f is *automatically* continuous at a.

The above result gives us many examples we can use:

Examples 14.4

A. f: $\mathbb{R} \to \mathbb{R}$; $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ for any $a_i \in \mathbb{R}$. That is, arbitrary polynomial functions are continuous.

B. f: $\mathbb{Z} \to \mathbb{R}$; $f(x)$ = the number of planets in the Andromeda galaxy with mass x times that of earth.

C. f: $\mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} x & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$ is continuous at exactly one point: 0.

Proof of C First, s'pose $a = 0$, then certainly a is a limit point of $\mathcal{D}(f)$ (every point is, in fact) whence it suffices, by Theorem 14.3, to show that $\lim_{x\to 0} f(x) = 0$. But, by definition of x, we have the inequality

$$0 \le |f(x)| \le x.$$

Since the left- and right-hand side functions both approach 0 as $x\to 0$, we obtain

$$\lim_{x\to 0} f(x) = 0$$

by the sandwich theorem, showing that f is continuous at $a = 0$.

Now s'pose $a \ne 0$. Then, again by Theorem 14.3, it suffices to show that $\lim_{x\to a} f(x)$ does not exist. For this we can use a sequential argument: By a very early result (Theorem 5.4) we can choose two sequences, $\{r_n\} \subset \mathbb{Q}$ and $\{i_n\} \subset \mathbb{R} - \mathbb{Q}$, both approaching a as $n\to+\infty$. But clearly, $f(r_n) = r_n$ and hence approaches a, whereas $f(i_n) = 0$ for al n, and hence approaches 0. Since $a \ne 0$, it follows that $\lim_{x\to a} f(x)$ does not exist. ✈

Parts of the following are consequences of earlier theorems.

Theorem 14.5 (Algebra of Continuous Functions)
Suppose that the domains of f, g, and h all contain a, and that all three functions are continuous at a. Then so are λf (λ constant) $f\pm g$, fg, fp for $p \in \mathbb{N}$, $f^{1/p}$ and f^g if $f(a) \ge 0$, and f/g if $g(a) \ne 0$.

More interestingly, we have the following[†]

Theorem 14.6 (Continuity of Composites)
Let f: $D\to\mathbb{R}$ and g: $E\to\mathbb{R}$ be such that $f(a) \in E$, and suppose that f is continuous at a and g is continuous at $f(a)$. Then $g\circ f$ is continuous at a.

Proof (and let us prove it directly for a change, and not fiddle around with sequences) First notice that the domain of $g\circ f$ is $f^{-1}(\mathcal{D}(g)) = f^{-1}(E)$, and that this contains the point a. **Now, let $\varepsilon > 0$.** Since g is continuous at $f(a)$, there exists $\Delta > 0$ such that

$x \in E$ and $|x - f(a)| < \Delta$ implies $|g(x) - g(f(a))| < \varepsilon$ .... (1)

But, since f is continuous at a, there exists $\delta > 0$ such that

$x \in D$ and $|x - a| < \delta$ implies $|f(x) - f(a)| < \Delta$. .... (2)

Now, s'pose $x \in f^{-1}(E)$ (the domain of $g\circ f$) **and $|x - a| < \delta$. Then**, by (2), $|f(x) - f(a)| < \Delta$, whence, by (1) , since now $f(x) \in E$, one has

$$|g(f(x)) - g(f(a))| < \varepsilon.$$

In other words,

$$|g\circ f(x) - g\circ f(a)| < \varepsilon,$$

as required. ❽

[†] I can't understand why Kosmala is so nervous about domains in his Theorem 4.1.9.

Examples 14.7 We may combine the above two theorems to prove that the following are continuous (also see the exercise set).

A. $f\colon \mathbb{R}\to\mathbb{R};\ f(x) = \dfrac{x}{x^2+4}$

B. $f\colon \mathbb{R}-\{0\}\to\mathbb{R};\ f(x) = (4x-2)\sin(x^2-3x)$

Theorem 14.8 (Sequential Criterion for Continuity)
The function $f\colon D\to\mathbb{R}$ is continuous at $a \in D$ iff, for every sequence $\{a_n\}$ of points in D converging to a, $f(a_n)$ converges to $f(a)$.

Proof in class.

Exercise Set 14

1. Let $f\colon \mathbb{R}\to\mathbb{R}$ given by $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational;} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$ Prove that f is discontinuous at every point.

2. Prove that, if $a \in \mathcal{D}(f)$ is also a limit point of $\mathcal{D}(f)$, then f is continuous at a if and only if $\lim\limits_{x\to a} |f(x) - f(a)| = 0$.

