Chapter G—Game Theory

G.1 Two-Person Zero Sum Games; Reduction by Dominance
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You’re the Expert—Harvesting Forests

Forest Lumber Inc. has a large plantation of Douglas fir trees. The company periodically harvests some of the trees and then replants. As a consultant to Forest Lumber Inc., you have been asked to advise the company at what age it should harvest its trees in order to maximize the value of its timber. How will you advise the company?

Introduction

It frequently happens that you are faced with having to make a decision or choose a best strategy from several possible choices. For instance, you might need to decide whether to invest in stocks or bonds, or you might need to choose an offensive play to use in a football game. In both of these examples, the result depends on something you cannot control. In the first case, your success partly depends on the future behavior of the economy. In the second case, it depends on the defensive strategy chosen by the opposing team.

We can model situations like these using **game theory**. We represent the various options and payoffs in a matrix and can then calculate the best single strategy or combination of strategies using matrix algebra and techniques from linear programming. Game theory is yet another illustration of the power of matrix algebra and linear programming.

Game theory is very new compared with most of the mathematics you learn. It was invented in the 1920's by the noted mathematicians Émile Borel (1871–1956) and John von Neumann (1903–1957). The connection with linear programming was discovered even more recently, in 1947, by von Neumann.

G.1 Two-Person Zero Sum Games; Reduction by Dominance

We have probably all played the simple game "Paper, Scissors, Rock" at some time in our lives. It goes as follows: There are two players—let us call them A and B—and at each turn, both players produce, by a gesture of the hand, either paper, a pair of scissors, or a rock. Rock beats scissors (since a rock can crush scissors), but is beaten by paper (since a rock can be covered by paper), while scissors beat paper (since scissors can cut paper). The round is a draw if both A and B show the same item. We could turn this into a betting game if, at each turn, we require the loser to pay the winner 1¢. For instance, if A shows a rock and B shows paper, then A pays B 1¢.

Paper, Scissors, Rock is an example of a **two-person zero sum game**. It is called a zero sum game because each player's loss is equal to the other player's gain.[1] We can represent this game by a matrix, called the **payoff matrix**.

$$\begin{array}{cc} & \mathbf{B} \\ & \begin{array}{ccc} p & s & r \end{array} \\ \mathbf{A} \quad \begin{array}{c} p \\ s \\ r \end{array} & \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array}$$

In this matrix, player A's options are listed on the left, while player B's options are listed on top. We think of A as playing the rows and B as playing the columns. Positive numbers indicate a win for the row player, while negative numbers indicate a loss for the row player. Thus, for example, the p,s entry represents the outcome if A plays p (paper) and B plays s (scissors). In this event, B wins, and the -1 entry there indicates that A loses 1¢. (If that entry were -2 instead, it would have meant that A loses 2¢.)

In each round of the game, each player's choice is called a **strategy**. Thus, if A chooses p, we refer to the p row as player A's strategy. In this section and the next, we are going to talk about **pure strategies**, whereby each player makes the same move at each round of the game. For example, if a player in the above game chooses to play scissors at each turn, then that player is using the pure strategy s.

Example 1 ***Paper, Scissors, Rock***

Two players, A and B, have decided to change the rules of the game "Paper, Scissors, Rock" by using instead the following payoff matrix:

$$\begin{array}{cc} & \mathbf{B} \\ & \begin{array}{ccc} p & s & r \end{array} \\ \mathbf{A} \quad \begin{array}{c} p \\ s \\ r \end{array} & \begin{bmatrix} -2 & -3 & 4 \\ 1 & -1 & 3 \\ 3 & 3 & 0 \end{bmatrix} \end{array}$$

[1] An example of a *non-zero sum game* would be one in which the government taxed the earnings of the winner. In that case the winner's gain would be less than the loser's loss.

If player B can't make up her mind whether to use paper or scissors as a pure strategy, what would you advise?

Solution First, remember that B plays the columns. Since we are comparing paper and scissors, we look at the p and s columns. It is in B's interest to have all payoffs as small as possible. That is, n is better than m if $n < m$. Comparing columns p and s, we see that all the entries in the s column are either less than (and thus better than), or equal to, the corresponding entries to their left. Thus, *no matter what A decides to do*, B will either be better off, or just as well off, playing s instead of p. We would therefore advise player B to stick to scissors rather than paper as a pure strategy.

Before we go on...

We say that the s column **dominates** the p column (or that the p column is **dominated by** the s column) since column s is $\leq$ column p (that is, the entries in column s are $\leq$ the corresponding entries in column p). If you were player B, it would be in your best interest to avoid using strategy p completely, since s is a better, or equally good, choice no matter what A does. Thus, if you are player B and wish to play to your advantage, you can simplify the matrix by eliminating columns that are dominated by others. This gives the following smaller matrix:

$$\mathbf{A}\quad \begin{array}{c} \\ p \\ s \\ r \end{array}\!\!\begin{array}{c} \mathbf{B} \\ \begin{array}{cc} s & r \end{array} \\ \begin{bmatrix} -3 & 4 \\ -1 & 3 \\ 3 & 0 \end{bmatrix} \end{array}$$

Notice that column r is not dominated by p or by s (why?).

What about player A? Since A plays the rows, and wishes to have the entries as large as possible, we say that one row **dominates** another row if its entries are $\geq$ the corresponding entries in the other row. Looking at what's left of the matrix, we see that none of the rows are dominated by any other row, so there are no rows that we can eliminate. In analyzing any game, we can begin by eliminating rows and columns that are dominated by others, making the game simpler. The procedure of eliminating dominated rows and columns is called **reduction by dominance.**

Reduction by Dominance

1. Check whether there is any row in the (remaining) matrix that is dominated by another row (this means that it is $\leq$ some other row). If there is one, delete it.
2. Check whether there is any column in the (remaining) matrix that is dominated by another column (this means that it is $\geq$ some other column). If there is one, delete it.
3. Repeat steps 1 and 2 in any order until there are no dominated rows or columns

Quick Example

In the payoff matrix

$$\mathbf{A}\ \begin{matrix} \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \overset{\mathbf{B}}{\begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & -2 & 1 \\ 2 & 3 & 4 \end{bmatrix}},$$

row 4 dominates both rows 2 and 3, so we eliminate both of these rows at once.

$$\mathbf{A}\ \begin{matrix} \\ 1 \\ 4 \end{matrix} \overset{\mathbf{B}}{\begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & -1 \\ 2 & 3 & 4 \end{bmatrix}}$$

Turning to the columns, we see that column 2 dominates column 3, so we eliminate column 3.

$$\mathbf{A}\ \begin{matrix} \\ 1 \\ 4 \end{matrix} \overset{\mathbf{B}}{\begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 3 \end{bmatrix}}$$

Looking again at the rows, we find that none of the rows is dominated by any of the others, and that the same is true for the columns. Thus the game cannot be reduced any further.[1]

This process might go on until you are left with a 1×1 matrix. If this is the case, then you are lucky indeed, and are left with a very simple game. The following example illustrates this possibility.

Example 2 *Football*

You are the head coach of the Alphas (Team A), and are attempting to come up with a strategy to deal with your rivals, the Betas (Team B). Team A is on offense, and Team B is on defense. You have five preferred plays, but are not sure which to select. You know, however, that Team B usually employs one of three defensive strategies. Over the years, you have diligently recorded the average yardage gained by your team for each combination of strategies used, and have come up with the following table.

[1] Just what we do when we arrive at this point will take us the rest of the chapter to answer!

		B 1	2	3
	1	0	−1	5
	2	7	5	10
A	3	15	−4	−5
	4	5	0	10
	5	−5	−10	10

Which of the five plays should you select?

Solution The table is a payoff matrix in disguise. We reduce the matrix by dominance. Looking at the rows, we notice that row 2 dominates rows 1, 4 and 5. Thus we can eliminate all three of those rows, getting

$$\mathbf{A}\quad \begin{matrix} \\ 2 \\ 3 \end{matrix} \overset{\mathbf{B}}{\begin{matrix} 1 & 2 & 3 \\ \end{matrix}} \begin{bmatrix} 7 & 5 & 10 \\ 15 & -4 & -5 \end{bmatrix}.$$

Turning to the columns, we see that column 1 is dominated by column 2, so we eliminate column 1.

$$\mathbf{A}\quad \begin{matrix} 2 \\ 3 \end{matrix} \overset{\mathbf{B}}{\overset{2 \quad 3}{\begin{bmatrix} 5 & 10 \\ -4 & -5 \end{bmatrix}}}.$$

Now we see that row 2 dominates row 3, so we eliminate row 3.

$$\mathbf{A}\quad 2 \quad \overset{\mathbf{B}}{\overset{2 \quad 3}{[5 \;\; 10]}}$$

Finally, column 3 is dominated by column 2, so we eliminate column 3, winding up with the following 1×1 game.

$$\mathbf{A}\quad 2 \quad \overset{\mathbf{B}}{\overset{2}{[5]}}$$

Since the only row remaining is row 2, you decide that the best play would be play #2, which should earn your team a five yard gain.

Before we go on...
Looking at the original chart, you might reason that it would be better to try play #3, since you would expect that play would earn your team 15 yards if Team B attempts play #1. On the other hand, Team B's coach would hardly be likely to recommend play #1 for Team B, since column 1 is dominated by column 2. In eliminating dominated rows and columns, we are therefore making the following assumption: *that each team's coach is planning the best possible move, and assuming at the same time that the rival coach is doing the same*. This is the **fundamental principle of game theory**, and we shall have more to say about it in the next section.

G.1 Exercises

Reduce the payoff matrices in Exercises 1–6 by dominance.

1. $\mathbf{A}\begin{matrix} \\ 1 \\ 2 \end{matrix}\overset{\mathbf{B}}{\begin{matrix} 1 & 2 & 3 \\ \end{matrix}} \begin{bmatrix} 1 & 1 & 10 \\ 2 & 3 & -4 \end{bmatrix}$

2. $\mathbf{A}\begin{matrix} \\ 1 \\ 2 \end{matrix}\overset{\mathbf{B}}{\begin{matrix} 1 & 2 & 3 \\ \end{matrix}} \begin{bmatrix} 2 & 0 & 10 \\ 15 & -4 & -5 \end{bmatrix}$

3. $\mathbf{A}\begin{matrix} \\ 1 \\ 2 \\ 3 \end{matrix}\overset{\mathbf{B}}{\begin{matrix} a & b & c \end{matrix}} \begin{bmatrix} 0 & -1 & -5 \\ -3 & -10 & 10 \\ 2 & 3 & -4 \end{bmatrix}$

4. $\mathbf{A}\begin{matrix} \\ 1 \\ 2 \\ 3 \end{matrix}\overset{\mathbf{B}}{\begin{matrix} a & b & c \end{matrix}} \begin{bmatrix} 2 & -4 & -9 \\ -1 & -2 & -3 \\ 5 & 0 & -1 \end{bmatrix}$

5. $\mathbf{A}\begin{matrix} \\ A \\ B \\ C \\ D \end{matrix}\overset{\mathbf{B}}{\begin{matrix} a & b & c \end{matrix}} \begin{bmatrix} 1 & -1 & -5 \\ 4 & 0 & 2 \\ 3 & -3 & 10 \\ 3 & -5 & -4 \end{bmatrix}$

6. $\mathbf{A}\begin{matrix} \\ A \\ B \\ C \\ D \end{matrix}\overset{\mathbf{B}}{\begin{matrix} a & b & c \end{matrix}} \begin{bmatrix} 2 & -4 & 9 \\ 1 & 1 & 0 \\ -1 & -2 & -3 \\ 1 & 1 & -1 \end{bmatrix}$

Applications

Set up the payoff matrix in each of Exercises 7–14.

7. ***Games to Pass the Time*** You and your friend have come up with the following simple game to pass the time: at each round, you simultaneously call “heads” or “tails.” If you have both called the same thing, your friend wins one point; if your calls differ, you win one point.

8. ***Games to Pass the Time*** Bored with the game in Exercise 7, you decide to use the following variation instead: If you both call “heads” your friend wins two points; if you both call “tails” your friend wins 1 point; if your calls differ, then you win two points if you called “heads” and one point if you called “tails.”

9. ***Marketing*** Your fast food outlet, Burger Express, has obtained a license to open branches in three closely situated South African cities: Brakpan, Nigel and Springs. Your market surveys show that the first two cities provide a potential market of 2,000 burgers a day, while the third provides a potential market of 1,000 burgers per day. Your company can finance an outlet in only one of those cities at the present time. Your main competitor, Burger Princess, has also obtained licenses for these cities, and is similarly planning to open only one outlet. If you both happen to locate in the same city, you will share the total business from all three cities equally, but if you locate in different cities, you will each get all the business in the city in which you have located plus half the business in the third city. (The payoff is the number of burgers you will sell minus the number of burgers your competitor will sell.)

10. ***Marketing*** Repeat Exercise 9 given that the potential sales markets in the three cities are: Brakpan, 2,500 per day; Nigel, 1,500 per day; Springs, 1,200 per day.

11. ***War Games*** You are deciding whether to invade Finland, Sweden, or Norway, and your opponent is simultaneously deciding which of these three countries to defend. If you invade a country that your opponent is defending, you will be defeated (payoff: -1), but if you invade a country your opponent is not defending, you will be successful (payoff: $+1$).

12. ***War games*** You must decide whether to attack your opponent by sea or air, and your opponent must simultaneously decide whether to mount an all-out air defense, an all-out coastal defense (against an attack from the sea), or a combined air and coastal defense. If there is no defense for your mode of attack, you win 100 points. If your attack is met by a shared air and coastal defense, you win 50 points. If your attack is met by an all-out defense, you lose 200 points.

13. ***Betting*** When you bet on a race-horse with odds of m-n, you stand to win m dollars for every bet of n dollars if your horse wins; for instance, if the horse you bet is running at 5-2 and wins, you will win $5 for every $2 you bet. (Thus a $2 bet will return $7.). Here are some actual odds from a 1992 race at Belmont Park, NY.[1] The favorite at 5-2 was Pleasant Tap. The second choice was Thunder Rumble at 7-2, while the third choice was Strike the Gold at 4-1. Assume you are making a $10 bet on one of these horses. The payoffs are your winnings. (If your horse does not win, you lose your entire bet. Of course, it is possible for none of your horses to win.)

14. ***Betting*** Referring to Exercise 13, suppose that just before the race, there has been frantic betting on Thunder Rumble, with the result that the odds have dropped to 2-5. The odds on the other two horses remain unchanged.

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[1] Source: *The New York Times,* September 18, 1992, p.B14.

15. ***Wrestling Tournaments*** City Community College (CCC) plans to host Midtown Military Academy (MMA) for a wrestling tournament. Each school has three wrestlers in the 190 lb. weight class: CCC has Boris, Sal and Josh, while MMA has Carlos, Brutus and Julius. Boris can beat Carlos and Brutus, Brutus can beat Josh and Sal, Julius can beat Josh, while the other combinations will result in an even match. Set up a payoff matrix, and use reduction by dominance to decide which wrestler each team should choose as their champion. Does one school have an advantage over the other?

16. ***Wrestling Tournaments*** One day before the wrestling tournament discussed in Exercise 15, Boris sustains a hamstring injury, and is replaced by Zak, who (unfortunately for CCC) can be beaten by both Carlos and Brutus. Set up the payoff matrix, and decide which wrestler each team should choose as their champion. Does one school have an advantage over the other?

17. ***Price Wars*** Computer Electronics, Inc., and the Gigantic Computer Store are planning to discount the price they charge for the HAL Laptop Computer, of which they are the only distributors. Since Computer Electronics provides a free warranty service, they can generally afford to charge more. A market survey provides the following data on the effects different pricing decisions will have on the market share enjoyed by Computer Electronics:

		GCS		
		$900	$1,000	$1,200
	$1,000	15%	60%	80%
CE	$1,200	15%	60%	60%
	$1,300	10%	20%	40%

What would you recommend Computer Electronics charge?

18. ***More Price Wars*** Repeat Exercise 17 using the following revised data:

		GCS		
		$900	$1,000	$1,200
	$1,000	20%	60%	60%
CE	$1,200	15%	60%	60%
	$1,300	10%	20%	40%

In general, why do price wars tend to force prices down?

19. ***The Battle of Rabaul-Lae***[1] In the Second World War, during the struggle for New Guinea, intelligence reports revealed that the Japanese were planning to move a troop and supply convoy from the port of Rabaul at the Eastern tip of New Britain to Lae, which lies just west of New Britain on New Guinea. It could travel either via a northern route which was plagued by poor visibility, or by a southern route, where the visibility was clear. General Kenney, who was the commander of the Allied Air Forces in the area, had the choice of concentrating reconnaissance aircraft on one route or the other and bombing the Japanese convoy once it was sighted. Kenney's staff drafted the following outcomes for his choices, where the payoffs are estimated days of bombing time:

		Japanese Commander's Strategies	
		Northern Route	**Southern Route**
Kenney's Strategies	**Northern Route**	2	2
	Southern Route	1	3

(a) What would you have recommended to General Kenney?
(b) What would you have recommended to the Japanese Commander?[2]

20. ***The Battle of Rabaul-Lae*** Referring to Exercise 19, suppose that General Kenney had a third alternative: Splitting his reconnaisance aircraft between the two routes, resulting in the following estimates:

		Japanese Commander's Strategies	
		Northern Route	**Southern Route**
	Northern Route	2	2
Kenney's Strategies	**Split Reconnaisance**	1.5	2.5
	Southern Route	1	3

(a) What would you have recommended to General Kenney?
(b) What would you have recommended to the Japanese Commander?

[1] As discussed in *Games and Decisions* by R.D. Luce and H. Raiffa, Section 11.3 (New York; Wiley, 1957). This is based on on an article in the *Journal of the Operations Research Society of America* **2** (1954) 365–385.
[2] The correct answers to parts (a) and (b) correspond to the actual decisions both commanders made.

G.2 Strictly Determined Games

In the preceding section, we saw that some games could not be reduced to 1×1 games using dominance. For example, consider the following payoff matrix.

$$\mathbf{A}\quad \begin{array}{c} \\ s \\ t \\ u \end{array}\overset{\displaystyle\mathbf{B}}{\begin{array}{ccc} p & q & r \end{array}} \begin{bmatrix} -4 & -3 & 3 \\ 2 & -1 & -2 \\ 1 & 0 & 2 \end{bmatrix}$$

This game cannot be reduced at all; no row is dominated by any other row, and no column is dominated by any other column. Thus we are faced with the following question: Once we have a payoff matrix that cannot be further reduced by dominance and is bigger than a 1×1 game, *how do we decide on a best pure strategy for each of the players?*

In order to answer this question, we shall make the following basic assumptions:

Fundamental Principles of Game Theory

1. Each player makes the best possible move.

2. Each player knows that his or her opponent is also making the best possible move.

Question Just what is meant by the "best possible move?"

Answer Look at the payoff matrix above, and pretend that you are player A (and thus playing the rows). As player A, you might at first be tempted to play s, since the largest possible payoff is the 3 in row s. But, since B is also planning to make the best possible move, B is hardly likely to play r and give you a win of three points, since B sees the 3 as a danger signal![1] Thus B will probably play p or q instead, causing you to lose. A little thought might convince you that your most judicious strategy would be to "cut your losses." By this, we mean the following: look for the *worst* (that is, smallest) possible payoff in each row, note it, and choose the row that gives the "least of all evils." In other words, select the row that gives the *largest* worst payoff. This is A's **optimal pure strategy under the minimax** (or **maximin**) **criterion**. It consists of selecting the strategy that maximizes the minimum possible payoff.

Optimal Pure Strategy (Minimax Criterion)

To locate the optimal pure strategy for the row player, first circle the **row minima:** the smallest payoff(s) in each row. Then select the largest row minimum. (If there are two or more largest row minima, choose either one.)

To locate the optimal pure strategy for the column player, first box the **column maxima:** the largest payoff in each column. Then select the smallest column maximum. (If there are two or more smallest column maxima, choose either one.)

[1] Remember that large entries are "good" for A, but terrible for B.

Note

Reduction by dominance does not effect the optimal pure strategies (why?). Thus, it is not necessary to first reduce by dominance when locating the optimal pure strategies.

Quick Example

$$\mathbf{A}\quad\begin{array}{c} \\ s \\ t \\ u\end{array}\begin{array}{c}\mathbf{B}\\ \begin{bmatrix} p & q & r \\ (-4) & -3 & 3 \\ 2 & -1 & (-2) \\ 1 & (0) & 2\end{bmatrix}\end{array}\begin{array}{c}\textbf{Row minima}\\ -4 \\ -2 \\ 0\ \text{(largest)}\end{array}$$

The largest of the row minima is 0, so the optimal pure strategy for player A is to play u.

$$\begin{array}{cc} & \begin{array}{ccc} p & q & r\end{array}\\ \mathbf{A}\quad\begin{array}{c} s \\ t \\ u\end{array} & \begin{bmatrix} -4 & -3 & \boxed{3} \\ \boxed{2} & -1 & -2 \\ 1 & \boxed{0} & 2\end{bmatrix}\\ \textbf{Column maxima} & \begin{array}{ccc} 2 & 0 & 3\end{array}\\ & \text{(smallest)}\end{array}$$

The smallest of column maxima is 0, so player B's optimal pure strategy is to play q.

Remember that the minimax criterion is based on the assumption that your opponent is very smart. This assumption does not always hold in management decision making, and then other criteria may be appropriate. For example, there is the “maximax” criterion, which maximizes the maximum possible payoff (also known as the “reckless” strategy), or the criterion that seeks to minimize “regret” (the difference between the payoff you get and the payoff you *would have gotten* if you had known beforehand what was going to happen).[1]

Example 1 *Optimal Pure Strategy*

Use the minimax criterion to find each player's optimal pure strategy in the “modified” paper, scissors, rock game from the last section.

$$\mathbf{A}\quad\begin{array}{c} \\ p \\ s \\ r\end{array}\begin{array}{c}\mathbf{B}\\ \begin{bmatrix} p & s & r \\ -2 & -3 & 4 \\ 1 & -1 & 3 \\ 3 & 3 & 0\end{bmatrix}\end{array}$$

Solution We *could* first reduce the game by dominance and eliminate column p (since it is dominated by column s) as we did in the last section. However, reduction by dominance

[1] See *Location in Space: Theoretical Perspectives in Economic Geography*, 3rd Edition, by Peter Dicken and Peter E. Lloyd, HarperCollins Publishers, 1990, pp. 276 ff.

won't affect the optimal strategies, so we'll leave it in. We now go ahead and circle the row minima and box the column maxima, getting

$$\mathbf{A}\ \begin{matrix} p \\ s \\ r \end{matrix} \overset{\mathbf{B}}{\overset{\begin{matrix} p & s & r \end{matrix}}{\begin{bmatrix} -2 & (-3) & \boxed{4} \\ 1 & (-1) & 3 \\ \boxed{3} & \boxed{3} & (0) \end{bmatrix}}}.$$

Since the largest of the circled payoffs is the 0 in row *r*, A's optimal pure strategy is to play *r* (rock). The smallest boxed payoffs are the 3s in columns *p* and *s*, so B's optimal pure strategy is to play either *p* (paper) or *s* (scissors).

Before we go on... If each of the players plays the optimal pure strategy, player A will always play *r*, and player B will always play *p* or *s*, and the result will be that player A will win 3¢ on each round of the game.

But is this the end of the story? In other words, does the minimax critereon give us the best possible moves for each player under the Fundamental Principle? Pretend again that you were player A. Since player B is very savvy (and we assume this of both players, because of the Fundamental Principle), B will reason that you will play the optimal strategy of *r*. In that case, B would be foolish to play her "optimal" strategy of *s*, and lose 3¢ on each round; she would play *r* instead, and not lose anything! But wait a minute! Since you are the equally savvy player A, you *know* that player B is going to reason that *r* is the better bet, so you had better play *p*, thereby winning 4¢ each round! But then player B, being "super-savvy," and reasoning that you have seen through the ruse and elected to play *p*, will decide instead to play *s*, thereby winning 3¢ on each round. Since you too are "super-savvy," you see through this at once, and, knowing that B will play *s*, you decide you had better play *r*, which was your original strategy! But then, player B . . . , and so we go around in circles.

Question It seems that the minimax approach gets us nowhere, so what is going on?
Answer The problem is that this game is not a "strictly determined" game; there is really *no pure strategy that can be guaranteed to be the best*, under the assumptions of the Fundamental Principles of game theory.

Question Just what is a "strictly determined" game?
Answer Take a look at the following definition.

Saddle Points; Strictly Determined Games
A **saddle point** is an entry that is simultaneously a row minimum and a column maximum. If a game has one or more saddle points, it is **strictly determined**.

To locate saddle points, circle the row minima and box the column maxima. The saddle points are those entries that are both circled and boxed. (If there are no saddle points, the game is not strictly determined.)

Notes

1. All saddle points will have the same payoff value, called the **value of the game.** A **fair** game has a value of zero; otherwise it is **unfair**, or **biased**.
2. Choosing the row and column through any saddle point gives optimal strategies for both players under the minimax criterion.

Quick Examples

1. In the following game, we have circled the row minima and boxed the column maxima

$$
\mathbf{A}\quad
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4
\end{array}
\overset{\displaystyle \mathbf{B}}{
\left[\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\boxed{4} & (-5) & -3 & 2 \\
0 & \boxed{(-1)} & \boxed{2} & \boxed{4} \\
1 & -3 & (-5) & 3 \\
\boxed{4} & \boxed{(-1)} & 1 & 0
\end{array}\right]}
$$

This process reveals two saddle points with payoff value -1. Thus, the game is strictly determined with value -1.

2. The "modified" paper, scissors, rock game we considered above has no saddle points, and is therefore not strictly determined.

$$
\mathbf{A}\quad
\begin{array}{c}
\\ p \\ s \\ r
\end{array}
\overset{\displaystyle \mathbf{B}}{
\left[\begin{array}{ccc}
p & s & r \\
-2 & (-3) & \boxed{4} \\
1 & (-1) & 3 \\
\boxed{3} & \boxed{3} & (0)
\end{array}\right]}
$$

Notes

1. If a game is strictly determined, then no matter how "savvy" the players are, their reasoning won't go around in circles; their best strategies are their optimal strategies, given by selecting rows and columns through saddle points (it doesn't matter which, since they all have the same value). In a strictly determined game, the optimal pure strategies for each player intersect in a saddle point.

2. If a game reduces to a 1×1 game by dominance, then it is strictly determined. Thus, if you are faced with a game that is not strictly determined, don't bother trying to reduce it by dominance; it won't result in a 1×1 game! In other words: *check first to see whether there are saddle points.* Reduction by dominance won't help us in this section, but will prove useful later on.

3. If a game is *not* strictly determined, then there is no best pure strategy for the players under the assumptions of the Fundamental Principles; applying these principles will lead us around in circles (as illustrated above).

Example 2 *Heating Costs*[1]

Now that it is mid-summer, you have prudently decided to begin planning for the coming winter. Your home heating oil tank, which is now empty, has a capacity of 200 gallons. Over the years, you have noticed that your winter heating oil consumption depends on the severity of the winter as follows:

mild winter: 100 gals.;
average winter: 150 gals.;
severe winter: 200 gals.

The price of oil also seems to fluctuate with the severity of the winters:

mild winter: \$1 per gal.;
average winter: \$1.50 per gal.;
severe winter: \$2 per gal.

You are trying to decide whether to stockpile 100 gals., 150 gals. or 200 gals. at the present price of \$1 per gallon. The problem is that, if you stockpile more than you need, the unused oil will probably go to waste, since you will be moving away next summer. Also—although your friends all assure you that you are simply being paranoid—you are convinced that Nature is conspiring against you, and can thus be regarded as an intelligent adversary. What should you do?

Solution The two players are Nature and You. Your three strategies are the sizes of your stockpile: 100 gals., 150 gals. and 200 gals. Nature's three strategies are the severity of the winter: mild, average and severe. As for the payoffs, we must interpret the question "what should you do?" Presumably, this means: "what should you do in order to save the most money?" Thus we shall take the payoffs to be the total costs of heating in the various contingencies. Since these are payoffs to Nature, they will have to be negative if you are playing the rows. Here is what the payoff matrix should look like (in chart form):

		Nature		
		Mild Winter	Average Winter	Severe Winter
	100 gals.			
You	150 gals.			
	200 gals.			

[1] This is a modified version of an example that appears in the classic book on game theory, *The Compleat Strategyst* by J.D. Williams (McGraw Hill Book Company, 1966).

Our task is now to fill in the payoffs. For this, we need to do a little cost accounting:
100 gal. stockpile in mild winter: no extra oil needed;
cost = 100 gals. @ $1 per gal. = $100
150 gal. stockpile in mild winter: no extra oil needed;
cost = 150 gals. @ $1 per gal. = $150
200 gal. stockpile in mild winter: no extra oil needed;
cost = 200 gals. @ $1 per gal. = $200
100 gal. stockpile in average winter: 50 extra gals. needed;
cost = 100 gals. @ $1 per gal. plus 50 gals. at $1.50 per gal = $175
150 gal. stockpile in average winter: no extra oil needed;
cost = 150 gals. @ $1 per gal. = $150
200 gal. stockpile in average winter: no extra oil needed;
cost = 200 gals. @ $1 per gal. = $200
100 gal. stockpile in severe winter: 100 extra gals. needed;
cost = 100 gals. @ $1 per gal. plus 100 gals. at $2 per gal. = $300
150 gal. stockpile in severe winter: 50 extra gals. needed;
cost = 150 gals. @ $1 per gal. plus 50 gals. at $2 per gal. = $250
200 gal. stockpile in severe winter: no extra oil needed;
cost = 200 gals. @ $1 per gal. = $200

The payoff matrix is thus:

		Nature		
		Mild Winter	Average Winter	Severe Winter
	100 gals.	−100	−175	−300
You	150 gals.	−150	−150	−250
	200 gals.	−200	−200	−200

We now circle the row minima and box the column maxima, getting:

		Nature		
		Mild Winter	Average Winter	Severe Winter
	100 gals.	[−100]	−175	(−300)
You	150 gals.	−150	[−150]	(−250)
	200 gals.	(−200)	(−200)	([−200])

This gives us a saddle point and shows that stockpiling 200 gals. is the optimal strategy.

Before we go on...
Question Would it not be wiser to stockpile 150 gals. and thus save $50 in the event of a mild or average winter?

Answer It would seem so, even though the principles of game theory tell us that stockpiling 200 gals. is the best strategy. This discrepancy between game theory and common sense arises from the fact that we have assumed that Nature will be actively planning its own optimal strategy in the game. In other words, we are assuming that Nature will be employing the Fundamental Principle of Game Theory. (Look at the comment about paranoia in the question.) If Nature is indeed planning to use an optimal strategy, we can forget about the possibility of a mild or average winter, and count on the worst!

G.2 Exercises

In Exercises 1–8, determine the optimal pure strategies under the minimax criterion for both players. Indicate whether the given game is strictly determined, and give its value if this is the case.