3. Prove that the function $f\colon \mathbb{R}\to\mathbb{R}$ given by $f(x) = \begin{cases} x\sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \end{cases}$ is continuous at every point. (You may assume continuity of the sine function.)

4. Prove: If f is continuous at a and if $f(a) > 0$, then there exists an interval $(a-\delta,\ a+\delta)$ such that $f(x) > 0$ for every $x \in (a-\delta,\ a+\delta)\cap\mathcal{D}(f)$. (That is, if a continuous function is positive at one point, it remains positive "nearby.")

5. Prove the following generalization of Theorem 14.8 (Sequential Criterion for Continuity): The function $f\colon D\to\mathbb{R}$ is continuous at $a \in D$ iff, for every *monotone* sequence $\{a_n\}$ of points in D converging to a, $f(a_n)$ converges to $f(a)$.

6. Show that the following function $f\colon \mathbb{R}\to\mathbb{R}$ is discontinuous at every rational number other than 0, and continuous at every irrational number.

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x \neq 0,\ x \text{ is rational, and } x = \frac{p}{q} \text{ in lowest terms with } p, q \in Z \\ 0 & \text{if } x \text{ is irrational or } x = 0 \end{cases}$$

7. Dirichlet's Function Returns Let $f\colon (0, 1)\to\mathbb{R}$ be given by

$$f(x) = \begin{cases} \dfrac{1}{q} & \text{if } x = \dfrac{p}{q} \text{ in lowest terms with } p, q \in \mathbb{N}; \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

By quoting the result of a certain previous exercise, prove in one line that $f(x)$ is continuous at every irrational number, but discontinuous at every rational number!

8. (For your eyes only; do not dare hand in: A theorem all the textbooks are too scared to state and prove:) **The Big Kahuna—Continuity of Closed Form Functions**
Definition We define a **closed form** function inductively as follows:

(1) $f\colon \mathbb{R}\to\mathbb{R};\ f(x) = x$, and $g\colon \mathbb{R}\to\mathbb{R};\ g(x) = 1$, $h\colon \mathbb{R}\to\mathbb{R};\ h(x) = \sin x$, and $s\colon \mathbb{R}\to\mathbb{R};\ s(x) = |x|$ are closed form functions.

(2) If f and fg are closed form functions, then so are the following: λf (λ constant), $f\pm g$, fg, f/g, $f\circ g$, f^{-1} (inverse of f, if it exists) and f^g, on their respective domains (which you can calculate).

(a) Prove that the following are closed form functions (domains are ignored)

(i) $f(x) = \dfrac{\sin x}{x^2 - 1}$ (ii) $g(x) = \tan x$

(iii) $g(x) = |2x-e^x|$ (iv) $h(x) = \sqrt{\dfrac{\sin^{-1}\sqrt{x^2 - 1}}{e^x}}$

(b) Why is the following true? (Give an outline of a proof, stating clearly where you are making assumptions not discussed in class.)

Theorem (Continuity of Closed Form Functions)
Closed form functions are continuous on their domains.

15 Theorems on Continuity

Recall that f is continuous *on a set A* if it is continuous at every point of that set.

Theorem 15.1 (Image of a closed interval is bounded)
If f is continuous on the closed interval $[a, b]$, then $f([a, b])$ is bounded.

Proof S'pose $f([a, b])$ was not bounded. Then, for every $n \in \mathbb{N}$, there would exist an $x_n \in [a, b]$ with $|f(x_n)| \geq n$. Then no subsequence of $\{f(x_n)\}$ is bounded, so that $\{f(x_n)\}$ contains no convergent subsequence, by Proposition 8.9. But, $\{x_n\}$, being a bounded sequence, has a subsequence $\{x_{n_r}\}$ converging to a limit $x \in [a, b]$, by the Bolzano-Weierstrass Theorem (Exercise Set 10 #6). But then, by Theorem 14.8, $f(x_{n_r})$ converges to $f(x)$ as $r \to +\infty$, contradicting the fact that $\{f(x_n)\}$ contains no convergent subsequence. ☠

Theorem 15.2 (Extreme Value Theorem)
If f is continuous on the closed interval $[a, b]$, then its image $f([a, b])$ contains a maximum and minimum element.

Note This says that there exist m and M in $[a, b]$ such that $f(m)$ is the minimum value of $f(x)$ for x in the interval $[a, b]$, and $f(M)$ is the maximum value of $f(x)$ for x in the interval $[a, b]$.