1. $$\mathbf{A}\ \begin{array}{c} \\ 1 \\ 2 \end{array}\overset{\mathbf{B}}{\begin{array}{cc} 1 & 2 \\ \hline \end{array}}\ \begin{bmatrix} 1 & 1 \\ 2 & -4 \end{bmatrix}$$

2. $$\mathbf{A}\ \begin{array}{c} 1 \\ 2 \end{array}\begin{bmatrix} -1 & 2 \\ 10 & -1 \end{bmatrix} \quad (\mathbf{B}\text{: columns } 1, 2)$$

3. $$\mathbf{A}\ \begin{array}{c} 1 \\ 2 \end{array}\begin{bmatrix} -2 & 1 & -3 \\ -2 & 3 & -2 \end{bmatrix} \quad (\mathbf{B}\text{: columns } 1, 2, 3)$$

4. $$\mathbf{A}\ \begin{array}{c} 1 \\ 2 \end{array}\begin{bmatrix} 2 & 0 & -2 \\ -1 & 3 & 0 \end{bmatrix} \quad (\mathbf{B}\text{: columns } 1, 2, 3)$$

5. $$\mathbf{A}\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} -3 & -5 & -5 \\ -3 & -3 & -1 \\ -5 & -10 & -4 \end{bmatrix} \quad (\mathbf{B}\text{: columns } a, b, c)$$

6. $$\mathbf{A}\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} 0 & 0 & 3 \\ -1 & -2 & -3 \\ 0 & 0 & 4 \end{bmatrix} \quad (\mathbf{B}\text{: columns } a, b, c)$$

7. $$\mathbf{A}\ \begin{array}{c} A \\ B \\ C \\ D \end{array}\begin{bmatrix} 1 & -1 & -5 \\ 4 & -4 & 2 \\ 3 & -3 & -10 \\ 5 & -5 & -4 \end{bmatrix} \quad (\mathbf{B}\text{: columns } a, b, c)$$

8. $$\mathbf{A}\ \begin{array}{c} A \\ B \\ C \\ D \end{array}\begin{bmatrix} -2 & -4 & 9 \\ 1 & 1 & 0 \\ -1 & -2 & -3 \\ 1 & 1 & -1 \end{bmatrix} \quad (\mathbf{B}\text{: columns } a, b, c)$$

Applications

9. *Hotel Expansion* June Fairweather, the owner of the Grand Hotel in Cancun, is contemplating adding an extra wing to accommodate a predicted surge in the number of vacationing college students. The problem is that she is not convinced by the optimistic predictions, and she recently had her accountants estimate how the Grand Hotel's net profits would be affected by the expansion, taking the tourist trade into account. Their report gives figures based on four possible changes in the tourist trade: a 20% drop, no change, a 20% increase, and a 50% increase,. The results are summarized in the following table which shows estimates of the annual increase in net profits:

	20% Drop	No Change	20% Increase	50% Increase
No Expansion	\$0	\$10,000	\$10,000	\$10,000
Expansion	−\$400,000	−\$300,000	\$10,000	\$100,000

(a) What is June Fairweather's optimal strategy?
(b) Would you advise June Fairweather to change her strategy if you were certain there will be at least a 20% increase in tourism?

10. ***Staff Cutbacks*** Frank Tempest manages a large snow-plow service in Manhattan, Kansas, and is alarmed by the recent weather trends; there have been no significant snowfalls since 1993. He is therefore contemplating laying off some of his workers, but is unsure about whether to lay off 5, 10, or 15 of his 50 workers. Being very methodical, he estimates his annual net profits based on four possible annual snowfall figures: 0 inches, 20 inches, 40 inches, and 60 inches. (He takes into account the fact that, if he is running a small operation in the face of a large annual snowfall, he will lose business to his competitors as he will be unable to discount on volume.)

	0 inches	20 inches	40 inches	60 inches
5 laid off	−\$500,000	−\$200,000	\$10,000	\$200,000
10 laid off	−\$200,000	\$0	\$0	\$0
15 laid off	−\$100,000	\$10,000	−\$200,000	−\$300,000

(a) Does the chart give a strictly determined game? What is his optimal strategy?
(b) How would your answers to part (a) change if he were to count on at least 40 inches of snow per year?

11. ***The Prisoner's Dilemma*** Slim Shady and Joe Rap have been arrested for grand theft auto, having been caught red-handed driving away in a stolen '98 Porsche. Although the police have more than enough evidence to convict them both, they feel that a confession would simplify the work of the prosecution. They decide to interrogate the prisoners separately. Slim and Joe are both told of the following plea-bargaining arrangement: if both confess, they will each receive a two-year sentence; if neither confesses, they will both receive 5-year sentences, and if only one confesses (and thus squeals on the other), he will receive a suspended sentence, while the other will receive a 10-year sentence. What should Slim do?

12. ***More Prisoners' Dilemmas*** Jane Good and Prudence Brown have been arrested for robbery, but the police lack sufficient evidence for a conviction, and so decide to interrogate them separately in the hope of extracting a confession. Both Jane and Prudence are told the following: if they both confess, they will each receive a 5-year sentence; if neither confesses, they will be released; if one confesses, she will receive a suspended sentence, while the other will receive a 10-year sentence. What should Jane do?

13. ***The Battle of Rabaul-Lae Revisited*** In Exercise 19 of the previous section, we had the following payoff matrix:

		Japanese Commander's Strategies: Northern Route	Japanese Commander's Strategies: Southern Route
Kenney's Strategies	**Northern Route**	2	2
	Southern Route	1	3

(a) Show that this is a strictly determined game. Show also that each commander could not have improved on his optimal strategy even if he knew in advance that his opponent was planning to use the optimal strategy.

(b) Would this conclusion still have applied in the event of the following payoff matrix?

		Japanese Commander's Strategies: Northern Route	Japanese Commander's Strategies: Southern Route
Kenney's Strategies	**Northern Route**	4	1
	Southern Route	2	3

14. ***The Battle of Rabaul-Lae Revisited*** In Exercise 20 of the previous section, we had the following payoff matrix:

		Japanese Commander's Strategies: Northern Route	Japanese Commander's Strategies: Southern Route
	Northern Route	2	2
Kenney's Strategies	**Split Reconnaisance**	1.5	2.5
	Southern Route	1	3

(a) Show that this is a strictly determined game. Show also that each commander could not have improved on his optimal strategy even if he knew in advance that his opponent was planning to use the optimal strategy.

(b) Would this conclusion still have applied in the event of the following payoff matrix?

		Japanese Commander's Strategies: Northern Route	Japanese Commander's Strategies: Southern Route
	Northern Route	5	1
Kenney's Strategies	**Split Reconnaisance**	2	3
	Southern Route	1	4

15. ***Medicine*** Following is part of a table that appeared in *The New York Times* (March 4, 1991).[1] The table shows the percentage of patients using one of three anti-stroke drugs who subsequently went on to experience a stroke:

[1] Source: *Third International Study of Infarct Survival* (ISIS-3), Oxford University.

	Drug Streptokinase	T.P.A.	Eminase
Strokes, probably from cerebral hemmorrhage	0.3%	0.7%	0.6%
Other strokes	0.8%	0.8%	0.9%

You would like to advise a patient to use one of these three drugs. Use this data to set up a payoff matrix, and hence determine what you would advise the patient. Does your answer lead to the lowest overall recurrence rate? If not, comment on why.

16. ***Medicine*** Repeat Exercise 15 in the (fictitious) event that the results had been as follows:

	Drug Streptokinase	T.P.A.	Eminase
Strokes, probably from cerebral hemmorrhage	0.3%	0.7%	0.3%
Other strokes	0.9%	0.8%	0.9%

17. ***Business Location***[1] Two hot dog sellers are deciding where they will locate along a stretch of highway. Customers are spread out evenly along this highway, and, assuming that the sellers sell their hot dogs at the same price, will prefer the closer seller. Suppose that there are 3 possible locations along the highway, and 40 potential sales. For the payoff matrix we look at the difference between seller I's sales and seller II's sales.

		Seller II's location a	b	c
Seller I's location	**a**	0	–30	0
	b	30	0	30
	c	0	–30	0

Is this game strictly determined? If so, what is the optimal strategy for each seller?

18. ***Business Location*** Repeat Exercise 17, except use 5 locations, which gives the following payoff matrix:

		Seller II's location a	b	c	d	e
Seller I's location	**a**	0	–30	–20	–10	0
	b	30	0	–10	0	10
	c	20	10	0	10	20
	d	10	0	–10	0	30
	e	0	–10	–20	–30	0

[1] Taken from an example in *Location in Space: Theoretical Perspectives in Economic Geography*, 3rd Edition, by P. Dicken and P.E. Lloyd, Harper & Row, 1990. This is an old problem, first analyzed by H.Hotelling, and analyzed using game theory by B.H. Stevens.

19. *Swords and Sorcerers* You are in command of three divisions of elves, and are planning an attack on two fortresses, the Dark Tower and Karnack, held jointly by two powerful divisions of trolls. You must decide how many divisions to send to each fortress in order to maximize your gains, while the enemy is making similar plans. (No divisions can be split.) You estimate that one troll division is equal in strength to two of your divisions. If x elf divisions encounter y troll divisions at a fortress, then the payoff for that fortress is 1 if $2x > y$ (since you would win the battle), 0 if $2x = y$, and -1 if $y > 2x$. The total payoff is then the sum of the payoffs for the two fortresses. Set up the payoff matrix, determine whether this game is strictly determined, and what your best strategy would be in the event that it is.

20. *Swords and Sorcerers* You are Manfred the Magician, and you can cast three kinds of spells: sleep spells, immobilization spells and mind-wipe spells. You are confronted by your dreadful rival, Mazdrud the Feared, who is capable of invoking four kinds of shields: a camouflage shield that blocks all but your mind-wipe spell (against which it is useless), an invisibility shield that gives him even odds against each of your spells, a heavy duty shield that blocks immobilization shields, is no use against sleep spells, and gives him even odds against mind-wipe spells, and an all-purpose shield, which is gives him even odds against your sleep spell, but is useless against the other two. Assuming you can cast only one of your spells, and that you will lose the battle if your spell is blocked, which should you employ? Is it a fair game?

Communication and Reasoning Exercises

21. Why is a saddle point called a “saddle point?” (Use a figure to illustrate your answer.)

22. Can the payoff in a saddle point ever be larger than all other payoffs in a game? Explain.

23. *Employment* One day, while browsing through an old *Statistical Abstract of the United States* you came across the following data which show the number of females employed (in thousands) in various categories according to their educational attainment.[1]

	Managerial/ Professional	Technical/Sales/ Adminisrative	Service	Precision Production	Operators/ Fabricators
Less than 4 years high school	260	1,084	2,016	258	1,398
4 years of high school only	2,426	9,511	3,596	569	2,126
1 to 3 years of college	2,690	5,075	1,080	156	353
At least 4 years of college	7,210	2,756	377	71	105

[1] Source: *Statistical Abstract of the United States 1991* (111th Ed.) U.S. Department of Commerce, Economics and Statistics Administration, and Bureau of the Census.

Since you had been studying game theory that day, the first thing you did was to search for a saddle point. Having found one, you concluded that, as a female, your best strategy in the job market is to forget about a college career. Find the flaw in this reasoning.

24. ***Employment*** The following data, from the same Statistical Abstract of the United States referred to in the preceding exercise, shows the number of males employed (in thousands) in various categories according to their educational attainment:

	Managerial/ Professional	**Technical/Sales/ Adminisrative**	**Service**	**Precision Production**	**Operators/ Fabricators**
Less than 4 years high school	476	643	1,022	2,268	3,074
4 years of high school only	2,393	3,469	1,793	5,788	5,348
1 to 3 years of college	2,748	2,928	946	2,200	1,443
At least 4 years of college	10,160	3,443	459	735	462

You would like to choose a career from the category above representing the largest number of employed males. Is the minimax criterion suitable? If not, formulate a criterion that is suitable, and use it to determine an optimal pure strategy.

25. Construct a 4×4 payoff matrix that contains exactly four saddle points.

26. Construct a 4×4 payoff matrix that contains exactly two saddle points in the same column.

27. Formulate an interesting application leading to a 3×3 payoff matrix with a single saddle point, and where your strategies might be: “buy gold,” “buy stocks,” and “buy bonds.”

28. Formulate an interesting application leading to 3×3 payoff matrix with no saddle points.

In the following strictly determined games, continue the following line of reasoning to demonstrate that there is no best pure strategy for the players if they follow the Fundamental Principle.

> If the row player plays the pure strategy 1, then the column player will play the pure strategy ___.
> This will cause the row player to switch to the pure strategy ___.
> This in turn will cause the column player to ...
> ...

29.

$$\begin{array}{cc} & \mathbf{B} \\ & \begin{array}{ccc} a & b & c \end{array} \\ \mathbf{A} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{ccc} -3 & -5 & -5 \\ -3 & -3 & -1 \\ 0 & -2 & -4 \end{array}\right] \end{array}$$

30.

$$\begin{array}{cc} & \mathbf{B} \\ & \begin{array}{ccc} a & b & c \end{array} \\ \mathbf{A} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{ccc} 0 & 3 & 3 \\ -1 & -2 & -3 \\ 3 & 2 & 4 \end{array}\right] \end{array}$$

In the following strictly determined games, complete the following line of reasoning to demonstrate that, if both players use the Fundamental Principle, then both eventually play strategies corresponding to a saddle point.

> If the row player plays the pure strategy *A*, then the column player will play the pure strategy ___.
> This will cause the row player to switch to the pure strategy ___.
> This in turn will cause the column player to ...

...

31.

$$\begin{array}{cc} & \mathbf{B} \\ & \begin{array}{ccc} a & b & c \end{array} \\ \mathbf{A} \begin{array}{c} A \\ B \\ C \\ D \end{array} & \left[\begin{array}{ccc} 1 & -4 & -5 \\ 4 & -4 & 2 \\ 3 & -3 & -2 \\ 5 & -5 & -4 \end{array}\right] \end{array}$$

32.

$$\begin{array}{cc} & \mathbf{B} \\ & \begin{array}{ccc} a & b & c \end{array} \\ \mathbf{A} \begin{array}{c} A \\ B \\ C \\ D \end{array} & \left[\begin{array}{ccc} -2 & -4 & 0 \\ 0 & 3 & -2 \\ -1 & -2 & -3 \\ 1 & 2 & 2 \end{array}\right] \end{array}$$

Does the line of reasoning used in the above exercises always “zero in” on a saddle point if the payoff matrix has one? Illustrate your answer by referring to the following payoff matrices.

33.

$$\begin{array}{cc} & \mathbf{B} \\ & \begin{array}{ccc} a & b & c \end{array} \\ \mathbf{A} \begin{array}{c} A \\ B \\ C \\ D \end{array} & \left[\begin{array}{ccc} 1 & -4 & -5 \\ -5 & -4 & 2 \\ 3 & -3 & -2 \\ 5 & -5 & -6 \end{array}\right] \end{array}$$

34.

$$\begin{array}{cc} & \mathbf{B} \\ & \begin{array}{ccc} a & b & c \end{array} \\ \mathbf{A} \begin{array}{c} A \\ B \\ C \\ D \end{array} & \left[\begin{array}{ccc} -2 & -4 & 9 \\ 1 & 2 & 0 \\ -1 & -2 & -3 \\ 1 & 1 & 2 \end{array}\right] \end{array}$$

35. What—if any—conclusions can you draw from Exercises 29–34?

G.3 Mixing Strategies

In the preceding sections we talked about finding the best pure strategy in a game. However, many games are not strictly determined and hence lack a best pure strategy according to the Fundamental Principle of game theory.

Question What do I do when I play a game that has no saddle points?
Answer The key is to realize that it is dangerous to stick to a single strategy. Take, as an example, the non-strictly determined game we encountered back in Section G.1.

$$\begin{array}{cc} & \textbf{B} \\ & \begin{array}{cc} s & r \end{array} \\ \textbf{A} \begin{array}{c} p \\ s \\ r \end{array} & \begin{bmatrix} -3 & 4 \\ -1 & 3 \\ 3 & 0 \end{bmatrix} \end{array}$$

If you are the row player and you stick to any one strategy, you will definitely lose: If you stick to p, player B will catch on and start playing s; if you stick to s, B will again play s, and if you stick to r, B will switch to r. Thus you seem to be in a “no-win” situation, even with those nice positive payoffs in the matrix. However, if you choose your row at random, perhaps you could confuse B and win some games.

Question So I'll switch strategies at random. Does this mean that I should play each of the three strategies one third of the time?
Answer Not necessarily. Strategy s looks better for you than p, since you would then avoid the danger of playing p versus s and losing 3 points. Perhaps it might be best to choose only between strategies r and s.

Question So how often should I choose each strategy?
Answer That is the $64,000 question, and we'll answer it in Section G.4. The important thing now is to recognize that, in some games, it is best to use a **mixed strategy**, in which, instead of sticking to a single strategy, you choose at random, perhaps choosing one strategy more often than another.

Question How do I choose strategies at random in this way?
Answer Suppose, for instance, you have decided to play p one sixth of the time, s one third of the time, and r the rest of the time. At each turn, roll a die. If you roll a 1, play p; since 1 comes up one sixth of the time, you will wind up playing p one sixth of the time. If you roll a 2 or a 3, play s, and if you roll a 4, 5 or 6, play r. If the fractions involved are fifths instead of sixths, you could use a five-sided die, or roll a regular die and ignore the outcome if it's a 6. Or, you could use a random number generator to produce random digits from 1 to 5. There are many ways to produce random numbers that we can use to choose rows in the proportions we wish.

Here is a convenient way to represent a mixed strategy. Suppose you decide to play *p* one sixth of the time, *s* one third of the time, and *r* one half of the time. You can represent this decision by the following row matrix:

$$S = \begin{matrix} & p & s & r \\ & [\frac{1}{6} & \frac{1}{3} & \frac{1}{2}] \end{matrix}$$

Note that the sum of the entries in this matrix is 1, since each time we must play some strategy.[1] We will call S the **matrix of the (mixed) strategy.** or simply the **mixed strategy.** Notice that we can also use a matrix to represent a *pure* strategy. For instance, the strategy of always playing *s* can be represented by the matrix

$$[0 \quad 1 \quad 0].$$

Now that we've settled for a mixed strategy approach, we must also assume that player B has decided likewise. For reasons we will see in a moment, we use a *column* to represent the column player's mixed strategy. B has two possible strategies, *s* and *r*. If, for instance, B decides to play *s* one third of the time, and *r* two thirds of the time, we represent B's strategy by the column matrix

$$\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

Now suppose A plays the mixed strategy $S = [\frac{1}{6} \quad \frac{1}{3} \quad \frac{1}{2}]$ and B plays

$$T = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}.$$

The following question may then occur to you (it occurred to us, at any rate).

Question If A and B use these mixed strategies, how much, on average, can A expect to win (or lose) on each round of the game?

Answer The average amount A can expect to win on each round of the game is called the **expected value** of the game for these mixed strategies. To calculate it, consider the following scenarios:

---

[1] If you have already studied probability theory, you will notice that the entries in this matrix are the *probabilities of your playing each of the strategies.* The sample space is $\{p, r, s\}$, and $P(p) = 1/6$; $P(s) = 1/3$, $P(r) = 1/2$. The sum must be 1 because of the fundamental rule that the sum of the probabilities of all the outcomes must add up to 1.

Scenario 1

A plays p, B plays s. Then the payoff matrix tells us that A loses 3 points. Since A only plays $p\ \frac{1}{6}$ of the time and B only plays $s\ \frac{1}{3}$ of the time, we expect this outcome to occur $\frac{1}{6} \times \frac{1}{3} = \frac{1}{18}$ of the time, in other words, only once in every 18 rounds. This gives a contribution of:

$$\frac{1}{6} \times \frac{1}{3} \times (-3) = -\frac{1}{3}$$

to the average value of the game.

Scenario 2

A plays s, B plays s. Then the payoff matrix tells us that A loses 1 point. Since A plays $s\ \frac{1}{3}$ of the time and B plays $s\ \frac{1}{3}$ of the time, we expect this outcome to occur $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$ of the time. This gives a contribution of:

$$\frac{1}{3} \times \frac{1}{3} \times (-1) = -\frac{1}{9}$$

to the value of the game.

We could continue like this and list the remaining 4 scenarios (there are 6 altogether, one for each entry in the payoff matrix), and then obtain the answer by adding up all the contributions. Here is a far more convenient way of doing this: *Calculate the product SPT of the matrices, where P is the payoff matrix!* This is exactly the same calculation:

$$SPT = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -3 & 4 \\ -1 & 3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{5}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix} = \frac{24}{18}.$$

If you write out the arithmetic involved, we have really calculated

$$\frac{1}{6} \times \frac{1}{3} \times (-3) + \frac{1}{3} \times \frac{1}{3} \times (-1) + \ldots .$$

This says that A can expect to win an average of 24/18 points at each round, or 24 points every 18 rounds. This is certainly better than A can do by sticking to any pure strategy (remember our analysis at the beginning of the section).

To summarize:

Calculating the Expected Value of a Game for Mixed Strategies *S* and *T*

1. Write the row player's mixed strategy as a row matrix S (checking that the entries add to 1).
2. Write the column player's mixed strategy as a column matrix T (again checking that the entries add to 1).
3. Calculate the product SPT, where P is the payoff matrix.

Quick Example

Consider the original version of “Paper, Scissors, Rock,”

$$\begin{array}{cc} & \mathbf{B} \\ & \begin{array}{ccc} p & s & r \end{array} \\ \mathbf{A} \quad \begin{array}{c} p \\ s \\ r \end{array} & \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array}.$$

Suppose that the row player plays *paper* half the time, and each of the other two strategies a quarter of the time, and the column player always plays *rock*. This gives

$$S = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}, \quad T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The expected value of the game is then

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \end{bmatrix}.$$

Thus, player A wins an average of 1/4 points per game.

Technology

As many calculators can do matrix algebra, you can avoid doing these calculations by hand by using technology just as in the chapter on matrix algebra. This is true in the remaining examples as well.

Example 1 ***Television Ratings Wars***

The commercial TV station CTV and the educational station ETV are competing for viewers in the Tuesday prime-time 9–10 PM time slot. CTV is trying to decide whether to show a sitcom, a docudrama, a talk show, or a movie, while ETV is thinking about either a nature documentary, a symphony concert, a ballet or an opera. A television rating company estimates the payoffs for the various alternatives as follows. (Each point indicates a shift of 1,000 viewers from one channel to the other; thus, for instance, a −2 indicates a loss of 2,000 viewers to ETV.)

		ETV			
		Nature Doc.	Symphony	Ballet	Opera
	Sitcom	2	1	−2	2
CTV	Docudrama	−1	1	−1	2
	Talk Show	−2	0	0	1
	Movie	3	1	−1	1

(a) Assuming that each channel chooses its option randomly[1] each Tuesday (with each possibility equally likely), how many viewers would CTV gain?

(b) If ETV shows a ballet half the time and a nature documentary the other half, and if CTV uses all options equally often, how many viewers would CTV gain?

(c) If CTV notices that ETV is showing a ballet half the time and a nature documentary the other half, what would CTV's best pure strategy be, and how many viewers would they gain if they followed it?

Solution

(a) If each channel chooses its options at random, each of them would come up a quarter of the time on average, so that the mixed strategies for CTV and ETV are, respectively,

$$S = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}, \quad T = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}.$$

Thus the expected value of the game is:

$$e = SPT$$

$$= \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 & 2 \\ -1 & 1 & -1 & 2 \\ -2 & 0 & 0 & 1 \\ 3 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{7}{16} \end{bmatrix}$$

[1] Although this may seem like an odd approach to programming, it might actually be a good idea; a channel could use a widely publicized weekly drawing to decide on the theme for their Tuesday evening show.

Since each point corresponds to a gain of 1,000 viewers, it follows that CTV can expect to gain 7,000/16 ≈ 438 additional viewers.

(b) Here, CTV's mixed strategy is the same as in part (a), while ETV's mixed strategy is

$$T = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}.$$

Thus the expected value of the game is

$$\begin{aligned} e &= SPT \\ &= \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 & 2 \\ -1 & 1 & -1 & 2 \\ -2 & 0 & 0 & 1 \\ 3 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \end{bmatrix}. \end{aligned}$$

Thus, CTV will lose an average of $\frac{1}{4}(1{,}000) = 250$ viewers.

(c) This time, we are given ETV's strategy $T = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$, but we are not given CTV's strategy S.

Thus, we take $S = [x \quad y \quad z \quad t]$ and must come up with values for x, y, z and t in order to make the expected value e of the game as large as possible. All we know about these unknowns is that they must be non-negative and must add to 1 (why?). First, we calculate e in terms of these unknowns:

$$\begin{aligned} e &= SPT \\ &= [x \quad y \quad z \quad t] \begin{bmatrix} 2 & 1 & -2 & 2 \\ -1 & 1 & -1 & 2 \\ -2 & 0 & 0 & 1 \\ 3 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}. \end{aligned}$$

Multiplying the two matrices on the right first gives:

$$e = [x \quad y \quad z \quad t]\begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} = -y - z + t.$$

Now we want this to be as large as possible. Since y and z are being subtracted, we would like them to be 0. Since t is being added, we would like it to be as large as possible, which is to say, $t = 1$. This leaves $x = 0$. (Recall that $x + y + z + t = 1$) Thus, CTV's best strategy is $S = [0 \quad 0 \quad 0 \quad 1]$. In other words, CTV should use the pure strategy of showing a movie every Tuesday evening. If it does so, the expected value will be $e = -y - z + t = 1$, so CTV can expect to gain 1,000 viewers.

Before We Go On... Part (c) illustrates the fact that, no matter what mixed strategy ETV selects, CTV can choose an appropriate counter-strategy in order to maximize its viewership. Since both players are playing for maximum gain, it would therefore be in ETV's best interest to select a mixed strategy that *minimizes* the effect of CTV's best counter-strategy. Finding this strategy, and also CTV's counter-strategy, is called *solving the game*. We will see a way of solving 2×2 games in the next example.

Notes

1. In order to save space, we will sometimes write column matrices in transpose form; for instance, we will write $[0 \quad -1 \quad -1 \quad 1]^T$ instead of $\begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}$.

2. If you are given your opponent's mixed strategy, your best strategy is the one that leads to the *highest average payoff* for you, since this is the expected value of the game.

In the "Before We Go On" discussion above we talked about "solving a game." Here is a more precise definition.

Solving a Game

An **optimal mixed strategy** is a mixed strategy that minimizes the potential loss against the opponent's best counter-strategy. Finding optimal mixed strategies for both players in a two-person zero sum game is called **solving the game**.

If a game happens to be strictly determined, then the optimal mixed strategies are pure strategies, obtained by selecting a saddle point.

Question What do you mean by "minimizing the potential loss against the opponent's best counter-strategy" ?

Answer Pretend that you are the row player. For every mixed strategy you try, there is a counter-strategy by your opponent that maximizes his or her gain. Your most prudent choice is thus to use that mixed strategy for which the effect of your opponent's best counter-strategy is best for you. This is your optimal mixed strategy (there my be more than one of these). Thus, if you use any *non-optimal* mixed strategy, the column player can do more damage to you than would be the case if you used an optimal strategy.

In the next section, we will show you how to solve an arbitrary game. In the meantime, here is a method of solving 2×2 games.

Example 2 *Solving a 2×2 Game*

Suppose that you are the row player in the game with payoff matrix

$$P = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}.$$

(a) What is your optimal mixed strategy?
(b) What is the column player's optimal mixed strategy?
(c) If each player uses the optimal mixed strategy, what is the expected value of the game?

Solution

(a) Since you have no idea of what to do, select a general strategy:

$$S = [x \quad y].$$

Since $x + y = 1$, we can replace y by $1-x$, so

$$S = [x \quad 1-x].$$

If the column player happens to choose the first column strategy, $T = [1\ 0]^T$, then the expected value of the game is

$$\begin{aligned} e &= SPT \\ &= \begin{bmatrix} x & 1-x \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= (-1)x + 2(1-x) \\ &= -3x + 2. \end{aligned}$$

If, on the other hand, the column player happens to choose the second column strategy, $T = [0\ 1]^T$, then the expected value of the game is

$$f = SPT$$

$$= \begin{bmatrix} x & 1-x \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= 3x + 0(1-x)$$
$$= 3x.$$

Since both e and f depend on x, we can graph them as in Figure 1:

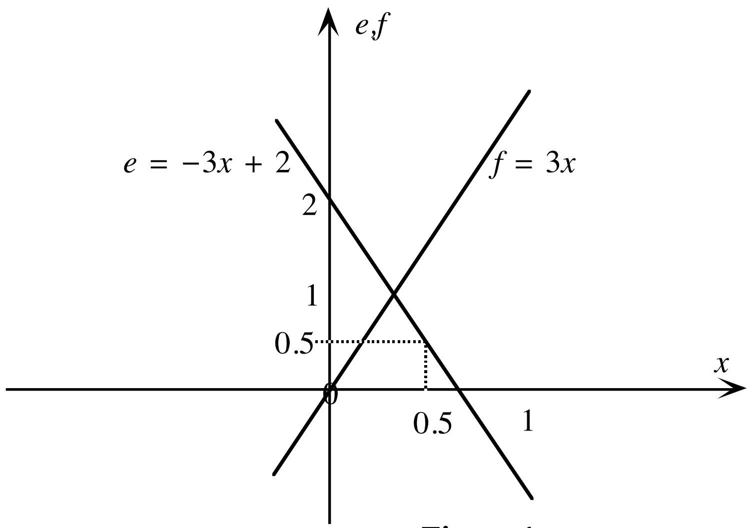

Figure 1

If, for instance, you happened to choose $x = 0.5$, then, since the graph of e is lower than the graph of f for that value of x, the worst possible outcome for you would occur if the column player plays the first strategy, in which case the outcome of the game would be 0.5. If instead you decided to choose $x = 0$, then since the graph of f is the lower of the two graphs when $x= 0$, the worst possible outcome for you would be if the column player played the second strategy, giving an outcome of 0. Since you are free to choose x to be any value between 0 and 1, the worst possible outcomes are shown by the bold portion of the graph (Figure 2).

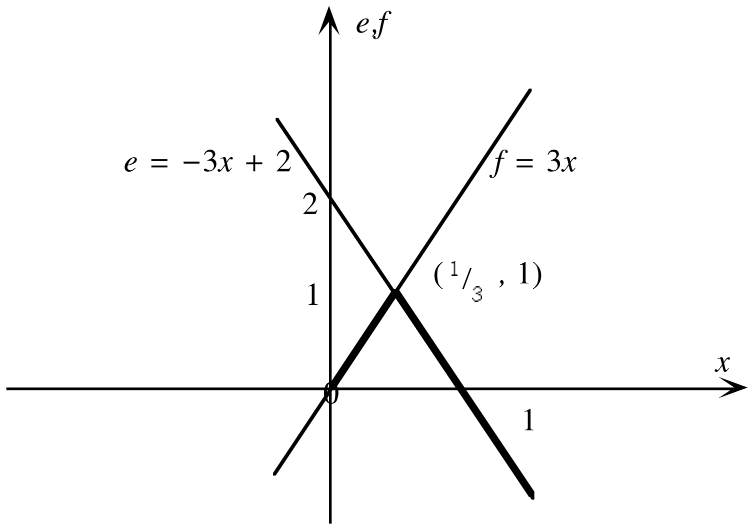

Figure 2

Since you are trying to make the worst possible outcome as large as possible (that is, to minimize damages), you are seeking the point on the bold portion of the graph that is highest. This is the intersection point of the two lines. To calculate its coordinates, we equate the two functions of *x*:

$$-3x + 2 = 3x,$$

giving $6x = 2,$

or $x = \frac{1}{3}.$

The *e* (or *f*) coordinate is then obtained by substituting $x = 1/3$ into the expression for *e* (or *f*) giving:

$$\begin{aligned} e &= -3x + 2 \\ &= -3\left(\tfrac{1}{3}\right) + 2 \\ &= 1. \end{aligned}$$

Since the intersection point is $(1/3,\ 1)$, you conclude that your best strategy is to take $x = 1/3$, giving an expected value of 1. Thus the row player's optimal mixed strategy is:

$$S = \left[\tfrac{1}{3} \quad \tfrac{2}{3}\right].$$

(b) To obtain the best strategy for the column player, we must reverse roles: start with the column player using the strategy

$$T = \begin{bmatrix} x \\ 1-x \end{bmatrix},$$

and calculate the expected values for the two pure row strategies:

$$\begin{aligned} e &= SPT \\ &= [1 \quad 0]\begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ 1-x \end{bmatrix} \\ &= (-1)x + 3(1-x) \\ &= -4x + 3 \end{aligned}$$

and

$$\begin{aligned} f &= SPT \\ &= [0 \quad 1]\begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ 1-x \end{bmatrix} \\ &= 2x + 0(1-x) \\ &= 2x. \end{aligned}$$

As with the row player, we know that the column player's best strategy will correspond to the intersection of the graphs of *f* and *g* (Figure 3).

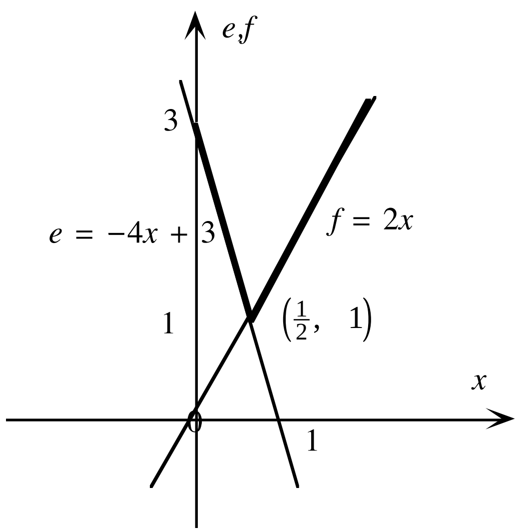

Figure 3

(Why is the upper edge heavy, rather than the lower edge?) The graphs intersect when

$$2x = -4x + 3,$$

or

$$x = \frac{1}{2}.$$

Thus the column player's optimal mixed strategy is $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.