Proof By Theorem 15.1, $\text{Im} f = f([a, b])$ is a bounded set. Thus all we need to show is that $\text{Im} f$ contains both its supremum and infimum, for then these will be maximum and minimum elements respectively.[††] Thus let $s = \sup(\text{Im} f)$. We must show it is an element of $\text{Im} f$. Then, since $s = \sup(\text{Im} f)$, there exists a sequence $f(x_n)$ of points in $\text{Im} f$ converging to s (by Exercise Set 8 #3). By B-W, the sequence $\{x_n\}$ has a subsequence $\{x_{n_r}\}$ converging to a point x in $[a, b]$. Thus, by continuity of f, we have $f(x_{n_r}) \to f(x)$. But since $\{f(x_{n_r})\}$ is a subsequence of $\{f(x_n)\}$, it also converges to s, so that $s = f(x)$ (by uniqueness of limits) showing it is in $\text{Im} f$, as required. The proof for inf is exactly the same. ✎

Theorem 15.3 (Bolzano's Intermediate Value Theorem)
If f is continuous on the closed interval $[a, b]$, and if k is any real number between $f(a)$ and $f(b)$, then there exists (at least one) c with $a \leq c \leq b$ such that $f(c) = k$.

Wade gives a proof using sequences and quoting all sorts of theorems. We give a direct proof using nothing more than the definition of a continuous function and our bare hands.

Proof Assume wlog that $f(a) \leq k \leq f(b)$, and let

$$E = \{x \in [a, b] \mid f(x) \leq k\}.$$

[††] Our proof will use a sequence. Also see Kosmala's cute proof that uses a function, on p. 169.

(Note that E may be disconnected.) Since $[a, b]$ is bounded above by b, so is the subset E. Since $a \in E$, E is nonempty. Thus $c = \sup E$ exists and $a \le c \le b$. There are now three possibilities: $f(c) < k$, $f(c) > k$, and $f(c) = k$.

Case 1. $f(c) < k$ Since $f(b) \ge k$, one cannot have $c = b$, so that $c < b$. Since f is continuous at c, choosing $\varepsilon < k-f(c)$, there is a $\delta > 0$ such that $x < c+\delta$ implies $f(x) < f(c)+\varepsilon < k$. In particular, with $d = \min\{b, c+\delta/2\} > c$, one has $f(d) < k$, so that $d \in E$, contradicting the fact that c is an upper bound of E.

Case 2. $f(c) > k$. Since $f(a) \le k$, one cannot have $c = a$, so that $c > a$. Since f is continuous at c, choosing $\varepsilon < f(c)-k$, there is a $\delta > 0$ such that $x > c-\delta$ implies $f(x) > f(c)-\varepsilon > k$. In particular, with $d = \max\{a, c-\delta/2\} < c$, one has $f(x) > k$, for every x in the interval $[d, c)$, so that $[d, c) \cap E = \emptyset$, and so d is also an upper bound of E, contradicting the fact that c is the *least* upper bound of E.

Since Cases 1 and 2 do not hold, it must be the case that $f(c) = k$, and we are done. ➹

Corollary 15.4

The image of any interval under a continuous function (whose domain includes that interval) is an interval.*

Proof In the exercise set, you will prove that a subset J of $\mathbb{R}$ is an interval iff it has the following property:

$$a \in J,\ b \in J \text{ and } a \le c \le b \Rightarrow c \in J.$$

Now, if J is any interval contained in $\mathcal{D}(f)$, and if $A \in f(J)$, $B \in f(J)$ and $A \le C \le B$, we can choose $a \in J$ with $f(a) = A$ and $b \in J$ with $f(b) = B$, so we are in the Intermediate Value Theorem, since f is continuous on $[a, b]$. Hence, by the IVT, $C \in f(J)$ as well, showing that it is an interval. ✠

Corollary 15.5 (Brouwer Fixed Point Theorem)

If f: $[a, b] \to [a, b]$ is any continuous function, then there exists (at least one) $c \in [a, b]$ such that $f(c) = c$.

Proof Let g: $[a, b] \to \mathbb{R}$ be given by $g(x) = f(x) - x$. Then g is continuous, with

$g(a) = f(a) - a \ge 0$ (since $\operatorname{Im} f \subset [a, b]$), and

$g(b) = f(b) - b \le 0$ (for the same reason).

Thus, applying the IVT with $k = 0$ gives the required number c. ❇

Theorem 15.6 (Continuity of Inverse)

If f: $[a, b] \to [c, d]$ is any strictly monotone continuous function with image $[c, d]$,†† then f is invertible, and f^{-1}: $[c, d] \to [a, b]$ is continuous.