(c) We can now calculate the expected value of the game as usual:

$$e = SPT$$

$$= [\tfrac{1}{3} \quad \tfrac{2}{3}] \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \tfrac{1}{2} \\ \tfrac{1}{2} \end{bmatrix}$$

$$= 1.$$

G.3 Exercises

In Exercises 1–6, calculate the expected value of the game with payoff matrix

$$P = \begin{bmatrix} 2 & 0 & -1 & 2 \\ -1 & 0 & 0 & -2 \\ -2 & 0 & 0 & 1 \\ 3 & 1 & -1 & 1 \end{bmatrix},$$

using the mixed strategies supplied.

1. $S = [0 \ \ 1 \ \ 0 \ \ 0]$, $T = [1 \ \ 0 \ \ 0 \ \ 0]^T$

2. $S = [0 \ \ 0 \ \ 0 \ \ 1]$, $T = [0 \ \ 1 \ \ 0 \ \ 0]^T$

3. $S = [\tfrac{1}{2} \ \ \tfrac{1}{2} \ \ 0 \ \ 0]$, $T = [0 \ \ 0 \ \ \tfrac{1}{2} \ \ \tfrac{1}{2}]^T$

4. $S = [0 \ \ \tfrac{1}{2} \ \ 0 \ \ \tfrac{1}{2}]$, $T = [\tfrac{1}{2} \ \ \tfrac{1}{2} \ \ 0 \ \ 0]^T$

5. $S = [\tfrac{1}{4} \ \ \tfrac{1}{4} \ \ \tfrac{1}{4} \ \ \tfrac{1}{4}]$, $T = [\tfrac{1}{4} \ \ \tfrac{1}{4} \ \ \tfrac{1}{4} \ \ \tfrac{1}{4}]^T$

6. $S = [\tfrac{1}{4} \ \ \tfrac{1}{4} \ \ \tfrac{1}{4} \ \ \tfrac{1}{4}]$, $T = [0 \ \ 0 \ \ \tfrac{1}{2} \ \ \tfrac{1}{2}]^T$

In Exercises 7–10, decide which pure strategy (or strategies) the row player should use in order to maximize the expected value of the game

$$P = \begin{bmatrix} 0 & -1 & 5 \\ 2 & -2 & 4 \\ 0 & 3 & 0 \\ 1 & 0 & -5 \end{bmatrix}$$

if the column player uses the given column strategies. Express your answer as a row matrix.

7. $T = [\tfrac{1}{3} \ \ \tfrac{2}{3} \ \ 0]^T$

8. $T = [\tfrac{1}{3} \ \ \tfrac{1}{3} \ \ \tfrac{1}{3}]^T$

9. $T = [\tfrac{1}{4} \ \ \tfrac{1}{4} \ \ \tfrac{1}{2}]^T$

10. $T = [0 \ \ \tfrac{1}{3} \ \ \tfrac{2}{3}]^T$

In Exercises 11–14, find: **(a)** the optimal mixed row strategy; **(b)** the optimal mixed column strategy, and **(c)** the expected value of the game in the event that each player uses his or her optimal mixed strategy.

11. $P = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}^T$

12. $P = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$

13. $P = \begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}$

14. $P = \begin{bmatrix} -2 & -1 \\ -1 & -3 \end{bmatrix}$

Applications

15. ***Factory Location***[1] A manufacturer of electrical machinery is located in a cramped, though low-rent, factory close to the center of a large city. The firm needs to expand, and it could do so in one of three ways: (1) remain where it is and install new equipment, (2) move to a suburban site in the same city, or (3) relocate in a different part of the country where labor is cheaper. Its decision will be influenced by the fact that one of the following will happen: (I) the government may introduce a program of equipment grants, (II) a new suburban highway may be built, or (III) the government may institute a policy of financial help to companies who move into regions of high unemployment. The value to the company of each combination is given in the following payoff matrix.

		Government's Options		
		I	II	III
Manufacturer's Options	1	200	155	145
	2	130	220	130
	3	118	118	225

If the manufacturer judges that there is a 20% probability that the government will go with option I, a 50% probability that they will go with option II, and a 30% probability that they will go with option III, what is the manufacturer's best option?

16. ***Crop Choice***[2] A farmer has a choice of growing wheat, barley, or rice. His success will depend on the weather, which could be dry, average, or wet. His payoff matrix is as follows.

		Weather		
		dry	average	wet
Crop Choices	wheat	23	18	10
	barley	13	16	20
	rice	11	20	21

If the probability that the weather will be dry is 10% , the probability that it will be average is 60%, and the probability that it will be wet is 30%, what is the farmer's best choice of crop?

17. ***Study Techniques*** Your mathematics test is tomorrow, and will cover game theory, linear programming and matrix algebra. You have decided to do an "all-niter" and must determine how to allocate your eight hours of study time among the three topics. If you were to spend the entire eight hours on any one of these topics (thus using a pure strategy) you feel

[1] Adapted from an example in *Location in Space: Theoretical Perspectives in Economic Geography* by P. Dicken and P.E. Lloyd, Harper & Row, 1990.
[2] Ibid.

confident that you will earn a 90% score on that portion of the test, but will not do so well on the other topics. You have come up with the following table, where the entries are your expected scores. (The fact that, for example, linear programming and matrix algebra are used in game theory is reflected in these numbers).

Your strategies ↓		**Test**	
	Game Theory	Linear Programming	Matrix Algebra
Game Theory	90	70	70
Linear Programming	40	90	40
Matrix Algebra	60	40	90

You have been told that the test will be weigted as follows: game theory: 25%; linear programming: 50%; matrix algebra: 25%.

(a) If you spend 25% of the night on game theory, 50% on linear programming and 25% on matrix algebra, what score do you expect to get on the test?

(b) Is it possible to improve on this by altering your study schedule? If so, what is the highest score you can expect on the test?

18. ***Study Techniques*** Your friend Joe has been spending all of his time in fraternity activities, and thus knows absolutely nothing about any of the three topics on tomorrow's math test. (See Exercise 17.) Since you are recognized as an expert on the use of game theory to solve study problems, he has turned to you for advice as to how to spend his "all-niter." As the following table shows, his situation is not so rosy. (Since he knows no linear programming or matrix algebra, the table shows, for instance, that studying game theory all night will not be much use in preparing him for this topic.)

Joe's strategies ↓		**Test**	
	Game Theory	Linear Programming	Matrix Algebra
Game Theory	30	0	20
Linear Programming	0	70	0
Matrix Algebra	0	0	70

Assuming that the test will be weighted as described in Exercise 17., what are the answers to parts (a) and (b) as they apply to Joe?

19. ***Animal Husbandry***[1] Reindeer herdsmen in Finland must frequently decide whether to slaughter reindeer that are expected to calve the following season, or to leave them in the wild in the expectation of future profit from their calves. If a cow is left in the wild, there are three likely outcomes: (1) the cow and its calf will be found the next season; (2) the cow and

[1] Adapted from "Statistical Husbandry; Chance, Probability, and Choice in a Reindeer Management Economy" by T. Ingold, *Science* Vol. 32, 1986, pp. 1029-1038.

its calf will be alive, but not found the next season; (3) the cow will die prior to calving. (There are other possibilities as well, such as: the cow lives but the calf dies, the calf lives but the cow dies, etc., but these are comparatively rare, and are thus ignored.) Using estimates of the value of a cow and of a calf, the future values of each, T. Ingold came up with the following payoff matrix, where the payoffs represent the revenue per deer in dollars:

	Cow & Calf Found	**Alive & Not Found**	**Cow Dies**
Leave in Wild	300	100	0
Slaughter	200	200	200

(a) Assuming that 50% of the cows left in the wild will be found, 40% will be alive and not found, and 10% of them will die, find the average revenue per cow if a herdsman slaughters half the reindeer that are expected to calve the following season.

(b) What is the best pure strategy for a reindeer farmer, given that 70% of the cows left in the wild will be found, 25% will be alive and not found, and 5% of them will die?

20. ***More Animal Husbandry*** Referring to the situation in Exercise 19, we consider a third option open to a reindeer herdsman: that the herdsman keep the cow in captivity until it calves, at an estimated average cost of $50. This leads to the following larger payoff matrix:

	Cow & Calf Found	**Alive & Not Found**	**Cow Dies**
Hold in Captivity	250	250	0
Leave in Wild	300	100	0
Slaughter	200	200	200

(a) If 60% of cows left in the wild are found the next season, 30% of them are alive but not found, and if 10% of all expectant cows die, find the average revenue per cow if a herdsman holds half of them captive and slaughters the other half.

(b) What is the best pure strategy for a reindeer farmer, given that 70% of the cows left in the wild will be found, 25% will be alive and not found, and 5% of them will die?

21. ***Management*** Your hospital currently employs no specialists in the fields of infectious diseases, oncology or rheumatology, but permits you to keep a single specialist in one of these fields on call at any one time. If a patient requires the services of a physician not contracted by your hospital, you are required to share the revenues with another hospital. The following table shows the estimated total monthly hospital revenue in millions of dollars based on a turnaround of 200 patients per month:[1]

[1] The diagonal entries are based on actual figures published in the Chicago Tribune (March 29, 1993, Section 4 p.1) Source: Lutheran General Health System, Argus Associates, Inc., Chicago Tribune/Stephen Ravenscraft and Celeste Schaefer.

Your strategies ↓	**Patient's Requirements** Infectious diseases	Oncology	Rheumatology
Infectious diseases	2.7886	0.8	0.6
Oncology	0.8	2.3188	0.9
Rheumatology	0.8	1.0	2.1748

(a) Assume that one month you decide to keep an infectious disease specialist on call 37.5% of the time, an oncologist 40% of the time and a rheumatologist 22.5% of the time. During that month, your hospital experiences the following caseload: infectious diseases: 50; oncology: 75; rheumatology: 75. What is your expected revenue?

(b) Given the montly caseload distribution in part (a), what would your best pure strategy be?

22. ***Management*** Referring to Exercise 21, you have the following data on monthly revenues from cardiologists, physical rehabilitation specialists and pulmonary specialists.[1]

Your strategies ↓	**Patient's Requirements** Cardiologist	Physical Rehabilitation	Pulmonary
Cardiologist	2.56	0.6	0.9
Physical Rehab.	0.3	3.8348	0.5
Pulmonary	1.0	0.6	2.4346

(a) Assume that one month you decide to keep a cardiologist on call 22.5% of the time, a physical rehabilitation specialist 40% of the time and a pulmonary specialist 37.5% of the time. During that month, your hospital experiences the following caseload: heart diseases: 75; physical rehabilitation: 50; pulmonary: 75. What is your expected revenue?

(b) Given the montly caseload distribution in part (a), what would your best pure strategy be?

23. ***Teaching Techniques*** College professors who are not yet tenured must frequently play an intricate game against their students. They can decide to make their courses extremely easy, at the risk of boring the better-prepared students; they can aim their courses at the "average student," running the risk of disenchanting the students at both ends of the spectrum; or they can make their courses extremely challenging, thus leaving the less-prepared students "in the dust," so to speak. The ratings they receive for each strategy depend on the number of students in each of the three categories: less-prepared, average, well-prepared. Professor Ogre spent years analyzing the feedback from students in the various categories, and came up with the following payoff matrix. (The payoffs are the scores awarded in the student evaluation on a scale of −2 (highly negative) to +2 (highly positive).)

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[1] *Ibid.*

		Student		
		Less Prepared	Average Student	Well Prepared
Level	Easy	2	0	−2
	Average	−1	2	0
	Difficult	−2	1	2

(a) If half the students in Prof. Ogre's classes are average, a quarter are less prepared, and a quarter are well prepared, what would his average approval rating be if he varied the level of his presentation in such a way as to use each level equally often?

(b) If 50% of the students in Prof. Ogre's classes are average, 30% are less prepared, and 20% are well prepared, at what level should Prof. Ogre present his classes?

(c) What kind of student would be *least* impressed with Prof. Ogre were he to vary the level of his presentation in such a way as to use each level equally often?

24. ***Textbook Writing*** You are writing a college level textbook on finite mathematics, and are trying to come up with the best combination of word problems. Over the years, you have accumulated a collection of amusing problems, serious applications, long complicated problems and "generic" problems[1]. Before your book is published, it must be scrutinized by several reviewers who, it seems, are never satisfied with the mix you use. You estimate that there are three kinds of reviewers: the "no-nonsense" types who prefer applications and generic problems, the "dead serious" types, who feel that a college-level text should be contain little or no humor and lots of complicated problems, and the "laid-back" types, who believe that learning best takes place in a light-hearted atmosphere bordering on anarchy. You have drawn up the following chart, where the payoffs represent the reactions of reviewers on a scale of −10 (ballistic) to +10 (ecstatic):

		Reviewers		
		No-Nonsense	Dead Serious	Laid-Back
You	Amusing	−5	−10	10
	Serious	5	3	0
	Long	−5	5	3
	Generic	5	3	−10

(a) Your first draft of the book contained no generic problems, and equal numbers of the other categories. If half the reviewers of your book were "dead serious" and the rest were equally divided between the "no-nonsense" and "laid-back" types, what score would you expect.?

(b) In your second draft of the book, you tried to balance the content by including some generic problems and eliminating several amusing ones, and wound up with a mix of

[1] of the following type: "An oil company has three refineries: A, B and C, each of which uses three processes: P_1, P_2, and P_3. Process P_1 uses 100 units of chemical C_1 and costs \$100 per day . . "

which one eighth were amusing, one quarter were serious, three eighths were long and a quarter were generic. What kind of reviewer would be *least* impressed by this mix?

(c) What kind of reviewer would be *most* impressed by the mix in your second draft?

25. ***Production Planning***[1] You are the production manager of a large pharmaceutical company, and are planning production runs of a new antibiotic. Due to the fact that the ingredients for the antibiotic come in fixed amounts, you are constrained to produce the drug in batches of 100,000 capsules. One of your customers has a contract to purchase 1, 2 or 3 batches of the drug every month. Unfortunately, part of the manufacturing process involves culturing a special mold for two weeks, so you must decide on the number of batches to make before you receive the customer's order. If your production for the coming month falls short of your customer's order, you will have to make up the difference by buying batches of the drug from a competitor. If, on the other hand, you produce more batches than the customer orders, you will have to discard the surplus because of the very short shelf-life of the product. Your manufacturing costs are $150 per batch, and the antibiotic sells at the fixed contract price of $200 per batch. You can buy the product from your competitor at a cost of $250 per batch. Based on past experience, you have found that your customer orders 1 batch 3/10 of the time, 2 batches half the time, and 3 batches 1/5 of the time.

(a) How many batches should you manufacture for the coming month in order to maximize your expected profit?

(b) How much profit should you expect the company to earn?

26. ***Production Planning*** Repeat Exercise 25 in the event that your customer orders 1 batch 3/10 of the time, 2 batches 3/10 of the time, and 3 batches 2/5 of the time.

27. ***Advertising*** The Softex Shampoo Company is considering how to split its advertising budget between ads on two radio stations: WISH and WASH. Its main competitor, Splish Shampoo, Inc. has found out about this, and is considering countering Softex's ads with its own. (proposed jingle: *Softex*, *Shmoftex; Splash with Splish*) Softex has calculated that, were it to devote its entire advertising budget to ads on WISH, it would increase revenues in the coming month by $100,000 in the event that Splish was running all its ads on the less popular WASH, but would lose $20,000 in revenues if Splish ran its ads on WISH. If, on the other hand, it devoted its entire budget to WASH ads, it would neither increase nor decrease revenues in the event that Splish was running all its ads on the more popular WISH, and would in fact lose $50,000 in revenues if Splish ran its ads on WASH. What should Softex do, and what effect will this have on revenues?