Proof Since f is strictly monotone, it is injective. Since its image is $[c, d]$ it is surjective. Hence it is invertible. Further, f^{-1} is also strictly monotone. To show continuity, let us

---

* By an interval, we mean any subset of the form $[a, b]$, $[a, b)$, $(a, b]$, or (a, b) with $a \le b$ and a, b, or both permitted to be infinite if not included in the interval. Note that this also allows degernate intervals such as $[2, 2] = \{2\}$, and $(2, 2) = \emptyset$. Kosmala does not allow such things as intervals, and so his Corollary 4.3.6 is rather contorted. (also, it considers only the case of closed intervals, which our (much better) corollary is the most general possible.

†† The image of $[a, b]$ under a continuous function must, byu Theorem 15.4, be an interval, so there is no loss of generality here.

use the sequential criterion as generalized in Exercise Set 14 #5: If $y_n \to y$ is a monotone convergent sequence in $[c, d]$, then $f^{-1}(y_n)$ is also a monotone sequence which, being bounded, it convergent to x, say. But then its image under f must converge to $f(x)$. Since it already converges to y, we must have $y = f(x)$, so that $x = f^{-1}(y)$. Thus, $f^{-1}(y_n)$ converges to $f^{-1}(y)$ as desired. ▼

Example 15.7 (Existence, continuity, and uniqueness of the *n*th root function)
Exercise Set 15 below contains part of it. Rest in class.

Exercise Set 15

1. **(a)** Where would the proof of Theorem 15.1 break down if $[a, b]$ was not given as a closed interval? (Don't just say: "Because then we couldn't apply result so-and-so, which requires a closed interval." You should indicate exactly where *its* proof breaks down.)
(b) Give an example to show that Theorem 15.1 fails if $[a, b]$ is replaced by an open interval.

2. Give an example to show that Theorem 15.2 fails if the closed interval $[a, b]$ is replaced by an open interval.

3. Prove that a subset J of $\mathbb{R}$ is an interval iff it has the following property:
$$a \in J,\ b \in J \text{ and } a \le c \le b \Rightarrow c \in J.$$

4. Give an example of a function with three discontinuities on $[a, b]$ but with the conclusion of IVT still holding for f.

5. Use the Intermediate Value Theorem to prove that every positive real number has a unique nth root, where $n = 2, 3, 4, \ldots$

6. Produce a new proof of the Brouwer fixed point theorem that does not require the IVT. (It goes roughly like this: "Define a sequence by $x_1 = a$, and $x_{n+1} = f(x_n)$. Then pass to a limit to get a fixed point.") Supply the details, being careful about convergence.

16. Uniform Continuity

(§3.4 in text)

Recall that f is continuous on the set D if, given $\varepsilon > 0$, and given $a \in A$, there exists $\delta > 0$, **depending on both a and ε**, such that

$$x \in D \text{ and } |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Formally,

$$\forall a \in D\ \forall \varepsilon > 0\ \exists \delta > 0\ [\forall x \in D\ [\ |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon]\ ].$$

If δ depends only on ε, we say that f is *uniformly continuous*. Formally,

Definition 1.1 f: $D \longrightarrow \mathbb{R}$ is **uniformly continuous** if given $\varepsilon > 0$, there exists $\delta > 0$ (depending only on ε) such that, for every a and $x \in D$,

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

In other words, we can choose the ε before we even mention the point a of continuity. Formally,

$$\forall \varepsilon > 0\ \exists \delta > 0\ [\forall a \in D\ \forall x \in D\ [\ |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon]\ ].$$

See the difference?

Note It follows that f: $D \longrightarrow \mathbb{R}$ is *not* uniformly continuous if there exists $\varepsilon > 0$ such that, for every $\delta > 0$ (no matter how small) there exist points a and x with $|x - a| < \delta$, but $|f(x) - f(a)| \geq \varepsilon$. This definition gives us the following way of showing that a function is *not* uniformly continuous.

Showing That f is Not Uniformly Continuous

1. Choose a suitable $\varepsilon > 0$ (often $\varepsilon = 1$ will do).

2. Let $\delta > 0$ be given. Choose a and x within δ of each other such that $|f(x) - f(a)| \geq \varepsilon$. (This often means trying to force $|f(x) - f(a)|$ to be larger than a multiple of $|x - a|$.)

Sequence Method (See Exercise Set 1 # 1)

1. Choose a suitable $\varepsilon > 0$ (often $\varepsilon = 1$ will do).

2. Find two sequences $\{x_n\}$ and $\{y_n\}$ with $|x_n - y_n| \longrightarrow 0$, but with $|f(x_n) - f(y_n)| \geq \varepsilon$.

Examples 16.2

A. f: $\mathbb{R} \longrightarrow \mathbb{R}$; $f(x) = x^2$ is *not* uniformly continuous. For, with $\varepsilon = 1$, no matter how small the positive number δ is, choose $a = {}^1/_\delta$ and $x = a + {}^\delta/_2$. Then $|x - a| = {}^\delta/_2 < \delta$, but

$$|f(x) - f(a)| = |x^2 - a^2| = |x + a| \cdot |x - a| > \left(\frac{1}{\delta} + \frac{1}{\delta}\right)\frac{\delta}{2} = 1 = \varepsilon.$$

B. f: $\mathbb{R}-\{0\}\longrightarrow\mathbb{R}$; $f(x) = 1/x$ is not uniformly continuous. For, with $\varepsilon = 1$, no matter how small the positive number δ is (and we can assume $\delta \leq {}^1/_2$*), choose a with $a = \delta$ and $x = {}^\delta/_2$. Then $|x - a| = {}^\delta/_2 < \delta$, but

$$|f(x) - f(a)| = \left|\frac{1}{x} - \frac{1}{a}\right| = \frac{|a - x|}{ax} > \frac{\delta/2}{\delta^2} = \frac{1}{2\delta} \geq 1 = \varepsilon.$$

(The last inequality follows since $\delta < {}^1/_2$.)

Many of the common examples of functions that *are* uniformly continuous are, in fact, more than just uniformly continuous:

Definition 16.3 The function f: $D\longrightarrow\mathbb{R}$ is **Lipschitz** (or **Lipschitzian**) if there exists $L >$ 0 such that, for all $x, y \in D$, one has

$$|f(x) - f(y)| \leq L|x - y|.$$

Examples 16.4 (of Lipschitz Functions)

A. f: $\mathbb{R}\longrightarrow\mathbb{R}$; $f(x) = mx + b$ (m, b arbitrary) one can take $L = m$ (unless $m = 0$, in which case, take $L = 666$). Thus all linear functions are Lipschitz.

B. f: $\mathbb{R}\longrightarrow\mathbb{R}$; $f(x) = \sin x$ is Lipschitz , since for all x and y, one has

$$|\sin x - \sin y| \leq |x - y|,$$

so we can take $L = 1$. (The inequality is an immediate consequence of the Mean Value Theorem.)

Proposition 16.5 (Lipschitz Functions)
Lipschitz functions are uniformly continuous.

Proof Just choose $\delta = \varepsilon/L$. ❖

Corollary 16.6 (Bounded Derivative)
If I is an interval and f: $I\longrightarrow\mathbb{R}$ is differentiable and differentiable on Int(I) with f' bounded, then f is uniformly continuous.

Proof follows from the MVT and Proposition 16.5. ❖

In addition to Lipschitz functions, we have the following result.

Proposition 16.7 (Functions with Compact Domain)
If f: $D\longrightarrow\mathbb{R}$ is continuous with D compact, then f is uniformly continuous.

Proof If f were not uniformly continuous, then there would exist $\varepsilon>0$ and two sequences (x_n) and (y_n) in D with $|x_n - y_n| < 1/n$ but with $|f(x_n) - f(y_n)| \geq \varepsilon$. for all n. But D is compact, showing that (x_n) has a convergent subsequence (x_{n_r}). The corresponding subsequence (y_{n_r}) in turn has a convergent subsequence, which we shall write as $(y_{\phi(n)})$ both $(x_{\phi(n)})$ and $(y_{\phi(n)})$ converge to x and y, say, with $|x-y| = 0$ by an easy triangle

* If not, then choose $\delta' = \min\{1/2, \delta\}$ and use this in place of δ.

inequality argument. But f is continuous at x and y, and hence sequentially continuous there. So,

$$|f(x_{\phi(n)}) - f(y_{\phi(n)})| \le |f(x_{\phi(n)}) - f(x)| + |f(x) - f(y)| + |f(y) - f(y_{\phi(n)})|$$
$$\longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

contradicting the fact that it is $\ge \varepsilon$. ❖

Exercise Set 16

1. Prove that f: $D \longrightarrow \mathbb{R}$ is *not* uniformly continuous iff there exists $\varepsilon > 0$ and two sequences $\{x_n\}$ and $\{y_n\}$ with $|x_n - y_n| \longrightarrow 0$, but with $|f(x_n) - f(y_n)) \ge \varepsilon$.

2. Give an example of f: $B \longrightarrow \mathbb{R}$ with B bounded and f continuous but not uniformly continuous.

3. Give an example of a non-Lipschitz continuous function f: $D \longrightarrow \mathbb{R}$ with D compact. (Hint: if a function has an "infinite" derivative at some point….)