28. ***Labor Negotiations*** The management team of the Abstract Concrete Company is negotiating a three-year contract with the labor unions at one of its plants, and is trying to decide on its offer for a salary increase. If it offers a 5% increase and the unions accept the offer, Abstract Concrete will gain $20 million in projected profits in the coming year, but if labor rejects the offer, the management team predicts that they will be forced to increase the offer to the union demand of 15%, thus halving the projected profits. If Abstract Concrete

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[1] Loosely based on a decision analysis exercise in *An Introduction to Management Science* (6th Ed.) by D.R. Anderson, D.J. Sweeny and T.A. Williams (West, New York)

offers a 15% increase, the company will earn $10 million in profits over the coming year if the unions accept. If the unions reject, they will probably go out on strike (since management has set 15% as its upper limit) and management has decided that they can then in fact gain $12 million in profits by selling out the defunct plant in retaliation. What intermediate percentage should they offer?

Communication and Reasoning Exercises

29. Explain why it is in the interests of neither player to use a fixed strategy in a game that is not strictly determined.

30. Describe a situation in which a both a mixed strategy and a fixed strategy are equally effective.

31. Design a 2×2 game such that, if the column player uses the mixed strategy $[0.3 \quad 0.7]^T$, then the row player's best strategy is the fixed strategy [0 1].

32. Design a 2×2 game such that, if the row player uses the mixed strategy $[0.6 \quad 0.4]$, then the column player's best strategy is the fixed strategy $[0 \quad 1]^T$.

33. According to the Principles of Game Theory, your opponent always makes the best possible moves, and assumes that you are doing the same. However, in many real life situations, the opponent may be “Nature,” and not actively “plotting against you.” In such cases, you can assume that the opponent will always use a given mixed strategy (for instance, raining 30% of the time, dry 70% of the time). Explain why your optimal strategy in such a situation can always be taken to be a fixed strategy, regardless of whether the game is strictly determined.

34. Explain what is wrong with a decision to play the mixed strategy [0.5 0.5] by alternating the two strategies: play the first strategy on the odd-numbered moves and the second strategy on the even-numbered moves. Illustrate your argument by using a game in which your best strategy is [0.5 0.5].

G.4 Solving Games with the Simplex Method

Recall that to *solve a game* means to find for each player a mixed strategy that minimizes the potential loss against the opponent's best counter-strategy. If a game is strictly determined, the optimal mixed strategies are the pure strategies determined by selecting a saddle point. We have also seen in the preceding section how to solve arbitrary 2×2 games. But not all games are 2×2 games. To solve a larger game turns out to be a linear programming problem.

Here is a reassuring fact:

No matter what zero sum game is being played, there is at least one optimal mixed strategy for each player. In other words, every two-person zero-sum game can be solved.

In a strictly determined game, the use of an optimal pure strategy will minimize a player's potential losses. In a *non* strictly determined game, a player can do better by using a mixed strategy. This is illustrated by the following example.

Example 1 *Optimal Mixed Strategy*

Consider once again the game "paper, scissors, rock." The payoff matrix is

$$P = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

(The moves are p, s, r, in that order.) If you are the row player and use any pure strategy—it doesn't matter which because of the symmetry of the game—then your potential loss is one point on each round of the game (since the smallest entry in each row is -1). On the other hand, if you use the mixed strategy $S = [\frac{1}{3} \ \frac{1}{3} \ \frac{1}{3}]$ by playing p, s and r equally often, then no matter what strategy the column player uses, the expected value of the game is

$$\begin{aligned} e &= [\tfrac{1}{3} \ \tfrac{1}{3} \ \tfrac{1}{3}] \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ &= [0 \ \ 0 \ \ 0] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0. \end{aligned}$$

Since a draw is better than a loss, it follows that using the mixed strategy $[\frac{1}{3} \ \frac{1}{3} \ \frac{1}{3}]$ is more advantageous than using any pure strategy.

Before we go on... We claim that the mixed strategy $S = [\frac{1}{3} \ \ \frac{1}{3} \ \ \frac{1}{3}]$ is the only *optimal* mixed strategy.

Question Why is this so?
Answer To see why this strategy is better than any other mixed strategy, suppose you tried another mixed strategy, like $S = [\frac{1}{2} \ \ \frac{1}{4} \ \ \frac{1}{4}]$. In any mixed strategy other than [4 4 4], one move will be played more often than some other; in this case p will be played more often than any other. Here is a counter-strategy that the column player can use against S: play the pure strategy s all the time. Since scissors beat paper, the column player will tend to win more often than lose. In fact, the expected value is

$$e = [\tfrac{1}{2} \ \ \tfrac{1}{4} \ \ \tfrac{1}{4}]\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -\frac{1}{4}\,.$$

So, the worst that can happen to you playing $[\frac{1}{3} \ \ \frac{1}{3} \ \ \frac{1}{3}]$ is better than the worst that can happen if you play $[\frac{1}{2} \ \ \frac{1}{4} \ \ \frac{1}{4}]$. The same will be true for any other mixed strategy with unequal proportions, so [4 4 4] is your optimal mixed strategy.

Not all games can be analyzed by such "common sense" methods, so we now describe a method of solving a game using the simplex method. Using the simplex method makes some sense, since we are looking to maximize a quantity—the worst expected payoff—subject to certain constraints, such as: the entries in the desired mixed strategy cannot exceed 1 and the entries in the opponent's mixed strategy cannot exceed 1. At the end of this section we shall explain in more detail why the following works.

Note
Just as when we used the simplex method in linear programming, you may find the various technologies (graphing calculators, spreadsheets or special purpose software) useful in your calculations.

Example 2 *Solving a Game using the Simplex Mathod*

Solve the game with payoff matrix

	1	2	3	4
1	0	1	−1	1
2	−1	−2	0	2
3	0	0	−2	1
4	0	−2	−1	1

Solution

Step 1* (Optional, but highly recommended) *Reduce the payoff matrix by dominance.

Looking at the matrix, we notice that rows 3 and 4 are dominated by row 1, so we eliminate them, obtaining

$$\begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & -1 & 1 \\ -1 & -2 & 0 & 2 \end{bmatrix} \end{array}.$$

Next, we can eliminate column 4, since it is dominated by column 1.

$$\begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & -1 \\ -1 & -2 & 0 \end{bmatrix} \end{array}$$

This is as far as reduction by dominance can take us.

Step 2 Convert to a payoff matrix with no negative entries by adding a suitable fixed number to all the entries.

If we add 2 to all the entries, we will eliminate all negative payoffs. Notice that this won't affect the analysis of the game in any way; the only thing that is affected is the expected value of the game, which will be increased by 2. Adding 2 to all the payoffs gives the new matrix,

$$\begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix} \end{array}.$$

Step 3 Solve the associated standard linear programming problem:

Maximize $p = x + y + z$ *subject to*

$$\begin{aligned} 2x + 3y + z &\le 1 \\ x \qquad + 2z &\le 1 \\ x \ge 0,\ y \ge 0,\ z &\ge 0. \end{aligned}$$

The number of variables and the coefficients of the constraints depend on the payoff matrix; there is one variable for each column. We always take the objective function to be the sum of the variables, and the right hand sides of the constraints are always 1.

We now use the simplex method. The first tableau is the following.

	x	y	z	s	t	p	
s	2	3	1	1	0	0	1
t	1	0	2	0	1	0	1
p	−1	−1	−1	0	0	1	0

Notice that the payoff matrix appears in the top left part of the tableau. We now proceed to the solution as usual:

	x	y	z	s	t	p		
s	**[2]**	3	1	1	0	0	1	
t	1	0	2	0	1	0	1	$2R_2 - R_1$
p	−1	−1	−1	0	0	1	0	$2R_3 + R_1$

	x	y	z	s	t	p		
x	2	3	1	1	0	0	1	$3R_1 - R_2$
t	0	−3	**[3]**	−1	2	0	1	
p	0	1	−1	1	0	2	1	$3R_3 + R_2$

	x	y	z	s	t	p	
x	6	12	0	4	−2	0	2
z	0	−3	3	−1	2	0	1
p	0	0	0	2	2	6	4

Here is how we read off the optimal strategies.

Step 4 Calculate the optimal strategies

Column Strategy

1. *Express the solution to the linear programming problem as a column vector.*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix}$$

2. ***Normalize*** *by dividing each entry in the solution vector by the value of p (which is also the sum of the values of the variables).*

Here, $p = \frac{2}{3}$, so we get

$$\begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

The entries of the column will now add up to 1.

3. *Insert zeros corresponding to the columns deleted when we reduced by dominance:*
Recalling that we deleted column 4, we insert a zero in the fourth position, getting the column player's optimal strategy: $[\frac{1}{2}\ \ 0\ \ \frac{1}{2}\ \ 0]^T$.

Row Strategy
1. *Read off the entries under the slack variables in the bottom row of the final tableau.*

$$[2\ \ 2]$$

2. ***Normalize*** *by dividing each entry in the vector by the sum of the entries:*
The sum of the entries is 4, so we get

$$[\tfrac{1}{2}\ \ \tfrac{1}{2}].$$

3. *Insert zeros corresponding to the rows deleted when we reduced by dominance:*
Recalling that we deleted rows 3 and 4, we insert zeros in the third and fourth positions, getting the row player's optimal strategy: $[\frac{1}{2}\ \ \frac{1}{2}\ \ 0\ \ 0]$.

Value of the Game: This is given by the formula

$$e = \frac{1}{p} - k,$$

where k is the number we originally added to the entries in the payoff matrix to make them non-negative.

Here, $p = \frac{2}{3}$, so $\frac{1}{p} = \frac{3}{2}$, and $k = 2$, hence the value of the game is

$$e = \frac{3}{2} - 2 = -\frac{1}{2}.$$

Here is a summary of the procedure.

Solving a Matrix Game

First: Check for saddle points. If there is one, you can solve the game by selecting each player's optimal pure strategy. Otherwise, continue with the following steps.

Step 1 Reduce the payoff matrix by dominance.
Step 2 Add a fixed number *k* to each of the entries so that they all become non-negative.
Step 3 Set up and solve the associated linear programming problem using the simplex method.
Step 4 Find the optimal strategies and the expected value as follows.

Column Strategy

1. Express the solution to the linear programming problem as a column vector.
2. Normalize by dividing each entry of the solution vector by p (which is also the sum of the values of the variables).
3. Insert zeros in positions corresponding to the columns deleted during reduction.

Row Strategy

1. List the entries under the slack variables in the bottom row of the final tableau in vector form.
2. Normalize by dividing each entry of the solution vector by the sum of the entries.
3. Insert zeros in positions corresponding to the rows deleted during reduction.

Value of the Game: $e = \frac{1}{p} - k$

Once again, we say that a game is **fair** if its value is 0. Otherwise it is **unfair** or **biased**.

Example 3 ***Restaurant Inspectors***

You manage two restaurants, Tender Steaks Inn (TSI) and Break For a Steak (BFS). Even though you run the establishments impeccably, you suspect that the Department of Health is out to get you, as they have been sending inspectors to both your restaurants on a daily basis and fining you for minor infractions. To ensure that a restaurant is operating without minor health violations seems to be a full time job, and you can spare only one employee to act as a full time “trouble-shooter.” Thus, you must decide on a daily basis which restaurant you want “covered” by your trouble-shooter. Unfortunately, the Department of Health has two inspectors. On some days, both arrive at the same restaurant while on other days one arrives at each. If both inspectors arrive at a restaurant that isn’t covered that day, the fine tends to be larger, since two inspectors are better at spotting minor infractions than one. The average fines you have been getting are shown in the following matrix.

		Health Inspectors		
		Both at BFS	**Both at TSI**	**One at Each**
Trouble-shooter at	**TSI**	$8,000	0	$2,000
	BFS	0	$10,000	$4,000

What fraction of the time should your trouble-shooter be at each restaurant in order to minimize your expected fine? If you employ this strategy, what daily fine do you expect to pay?

Solution The matrix is not quite the payoff matrix since fines, being penalties, should be represented as negative numbers. Thus the actual payoff matrix is

$$P = \begin{bmatrix} -8{,}000 & 0 & -2{,}000 \\ 0 & -10{,}000 & -4{,}000 \end{bmatrix}.$$

We first check for saddle points, and notice that there are none. Further, there are no dominated rows or columns. Thus we turn to the simplex method.

Step 2 (Making the entries non-negative) We must add at least 10,000 to each entry to accomplish this, so we take $k = 10{,}000$, getting the new payoff matrix.

$$P = \begin{bmatrix} 2{,}000 & 10{,}000 & 8{,}000 \\ 10{,}000 & 0 & 6{,}000 \end{bmatrix}$$

Step 3 We set up the linear programming problem.

$$\begin{aligned} &\textit{Maximize } p = x + y + z \textit{ subject to} \\ &2{,}000x + 10{,}000y + 8{,}000z \le 1 \\ &10{,}000x \qquad\qquad\quad + 6{,}000z \le 1 \\ &x \ge 0,\ y \ge 0,\ z \ge 0. \end{aligned}$$

We now use the simplex method to solve this problem.

	x	y	z	s	t	p		
s	2,000	10,000	8,000	1	0	0	1	$5R_1 - R_2$
t	[10,000]	0	6,000	0	1	0	1	
p	−1	−1	−1	0	0	1	0	$10{,}000R_3 + R_2$

	x	y	z	s	t	p		
s	0	50,000	34,000	5	−1	0	4	
x	10,000	0	6,000	0	1	0	1	
p	0	−10,000	−4,000	0	1	10,000	1	$5R_3 + R_1$

	x	y	z	s	t	p	
y	0	50,000	34,000	5	−1	0	4
x	10,000	0	6.000	0	1	0	1
p	0	0	14,000	5	4	50,000	9

Step 4 (Solution)
Column Strategy The solution vector is

$$[x \;\; y \;\; z]^T = [\tfrac{1}{10{,}000} \;\; \tfrac{1}{12{,}500} \;\; 0]^T$$

We normalize by dividing each entry by $p = 9/50{,}000$, getting the column mixed strategy

$$T = [\tfrac{5}{9} \;\; \tfrac{4}{9} \;\; 0]^T.$$

Row Strategy The values in the bottom row under the slack variables give the vector

$$[5 \;\; 4].$$

To normalize, divide by the sum, 9, obtaining the row strategy

$$S = [\tfrac{5}{9} \;\; \tfrac{4}{9}]$$

Value of the Game This is

$$e = \frac{1}{p} - k = \frac{50{,}000}{9} - 10{,}000 = -\frac{40{,}000}{9} \approx -4{,}444.44.$$

In other words, you should have your trouble shooter at TSI 5/9 of the time, and at BFS 4/9 of the time, and you can expect your fine to average $4,444.44 per day.

Before we go on... This strategy does seem to make sense, since the fines levied against TSI are larger than those against BFS. But notice that we could hardly have come up with these exact proportions without solving the game.

We still owe you an explanation of why this method works. The main point is to understand how we turn a game into a linear programming example. Go back to Example 2, Step 3. At this point, we had reduced the game to

$$P = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$

We now need to find the optimal strategies for the row and column players, which we shall call S^* and T^* respectively, and the value of the game, $e = S^*PT^*$.

We first concentrate on finding the column player's optimal strategy, $T^* = [u \quad v \quad w]^T$, where u, v and w are the unknowns we must find. Now, since e is the value of the game and T^* is optimal, we have $SPT^* \le e$ for every S. That is,

$$S \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \le e$$

for every S. In particular,

$$[1 \ \ 0] \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = [2 \ \ 3 \ \ 1] \begin{bmatrix} u \\ v \\ w \end{bmatrix} = 2u + 3v + w \le e$$

and

$$[0 \ \ 1] \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = [1 \ \ 0 \ \ 2] \begin{bmatrix} u \\ v \\ w \end{bmatrix} = u + 2w \le e.$$

We would like to think of these as constraints on the variables u, v and w, but they also involve the unknown value e. To fix this up, we divide by e (which we know to be positive because we made all the payoffs ≥ 0). This gives

$$2\left(\frac{u}{e}\right) + 3\left(\frac{v}{e}\right) + \frac{w}{e} \le 1$$

and

$$\frac{u}{e} + 2\left(\frac{w}{e}\right) \le 1.$$

If we let $x = u/e$, $y = v/e$, and $z = w/e$, we can write

$$\begin{aligned} 2x + 3y + z &\le 1 \\ x + 2z &\le 1. \end{aligned}$$

These look like constraints in a linear programming problem, so we're making progress. A linear programming problem also needs an objective function, and from the column player's point of view the objective is to minimize e, the value of the game. In order to write this objective in terms of x, y and z, we use a fact we haven't used yet: that $u + v + w = 1$. Dividing by e gives us

$$\frac{u+v+w}{e} = \frac{1}{e},$$

$$\text{or} \quad \frac{u}{e} + \frac{v}{e} + \frac{w}{e} = \frac{1}{e},$$

$$\text{or} \quad x + y + z = \frac{1}{e}.$$

Thus, if we were to *maximize* $x + y + z$, we would have the effect of minimizing e, which is exactly what the column player wants to do. So, we get the following linear programming problem.

$$\begin{array}{ll} \text{Maximize} & p = x + y + z \\ \text{subject to} & 2x + 3y + z \le 1 \\ & x + 2z \le 1 \\ \text{and} & x \ge 0,\ y \ge 0,\ z \ge 0. \end{array}$$

Note that $p = 1/e$. Why can we say that x, y and z should be ≥ 0? Because $x = u/e$ and both u and e should be non-negative, and similarly for y and z. Once we solve this linear programming problem, we can find the value of the game by computing $e = 1/p$, and the column player's optimal mixed strategy by computing $u = x \cdot e = x/p$, $v = y/p$, and $w = z/p$. This is the procedure we have used in the examples.