17 The Derivative

Definition 17.1 Suppose that f: $D\to\mathbb{R}$, and a is also a limit point of D. Then the **derivative** of f at a is defined by

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} ,$$

provided this limit is finite, and we say that f is **differentiable at a**. *The function f* is **differentiable on $A \subset D$** if it is differentiable at each point of A.

Note Here we require that a must be *both* a point of the domain and a limit point.

Examples

A. f: $\mathbb{R}\to\mathbb{R}$; $f(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is differentiable at 0, since the difference quotient does approach zero.

B. f: $\mathbb{R}\to\mathbb{R}$; $f(x) = x^3$ is not differentiable at 0.

C. f: $\mathbb{R}\to\mathbb{R}$; $f(x) = |x|$ is not differentiable at 0.

D. f: $\mathbb{R}\to\mathbb{R}$; $f(x) = 1/x$ is differentiable at each point of its domain.

Notes 17.2

1. Using the substitution $x = a+h$, we get the more familiar-looking derivative formula

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} .$$

2. If f is differentiable on $A \subset \mathcal{D}(f)$, then it determines a function f': $A\to\mathbb{R}$, called the **derivative of f.**

3. There are various notations for the derivative.

Proposition 17.3 (Differentiability Implies Continuity)
If f: $D\to\mathbb{R}$ is differentiable at a, then it is continuous at a.

Proof We use the criterion in Exercise Set 14 #2, and show that $\lim_{x\to a} |f(x) - f(a)| = 0$.

▲

Examples

A. Trick example: it might seem as though the function f: $\mathbb{R}\to\mathbb{R}$; $f(x) = \begin{cases} 2x + 1 & \text{if } x \leq 0 \\ x^2 + 2x & \text{if } x > 0 \end{cases}$ is differentiable at $x = 0$ with derivative $2x$, but it is not continuous there!

Exercise Set 17

1. Give the δ-ε definition of the derivative in terms of h. (Notes 17.2(1))

2. Wade, p. 89, #1 (some practice finding derivatives the old-fashioned way).

Also: Prove that $\frac{d}{dx}|x| = \frac{|x|}{x}$ at every point where the latter is defined.

3. Wade, p. 89, #4 (sine and cosine)
4. Prove the following:

(a) If f: $(a, b)\to\mathbb{R}$ is constant, then $f'(x) = 0$ for every $x \in (a, b)$.

(b) If $n \in \mathbb{N}$, then $\frac{d}{dx}(x^n) = nx^{n-1}$.

6. Dirichlet's Function Returns Yet Again Let f: $(0, 1)\to\mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms with } p, q \in \mathbb{N}; \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that $f(x)$ is not differentiable at any point. [Hint: We can dismiss rational points, by a previous exercise. The proof for irrational points seems a little tricky, so proceed as follows.

(a) Prove that, if x is irrational, and $f'(x)$ exists, then it must be zero. (Use a sequence argument.)

(b) Show that, if x is any irrational number and q is any integer ≥ 1, there exists a rational number of the form $\frac{p}{q}$ with $x < \frac{p}{q} < x + \frac{1}{q}$. [Hint for (a): multiply through by q and see what it says.]

(c) Now let x be any irrational number, choose a sequence of *prime* integers $q_n\to+\infty$, choose rational numbers $r_n = \frac{p_n}{q_n}$ with the property stated in part (a). Then $r_n\to x$ as $n\to+\infty$. Use this sequence together with part (b) to conclude that, if $f'(x)$ exists, then it is at least 1, contradicting (a).

End of Hint]

18 Properties of the Derivative

(§4.2 in book)

Theorem 18.1 (Properties of the Derivative)
If f and g are differentiable at $a \in \mathcal{D}(f)\cap\mathcal{D}(g)$, then so are $f\pm g$, $f\cdot g$, and f/g if $g(a) \neq 0$, and:

(a) $(f\pm g)'(a) = f'(a) \pm g'(a)$

(b) $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$

(c) $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$

Note Results (a) and (b) extend by induction to arbitrary finite sums and products.

Theorem 18.2 (Chain Rule)
Suppose that f: $D \to \mathbb{R}$ and g: $E \to \mathbb{R}$ are such that $f(a) \in E$ and is also a limit point of E. Suppose further that f is differentiable at a, and g is differentiable at $f(a)$. Then $g \circ f$ is differentiable at a, and

$$(g \circ f)'(a) = g'(f(a)) f'(a)$$

Proof[†] Let r: $\mathbb{R} \rightarrowtail \mathbb{R}$ be given by

$$r(x) = \frac{g \circ f(x) - g \circ f(a)}{x - a}.$$

Then we need to show that $\lim_{x \to a} r(x) = g'(f(a)) f'(a)$. To do this, define a new function h: $\mathbb{R} \rightarrowtail \mathbb{R}$ by

$$h(x) = \begin{cases} \dfrac{g(f(x)) - g(f(a))}{f(x) - f(a)} & \text{if } f(x) \neq f(a) \\ g'(f(a)) & \text{if } f(x) = f(a) \end{cases}$$

Then h is defined on $\mathcal{D}(f) \cap f^{-1}(\mathcal{D}(g))$, and has the following interesting properties:

(1) $r(x) = h(x) \cdot \dfrac{f(x) - f(a)}{x - a}$, whether or not $f(x) = f(a)$.

(2) $h(x) \to g'(f(a))$ as $x \to a$.

It now follows, taking the limit of (1) as $x \to a$, that $\lim_{x \to a} r(x) = g'(f(a)) f'(a)$, as required. ♣

Definition 18.3 The function f: $D \to \mathbb{R}$ has a **relative minimum** (resp. **maximum)** at $a \in D$ if there is a $\delta > 0$ such that $f(x) \geq f(a)$ (resp. $f(x) \leq f(a)$) whenever $x \in (a-\delta, a+\delta) \cap D$. We also use **relative extremum** to refer to both.

Examples 18.4
A. We look at 4 types of relative extrema: stationary points, singular points, discontinuity, and endpoints.
B. Not all endpoints need be: look at $x\sin(1/x)$ with domain expanded to $[0, +\infty)$.

Definition 18.5 The point a is an **interior point** of $D \subset \mathbb{R}$ if there is a $\delta > 0$ with $(a-\delta, a+\delta) \subset D$. In other words, there is a little open interval containing a that sits inside D.

Theorem 18.6 (Smooth Relative Extrema are Stationary)
If f is differentiable at the interior point a of $\mathcal{D}(f)$, and also has a local extremum at a, then $f'(a) = 0$.

Proof Assume wlog that f has a local maximum at a. Then $(f(x)-f(a))/(x-a)$ is positive if $x < a$ and negative if $x > a$. Since a is an interior point, of $\mathcal{D}(f)$, we can choose two sequences approaching a; $\{x_n\}$ from the left and $\{y_n\}$ from the right. Since f is differentiable at a, both $(f(x_n)-f(a))/(x_n-a)$ and $(f(y_n)-f(a))/(y_n-a)$ converge to $f'(a)$. But, by the properties of limits of sequences, they converge, respectively to a number ≥ 0 and ≤ 0, and hence 0, whence it must be that $f'(a) = 0$. ➪

[†] What follows is the “standard” proof. Kosmala's uses sequences and seems odd.

Theorem 18.7 (Rolle's Theorem)
Suppose that f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$. Then there exits $c \in (a, b)$ such that $f'(c) = 0$.

Proof If f is constant, then by Exercise Set 17 #4(a), the result is trivially true. Thus, assume f is not constant, and assume wlog that $f(x) > f(a)$ for some $x \in (a, b)$. By Theorem 15.2, there is a point c such that f attains the maximum at c. But, since c is an interior point of $\mathcal{D}(f)$, Theorem 18.6 tells us that $f'(c) = 0$, as required. ❖

Theorem 18.8 (Mean Value Theorem)
Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). Then there exits $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Proof Apply Rolle's theorem to the function g: $[a, b] \to \mathbb{R}$ given by

$$g(x) = f(x) - (x-a)\frac{f(b) - f(a)}{b - a} .$$ ➢

Corollary 18.9 (Positive Derivative Implies Increasing)
If f' exists and is positive on an interval, then f is increasing on that interval. Similarly, if f' exists and is negative on an interval, then f is decreasing on that interval

Proof S'pose f' is positive in the interval I. If the interval had width zero, the theorem holds vacuously. Thus, assume I has non-zero width. Then, if $a < b$ are any points in I, one has $(f(b) - f(a))/(b - a) > 0$, by the MVT applied to (a, b). But this says that $f(b) > f(a)$. The proof for decreasing f is similar. ➠

Theorem 18.10 (Non-zero Derivative Implies Invertible)
If f' exists and is always positive (or negative) on an open interval, then f^{-1} exists and is differentiable on that interval, and moreover,

$$(f^{-1})'(y) = \frac{1}{f'(x)} \qquad \text{for } y = f(x)$$

for every x in that interval

Proof By 18.9 we know that f is monotone, and hence (Theorem 15.6) that f^{-1} exists and is continuous. To establish differentiability of f^{-1}, let us choose a point $b = f(a)$ in the image of the interval and consider the ratio

$$\phi(y) = \frac{f^{-1}(y) - f^{-1}(b)}{y - b} = \frac{f^{-1}(y) - f^{-1}(b)}{f(f^{-1}(y)) - f(f^{-1}(b))} = \frac{1}{\left[\frac{f(f^{-1}(y)) - f(f^{-1}(b))}{f^{-1}(y) - f^{-1}(b)}\right]} .$$