Question What about the row player's strategy?
Answer If we analyze the problem in the same manner from the row player's point of view, we end up with the following linear programming problem.

$$\begin{array}{ll} \text{Minimize} & c = x + y \\ \text{subject to} & 2x + y \ge 1 \\ & 3x \ge 1 \\ & x + 2y \ge 1 \\ \text{and} & x \ge 0,\ y \ge 0. \end{array}$$

This is the **dual** problem to the one corresponding to the column player's strategy. As we saw in Section 4.5, we can read off the answer to the dual problem from the final tableau when we use the simplex method to solve the original problem. The way we read off the answer is the way we have done in the examples.

G.4 Exercises

Solve the games in Exercises 1–10 using whatever method is appropriate.

1. $P = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

2. $P = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$

3. $P = \begin{bmatrix} -1 & 1 & 2 \\ 2 & -1 & -2 \end{bmatrix}$

4. $P = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -2 \end{bmatrix}$

5. $P = \begin{bmatrix} 0 & -1 \\ -2 & 0 \\ 4 & -2 \end{bmatrix}$

6. $P = \begin{bmatrix} -2 & 0 \\ 0 & -2 \\ -3 & 2 \end{bmatrix}$

7. $P = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & -1 \\ 2 & 2 & 1 \end{bmatrix}$

8. $P = \begin{bmatrix} 0 & -1 & 2 \\ -3 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

9. $P = \begin{bmatrix} -1 & -1 & 1 & 1 \\ -2 & -2 & 0 & 2 \\ -1 & 1 & 2 & 2 \\ -3 & 0 & 0 & 3 \end{bmatrix}$

10. $P = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 3 \\ -1 & 3 & 3 & 4 \\ -2 & -2 & 4 & 5 \end{bmatrix}$

11. ***Political Campaigns*** Incumbent Senator Tax U. Spend (Democrat) and challenger Trickle R. Down (Republican) are contesting the Packard County seat. The election hinges on the swing vote in three cities: Littleville, Metropolis and Urbantown, where polls show them to be in a dead heat. The candidates have decided to spend the last four weeks prior to the election campaigning in those three cities. Their campaign strategists have come up with the following payoff matrix where the payoffs represent the number of potential votes gained or lost for each one-day campaign stop.

		T. Spend		
		Littleville	Metropolis	Urbantown
	Littleville	−200	−300	300
T. Down	Metropolis	−500	500	−100
	Urbantown	−500	0	0

What percentage of the time should each candidate spend at each city in order to maximize votes? If both candidates follow these strategies and make a campaign stop every day for the next four weeks, predict the resulting vote.

12. *Marketing Strategies* Your company's new computer database software product, Chevy 4-5-6, is competing with the established market leader, Zbase 7, in a saturated market. (Thus, for each new customer you get, Zbase 7 loses one, and vice versa.) Your company is planning to launch an ad campaign on radio and TV, and your corporate spies inform you that the marketer of Zbase 7 is planning a similar strategy and plans to spend as much on advertising as you do. You are as yet undecided on what percentage of Chevy 4-5-6's advertising budget should be spent on each of the two media. Your market analysts tell you that, if you and your competitor allocate your entire budgets to the same medium, you would probably lose 1,000 of your customers to Zbase 7. If Zbase 7 advertises only on TV and your company advertises only on radio, you would gain 500 customers from Zbase 7, and if you advertised only on TV and Zbase 7 advertised only on radio, your company would gain 1,500 of Zbase 7's customers. What should you do, and what do you expect the outcome to be?

13. *Factory Location*[1] (Similar to Exercise 15 in the preceding section) A manufacturer of electrical machinery is located in a cramped, though low-rent, factory close to the center of a large city. The firm needs to expand, and it could do so in one of three ways: (1) it could remain where it is and install new equipment, (2) it could move to a suburban cite in the same city, or (3) it could relocate in a different part of the country where labor is cheaper. Its decision will be influenced by the fact that one of the following will happen: (I) the government may introduce a program of equipment grants, (II) a new suburban highway may be built, or (III) the government may institute a policy of financial help to companies who move into regions of high unemployment. The value to the company of each combination is given in this payoff matrix:

		Government's Options		
		I	**II**	**III**
	1	200	155	145
Manufacturer's Options	**2**	130	220	130
	3	118	118	225

If the company does not know what the probabilities of the various government options are, what should it do?

14. *Crop Choice*[2] (Similar to Exercise 16 in the preceding section) A farmer has a choice of growing wheat, barley, or rice. His success will depend on the weather, which could be dry, average, or wet. His payoff matrix is as follows:

		Weather		
		Dry	**Average**	**Wet**
	Wheat	23	18	10
Crop Choices	**Barley**	13	16	20
	Rice	11	20	21

[1] Adapted from an example in *Location in Space: Theoretical Perspectives in Economic Geography* by P. Dicken and P.E. Lloyd, Harper & Row, 1990.
[2] Ibid.

If the farmer does not know what the weather will be, what *combination* of crops should he plant?

15. ***Study Techniques*** (Similar to Exercise 17 in the preceding section) Your mathematics test is tomorrow, and will cover game theory, linear programming and matrix algebra. You have decided to do an "all-niter" and must determine how to allocate your eight hours of study time among the three topics. If you were to spend the entire eight hours on any one of these topics (thus using a pure strategy) you feel confident that you will earn a 100% score on that portion of the test, but will not do so well on the other topics. You have come up with the following table, where the entries are your expected scores.

		Test	
Your strategies ↓	**Game Theory**	**Linear Programming**	**Matrix Algebra**
Game Theory	100	50	40
Linear Programming	50	100	50
Matrix Algebra	40	50	100

You have not been been told how tomorrow's test will be weighted. How much time should you spend on each topic, and what score can you expect?

16. ***Study Techniques*** (Similar to Exercise 18 in the preceding section.) Your friend Joe has been spending all of his time in fraternity activities, and thus knows absolutely nothing about any of the three topics on tomorrow's math test. (See Exercise 15.) Since you are recognized as an expert on the use of game theory to solve study problems, he has turned to you for advice as to how to spend his "all-niter." As the following table shows, his situation is not so rosy. (Since he knows no linear programming or matrix algebra, the table shows, for instance, that studying game theory all night will not be much use in preparing him for this topic.)

		Test	
Joe's strategies ↓	**Game Theory**	**Linear Programming**	**Matrix Algebra**
Game Theory	30	0	20
Linear Programming	0	70	0
Matrix Algebra	0	0	70

Since you have not been told how tomorrow's test will be weighted, how much time should Joe spend on each topic?

17. ***Morra Games*** A three-finger Morra game is a game in which two players simultaneously show one, two, or three fingers at each round. The outcome depends on a predetermined set of rules. Here is an interesting example: If the number of fingers shown by A and B differ by 1, then A loses one point. If they differ by more than 1, then the round is a draw. If they both show the same number of fingers, then A wins an amount equal to the sum of the fingers shown. Determine the optimal mixed strategy for each player, and also the expected value of the game.

18. ***Morra Games*** Referring to Exercise 17, consider the following rules for a three-finger Morra game: If the sum of the fingers shown is odd, then A wins an amount equal to that sum; if not, then B wins 3 points. Determine the optimal mixed strategy for each player, and also the expected value of the game.

19. ***Investments*** You have inherited $10,000 from a deceased aunt and are planning to invest all of it for a year. Your investment counselor has come up with three promising investments: Solid Trust Bonds, a municipal bond fund, Hi-Yield Mutual Fund, consisting mostly of stock issues, and Heavy Metal Portfolio, consisting mostly of precious metal stocks. She tells you that the income you can expect to earn depends largely on the overall economy and that, as far as her economic forecasting team can see, we are headed in one of three directions:
(1) recovery plus inflation, which should result in a 20% yield in precious metals funds, a 10% loss in the bond market, and a 10% yield in mutual stock funds;
(2) recession, which should result in a 30% yield in bond issues, a flat precious metals market and a 20% decline in mutual stock funds;
(3) inflation in the face of a flat economy, which should result in a 20% yield in precious metals funds, a flat bond market, and a 10% decline in mutual stock funds.

What should you do to maximize your income, and how much can you expect to earn if you follow that strategy?

20. ***Investments*** You are wondering what to do with your $100,000 lotto winnings when your broker calls you to let you know that he has two very interesting pharmaceutical stocks: BioTech Labs and Genetic Splicing Inc. Both have promising new drugs presently under scrutiny by the FDA, with a decision expected in the coming year. If the FDA approves BioTech's new arthritis drug within the next year, its stock will probably quadruple; if the drug is not approved, the stock will probably drop to one fourth of its present value. If the FDA approves Genetic Splicing's anti-virus drug, its stock is likely to double, but if the drug is not approved, its stock will probably be unaffected (due to the fact that it has other promising new drugs in the pipeline). Moreover, according to well-placed sources in the FDA, at least one of the two drugs will be approved this year. In what proportion should you split your winnings between these two stocks in order to maximize your income, and how much can you expect to earn in the coming year?

21. ***Production Planning*** (See Exercise 25 of the preceding section.) You are the production manager of a large pharmaceutical company, and are planning production runs of a new antibiotic. Due to the fact that the ingredients for the antibiotic come in fixed amounts, you are constrained to produce the drug in batches of 100,000 capsules. One of your customers has a contract to purchase 1, 2 or 3 batches of the drug every month. Unfortunately, part of the manufacturing process involves culturing a special mold for two weeks, so you must decide on the number of batches to make before you receive the customer's order. If your production for the coming month falls short of your customer's order, you will have to make up the difference by buying batches of the drug from a competitor. If, on the other hand, you produce more batches than the customer orders, you will have to discard the surplus because of the very short shelf-life of the product. Your manufacturing costs are $150 per batch, and the antibiotic sells at the fixed contract price of $200 per batch. You can buy the product from your competitor at a cost of $250 per batch. As the demand for the drug has tended to be erratic, you have no idea of how many batches the customer will order each month. What fraction of the time should you make 1, 2 and 3 batches of the drug? How much profit should you expect the company to earn in the long run?

22. ***Production Planning*** Repeat Exercise 21 in the event that your manufacturing costs are $100 per batch, the antibiotic sells at the fixed contract price of $300 per batch, and your competitor will supply the drug at $400 per batch.

23. ***Transportation*** Your New York based freight haul company owns a small freight train which can serve Chicago, Dallas, and Miami with one run per month. For the past three years, you have been shipping imported coal to the three cities. The return run is sometimes empty, though on occasion you have had the opportunity to transport a shipment of wheat from Chicago or Dallas back to New York. Your company makes a profit of $8,000 per shipment to Chicago, $9,000 per shipment to Dallas and $10,000 per shipment to Miami. In the event of a return shipment from either Chicago or Dallas, your profit is increased by $4,000. What percentage of shipments should you assign to each city, and what are your expected monthly profits?

24. ***Transportation*** Repeat Exercise 23 using the following data: Your company makes a profit of $5,000 per shipment to Chicago, $6,000 per shipment to Dallas and $10,000 per shipment to Miami. Each return shipment from Chicago result in an extra $10,000 profit, while each return shipment from Dallas increases your profit by $6,000.

The use of technology is required for Exercises 25–28.[1]

25. ***Military Strategy*** Colonel Blotto is a well-known game in military strategy.[2] Here is a version of this game: Colonel Blotto has four regiments under his command, while his opponent, Captain Kije, has three. The armies are to try to occupy two locations, and each commander must decide how many regiments to send to each location. The army that sends more regiments to a location captures that location as well as the other army's regiments. If both armies send the same number of regiments to a location, then there is a draw. The payoffs are one point for each location captured and one point for each regiment captured. Find the optimum strategy for each commander and also the value of the game.

[1] You can use, for example, the Pivot and Gauss-Jordan Tool or Simplex Method Tool at the RealWorld web site.

[2] See Samuel Karlin, Mathematical Methods and Theory in Games, Programming and Economics (Addison-Wesley, 1959)

26. ***Military Strategy*** Repeat the Colonel Blotto game in Execise 25, but this time assuming that, in the event that both armies send the same (non-zero) number of regiments to a location, Colonel Kije wins the location.

27. ***Management*** (Based on Exercise 21 of the preceding section) Your hospital currently employs no specialists in the fields of infectious diseases, oncology or rheumatology, but permits you to keep a single specialist in one of these fields on call at any one time. If a patient requires the services of a physician not contracted by your hospital, you are required to share the revenues with another hospital. The following table shows the estimated total monthly hospital revenue in millions of dollars based on a turnaround of 200 patients per month[1]:

	Patients' Requirements		
Your Strategies ↓	**Infectious Diseases**	**Oncology**	**Rheumatology**
Infectious Diseases	2.7886	0.8	0.6
Oncology	0.8	2.3188	0.9
Rheumatology	0.8	1.0	2.1748

(a) What is your best mixed strategy?
(b) What is your expected revenue if you use the best strategy?

28. ***Management*** (Based on Exercise 22 of the preceding section) Referring to Exercise 27, you have the following data on monthly revenues from cardiologists, physical rehabilitation specialists and pulmonary specialists[2]

	Patient's Requirements		
Your strategies ↓	cardiologist	physical rehabilitation	pulmonary
cardiologist	2.56	0.6	0.9
physical rehab.	0.3	3.8348	0.5
pulmonary	1.0	0.6	2.4346

(a) What is your best mixed strategy?
(b) What is your expected revenue if you use the best strategy?

Communication and Reasoning Exercises

29. In what way is the row player's optimal strategy the best one to use?

30. What is the role of randomness in implementing an optimal mixed strategy?

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[1] The diagonal entries are based on actual figures published in the Chicago Tribune (March 29, 1993, Section 4 p.1) Source: Lutheran General Health System, Argus Associates, Inc., Chicago Tribune/Stephen Ravenscraft and Celeste Schaefer.

[2] Ibid.

31. In Example 3, we found optimal row and column strategies. And yet we learned in the last section that, if the column player uses any mixed strategy, there is a pure strategy that the row player can use to maximize the payoff. Therefore, what is wrong with the row player using such a pure strategy instead of the optimal strategy [5/9 4/9]?

32. If you know that your opponent is playing his or her optimal mixed strategy, is there anything you can do about it to improve your expected payoff to better than the value of the game?

You're the Expert—Harvesting Forests

Douglas fir trees in a typical forest reach their maximum size in about 135 years. The following table shows the approximate volume in cubic feet of a typical Douglas fir during its growth period.[1]

Age	10	20	30	40	50	60	70	80	90	100	110	120	130	135
Vol.	694	1,912	3,558	5,536	7,750	10,104	12,502	14,848	17,046	19,000	20,614	21,792	22,438	22,514

Forest Lumber, Inc., has a large number of stands of Douglas fir trees. The company periodically harvests some of the trees and replants. As a consultant to Forest Lumber, you have been asked to advise the company at what age it should harvest its trees in order to maximize the value of its timber.

Your first impulse is to tell the company to wait until maximum growth is achieved—that is, to wait for 135 years—but you suspect that something may be wrong with this quick analysis. After all, if the company harvests younger trees, it can invest the proceeds and earn interest.