It suffices to prove that $\phi(y) \to 1/f'(a)$ as $y \to b$. using the sequential criterion, let $y_n \to b$. Then $f^{-1}(y_n) \to f^{-1}(b)$ by continuity of the inverse, whence the quantity in the denominator on the right must approach $1/f'(f^{-1}(b)) = 1/f'(a)$, as desired. ➲

Definition 18.11 The function f: $D \to \mathbb{R}$ is **Lipshitz** if there exists $L > 0$ such that

$$|f(x) - f(y)| \le L|x - y|$$

for every pair of points x, y in D.

Proposition 18.12 (Bounded Derivative Implies Lipshitz)
If f' is bounded on the set S, then f is Lipshitz on S

Proof Exercise Set

Next, we give another consequence of the Rolle's Theorem:

Proposition 18.13 (Cauchy's Mean Value Theorem)
If f and g are continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c)[g(b) - g(a)] = g'(c)[f(b)-f(a)]$$

Proof Apply Rolle's theorem to the function

$$h(x) = f(x)[g(b) - g(a)] - g(x)[f(b)-f(a)].$$ ✤

Finally, we give a useful consequence of the MVT

Proposition 18.14 (Bernoulli's Inequality)
Let $\alpha \in \mathbb{Q}$. Then, for any $\delta > -1$, one has

$$(1+\delta)^\alpha \le 1 + \alpha\delta \qquad \text{if } 0 < \alpha \le 1$$
$$(1+\delta)^\alpha \ge 1 + \alpha\delta \qquad \text{if } \alpha > 1$$

Proof
We can assume wlog that $\delta \neq 0$, since the result is trivially true when $\delta = 0$. By the power rule (Exercise4 below) the function $f(x) = x^\alpha$ is differentiable with derivative $f'(x) = \alpha x^{\alpha-1}$. Applying the MVT to this function on the interval $[1, 1+\delta]$ (if $\delta > 0$ or $[1+\delta, 1]$ in the case $\delta < 0$) we get

$$\alpha c^{\alpha-1} = \frac{f(1+\delta) - f(1)}{\delta},$$

for some c between 1 and $1+\delta$. Thus,

$$f(1+\delta) = f(1) + \delta\alpha c^{\alpha-1},$$

or

$$(1+\delta)^\alpha = 1 + \delta\alpha c^{\alpha-1}.$$

Case 1: $0 < \alpha \le 1$:
Subcase 1: If $\delta > 0$, since $\alpha-1 \le 0$, we have $c^{\alpha-1} \le 1$ (because $c > 1$), giving the result.
Subcase 2: If $\delta < 0$, we have $c^{\alpha-1} \ge 1$ and hence $\delta\alpha c^{\alpha-1} \le \delta\alpha$ (since δ is negative), giving the result again.
Case 2 is left as an exercise. ❑

On Line Discussion
1. Should not the inverse of the derivative be expected to be the derivative of the inverse? How does this relate to Theorem 18.10?
2. Do functions have to be differentiable to be Lipshitz?

Exercise Set 18
1. Give an example to show that the assumption that a be an interior point of $\mathcal{D}(f)$, is necessary.
2. Give an example to show that the assumption in Rolle's Theorem that f is differentiable in (a, b) is necessary.

3. Give an example of a function f with the property that f is differentiable on a neighborhood of some point a and $f'(a) > 0$, but with f neither increasing nor decreasing on any neighborhood of a. {Hint: look at some examples you've already seen of differentiable functions.]

4. Use the Chain rule to prove that, if $q \in \mathbb{Q}$, then $\frac{d}{dx}(x^q) = qx^{q-1}$.

5. (a) Prove the following converse to Exercise Set 17 #4(a):

Theorem 18.X4 (Zero Derivative Implies Constant)
If f is differentiable on the interval (a, b) and $f'(x) = 0$ for every $x \in (a, b)$, then f is constant on (a, b).

(b) Conclude: If F and G are any two differentiable functions on (a, b) such that $F'(x) = G'(x)$ for every $x \in (a, b)$, then $F(x) = G(x) + C$, where C is some constant.

6. Prove Proposition 18.12.

Reading Assignment: L'Hôpital's Rule (pp. 96-97)