After some thought, you decide to calculate the *discounted* value of a tree's timber, which takes this into account. This means discounting its value at the rate of interest that the company can expect from a cash investment. To calculate the discounted value of a fir tree at a rate of r% per year, divide its present value by the quantity $(1+r)^t$, where t is the age of the tree in years.

The company's financial analyst tells you that the company can earn a net interest of 2% under current conditions, but would like you to take into account the possibility that the interest rate will fluctuate between 2% and 4% in the coming years. You then get to work, estimating a value of $1 per cubic foot for the timber, and come up with several tables, one for each interest rate. Here is the table showing the discounted value of each tree if $r = 2\%$.[2]

Age	10	20	30	40	50	60	70	80	90	100	110	120	130	135
Value	569	1,287	1,964	2,507	2,897	3,080	3,126	3,045	2,868	2,623	2,334	2,024	1,710	1,554

Notice that the discounted value peaks at about 70 years, giving a maximum discounted value of $3,125 per tree.

Discount rates of 3% and 4% yield similar tables, but with lower discounted values, and peaking at different ages. In order to compare one table with another, you decide that it would be more meaningful to scale each table so that it peaks at a value of 100. You can do this by dividing each entry in a table by the largest value and then multiplying by 100, so that you are calculating the discounted value as a percentage of the maximum discounted value for that interest rate. Call it the *relative* discounted value.

Here is a table showing the relative discounted values using interest rates of r = 2, 3 and 4%. (You decide to ignore tree ages beyond 90 years, since all the relative values peak earlier.)

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[1] Source: Tom Tietenberg, *Environmental and Natural Resource Economics* (3rd. Ed.) HarperCollins, 1992, p. 282.

[2] Ibid.

Age		10	20	30	40	50	60	70	80	90
Relative	2%	18	41	63	80	92	99	100	97	92
value	3%	29	60	83	96	100	97	89	79	67
	4%	41	76	95	100	95	83	70	56	43

Now this is beginning to look like a zero-sum game! Your company's strategies are listed along the top as the waiting period between harvests, while the market's strategies appear as the various interest rates.

You are just about to begin solving this game when you notice that the payoffs are payoffs to Forest Lumber, so you should either make them all negative (since Forest Lumber is the column player) or you should take the transpose of the table, interchanging rows and columns. Since negative payoffs complicate the solution process, you decide to transpose the table instead, getting the following payoff matrix:

$$P = \begin{array}{c} \\ \mathbf{10} \\ \mathbf{20} \\ \mathbf{30} \\ \mathbf{40} \\ \mathbf{50} \\ \mathbf{60} \\ \mathbf{70} \\ \mathbf{80} \\ \mathbf{90} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{2\%} & \mathbf{3\%} & \mathbf{4\%} \end{array} \\ \begin{bmatrix} 18 & 29 & 41 \\ 41 & 60 & 76 \\ 63 & 83 & 95 \\ 80 & 96 & 100 \\ 92 & 100 & 95 \\ 99 & 97 & 83 \\ 100 & 89 & 70 \\ 97 & 79 & 56 \\ 92 & 67 & 43 \end{bmatrix} \end{array}$$

Since all the payoffs are positive, you begin by looking for dominated rows and columns, and you notice that you can immediately eliminate rows 10, 20, 30, 80 and 90:

$$\begin{array}{c} \\ \mathbf{40} \\ \mathbf{50} \\ \mathbf{60} \\ \mathbf{70} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{2\%} & \mathbf{3\%} & \mathbf{4\%} \end{array} \\ \begin{bmatrix} 80 & 96 & 100 \\ 92 & 100 & 95 \\ 99 & 97 & 83 \\ 100 & 89 & 70 \end{bmatrix} \end{array}$$

This is definitely a more manageable game!

You next set up the simplex tableau and proceed to solve it using your handy matrix software.

	x	y	z	s	t	u	v	p	
s	80	96	100	1	0	0	0	0	1
t	92	100	95	0	1	0	0	0	1
u	99	97	83	0	0	1	0	0	1
v	100	89	70	0	0	0	1	0	1
p	−1	−1	−1	0	0	0	0	1	0

	x	y	z	s	t	u	v	p	
s	0	124	220	5	0	0	−4	0	1
t	0	453	765	0	25	0	−23	0	2
u	0	889	1370	0	0	100	−99	0	1
x	100	89	70	0	0	0	1	0	1
p	0	−11	−30	0	0	0	0	100	0

	x	y	z	s	t	u	v	p	
s	0	−2570	0	685	0	−2200	1630	0	115
t	0	−11895	0	0	6850	−15300	8845	0	395
z	0	889	1370	0	0	100	−99	0	1
x	13700	5970	0	0	0	−700	830	0	130
p	0	1160	0	0	0	30	−160	13700	140

	x	y	z	s	t	u	v	p	
s	0	−668560	0	1211765	−2233100	1096000	0	0	74665
v	0	−11895	0	0	6850	−15300	8845	0	395
z	0	6685600	12117650	0	678150	−630200	0	0	47950
x	24235300	12535500	0	0	−1137100	1301500	0	0	260300
p	0	1671400	0	0	219200	41100	0	24235300	260300

(Quite a collection of telephone numbers!) Now that you are done with the simplex method part of the calculation, you decide to calculate only the row strategy, since the column strategy is of no relevance to the company. From the bottom row we obtain the vector

$$[0 \quad 219{,}200 \quad 41{,}100 \quad 0].$$

Normalizing this gives

$$[0 \quad 0.842 \quad 0.158 \quad 0].$$

You advise Forest Lumber to follow a mixed strategy: 84% of their harvesting should consist of 50-year old trees, and 16% of their harvesting should consist of 60-year old trees.

You are disappointed when the company spokesperson tells you that it will be impossible to follow your advice: their tree plantation consists of several stands and their policy is to completely clear a whole stand, rather than fractions of stands. After thinking about this for a while, you realize that 84% of 50-year old trees plus 16% of 60-year old trees averages to $.84\times50 + .16\times60 = 51.6$. Thus you advise them to harvest an entire stand of trees every 51.6 years.

You could also use your mixed strategy to calculate the expected discounted value of each tree they harvest if you know the interest rate. For example, if the interest rate is 2%, then the 2% discounted value (not relative value) table gives us:

$$0.842\times2{,}879 + 0.158\times3{,}080 \approx \$2{,}911 \text{ per tree.}$$

Exercises

1. If you knew that the interest rate would remain at 2% for the indefinite future, what would you recommend?

2. If you recommended that all the trees be harvested at 60 years, which of the three interest rates would be the worst for Forest Lumber Inc.?

3. If you knew that interest rates would fluctuate between 2% and 3% with both 2% and 3% equally likely, what would you recommend?

4. If you knew that interest rates would fluctuate between 2%, 3%, and 4%, with all equally likely, what would you recommend? Would this agree with your recommendation if the interest rate were to remain steady at 3%?

5. Find the optimal column strategy, and use it to calculate the expected interest rate, assuming that interest rates followed the optimal strategy.

6. Calculate the value of the game, and interpret your answer as follows: if, for instance, it is 93, it will mean that the average tree will be harvested at 93% of its peak value. Why can't you reasonably expect an answer of 100?

7. Does the use of relative discounted value for payoffs help Forest Lumber Inc. maximize revenue or maximize the number of trees harvested at peak value? Explain the possible distinction between these.

Answers to Odd-Numbered Exercises

G.1

1.

$$\mathbf{A}\ \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \mathbf{B} \\ \begin{array}{cc} 1 & 3 \end{array} \\ \begin{bmatrix} 1 & 10 \\ 2 & -4 \end{bmatrix} \end{array}$$

3.

$$\mathbf{A}\ \begin{array}{c} \\ 2 \\ 3 \end{array}\begin{array}{c} \mathbf{B} \\ \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} -3 & -10 & 10 \\ 2 & 3 & -4 \end{bmatrix} \end{array}$$

5.

$$\mathbf{A}\ B\ \begin{array}{c} \mathbf{B} \\ b \\ [\,0\,] \end{array}$$

7.

$$\textbf{You}\ \begin{array}{c} \\ H \\ T \end{array}\begin{array}{c} \textbf{Your Friend} \\ \begin{array}{cc} H & T \end{array} \\ \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \end{array}$$

9. B = Brakpan; N = Nigel; S = Springs

$$\textbf{You}\ \begin{array}{c} \\ \mathrm{B} \\ \mathrm{N} \\ \mathrm{S} \end{array}\begin{array}{c} \textbf{Your Opponent} \\ \begin{array}{ccc} \mathrm{B} & \mathrm{N} & \mathrm{S} \end{array} \\ \begin{bmatrix} 0 & 0 & 1{,}000 \\ 0 & 0 & 1{,}000 \\ -1{,}000 & -1{,}000 & 0 \end{bmatrix} \end{array}$$

11. F = Finland; S = Sweden; N = Norway

$$\textbf{You Invade}\ \begin{array}{c} \\ \mathrm{F} \\ \mathrm{S} \\ \mathrm{N} \end{array}\begin{array}{c} \textbf{Your Opponent Defends} \\ \begin{array}{ccc} \mathrm{F} & \mathrm{S} & \mathrm{N} \end{array} \\ \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \end{array}$$

13. P = PleasantTap; T = Thunder Rumble; S = Strike the Gold, N = None

$$\textbf{You Bet}\ \begin{array}{c} \\ \mathrm{P} \\ \mathrm{T} \\ \mathrm{S} \end{array}\begin{array}{c} \textbf{Winner} \\ \begin{array}{cccc} \mathrm{P} & \mathrm{T} & \mathrm{S} & \mathrm{N} \end{array} \\ \begin{bmatrix} 25 & -10 & -10 & -10 \\ -10 & 35 & -10 & -10 \\ -10 & -10 & 40 & -10 \end{bmatrix} \end{array}$$

15. Boris vs. Julius; evenly matched **17.** $1,200, since reduction yields two possibilities: $1,000 and $1,200, both resulting in a 15% market share, and the more they can charge for the same market, the better! **19. (a)** General Kennedy should reconnoiter the northern route. **(b)** The Japanese commander should use the northern route.

G.2

1. A should play 1; B should play 2; Strictly determined; Value = 1 **3.** A's optimal strategy is 2; B's optimal strategy is 1 or 3; Strictly determined; Value = −2 **5.** A's optimal strategy is 2; B's optimal strategy is *a* or *b*; Strictly determined; Value = −3 **7.** A's optimal strategy is *B*; B's optimal strategy is *b*; Not strictly determined **9. (a)** No expansion **(b)** There are two saddle points, giving June Fairweather the choice of either expanding or not expanding. Advise her to expand, since this has a better potential payoff in the event of a 50% increase in tourism. **11.** Confess **13. (a)** The entry in location 1,1 is a saddle point. If either commander had used a non-optimal strategy, the payoff would either have stayed the same or gotten worse, assuming that his opponent was using the optimal strategy. **(b)** Not strictly determined. If the Japanese commander knows that Kenney will use his optimal strategy (Southern Route), then he would do better to use the northern route, even though his optimal strategy is the southern route. On the other hand, if Kenney knows that the Japanese commander will use his optimal strategy (Southern Route), Kenney would do best to stick to his optimal strategy (Southern Route). **15.** The game is strictly deternimed with two saddle points, suggesting that the patient use either Streptokinase or T.P.A.. Of the two, you should recommend Streptokinase, since it is less likely to result in a repeated stroke from cerebral hemmorrhage. This does give the lowest overall recurrence rate:

1.1%. **17.** Strictly determined. Both sellers should locate at b. **19.** Payoff matrix: (D = Dark Tower; K = Karnack)

		Trolls		
		2 to D, 0 to K	1 to D, 1 to K	0 to D, 2 to K
Elves	3 to D, 0 to K	−1	0	0
	2 to D, 1 to K	0	−1	0
	1 to D, 2 to K	0	−1	0
	0 to D, 3 to K	0	1	−1

The game is not strictly determined.

21. Like a saddle point in a payoff matrix, the center of a saddle is a low point (minimum height) in one direction and a high point (maximum) in a perpendicular drection, **23.** Although there is a sadlle point in the 2,4 position, you would be wrong to use saddle points (based on the minimax criterion) to reach the conclusion that row strategy 2 is best. One reason is that the entries in the matrix do not reepresent payoffs, since high numbers of employees in an area do not necessarily represent benefit to the row player. Another reason for this is that there is no opponent deciding what your job will be in such a way as to force you into the least populated job.

25. $\begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

G.3

1. −1 **3.** −1/4 **5.** 3/16 **7.** [0 0 1 0] **9.** [1 0 0 0] **11.** S = [1/4 3/4]; T = [3/4 1/4]^t; e = −1/4 **13.** S = [3/4 1/4]; T = [3/4 1/4]^t; e = −5/4 **15.** Option 2: move to the suburbs. **17. (a)** 66% **(b)** Yes; spend the whole night studying game theory. This will give you an expected score of 75% **19. (a)** $150 **(b)** Leave them in the wild. **21. (a)** $1.20766 million **(b)** Keep a infectious disease specialist on call all the time. **23. (a)** 5/12 **(b)** Average **(c)** Less prepared students **25. (a)** 2 batches **(b)** $30 **27.** Allocate 5/17 of their budget to WISH and the rest (12/17) to WASH. Softex will lose approximately $8,824. **29.** If one player plays a pure strategy, the other player can always choose a pure strategy that makes the value of the game better for him or her than if the first player chose a good mixed strategy. See also the answers to Exercises 27 and 28 in the preceding section. **31.** An example is $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$.

(This is cheating a bit: the row player's best strategy is always [0 1].) **33.** Let P be the payoff matrix, and suppose that the column player (Nature) is using strategy T. The row player can reason as follows. Compute

$$[1\ \ 0\ \ 0\ \ \dots\ \ 0]PT = e_1,$$
$$[0\ \ 1\ \ 0\ \ \dots\ \ 0]PT = e_2,$$
$$\dots$$
$$[0\ \ 0\ \ 0\ \ \dots\ \ 1]PT = e_m.$$

These are the expected values of the game for each choice of pure strategy available to the row player. Suppose the largest of these is e_i. If the row player plays any mixed strategy, say $[x_1\ \ x_2\ \ \dots\ \ x_m]$, the expected value of the game will be $x_1e_1 + x_2e_2 + \dots + x_me_m$. This will be strictly smaller than e_i unless $x_i = 1$ and all the others are 0. That is, the row player's best strategy *assuming that the column player sticks with his strategy* T, is to play his ith pure strategy.

G.4

1. $S = [2/3 \quad 1/3]$; $T = [2/3 \quad 1/3]^T$; $e = -2/3$ **3.** $S = [3/5 \quad 2/5]$; $T = [2/5 \quad 3/5 \quad 0]^T$; $e = 1/5$ **5.** $S = [0 \quad 3/4 \quad 1/4]$; $T = [1/4 \quad 3/4]^T$; $e = -1/2$ **7.** $S = [0 \quad 0 \quad 1]$; $T = [0 \quad 0 \quad 1]^T$; $e = 1$ **9.** $S = [0 \quad 0 \quad 1 \quad 0]$; $T = [1 \quad 0 \quad 0 \quad 0]^T$; $e = -1$ **11.** T. Down should spend 10/11 or 90.9% of the time in Littleville, 1/11, or 9.09% of the time in Metropolis, and skip Urbantown. T. Spend should spend 8/11 or 72.73% of the time in Littleville, 3/11, or 27.27% of the time in Metropolis, and skip Urbantown. If they follow these strategies, T. Down can expect to lose by approximately 227 votes. **13.** Select Option 1 with a probability of 214/431, select Option 2 with a probability of 107/431, select Option 3 with as probability of 110/431 **15.** Devote 5/14 of the night to game theory, 4/14 to linear programming, 5/14 to matrix algebra, and you can expect to earn approximately 64% **17.** A should use the strategy [1/2 1/3 1/6], B should use the strategy $[1/2 \quad 1/3 \quad 1/6]^T$, and A can expect to win an average of 2/3 points per game. **19.** Invest $6,666,67 in precious metals funds, $3,333.33 in the bond market, and nothing in mutual stock funds. You can expect to have $11,000 at the end of the year. **21.** You should make 1 batch 3/5 of the time, 2 batches 2/5 of the time, and never make 3 batches. You can expect to lose an everage of $10 per month. **23.** Send all shipments to Miami for an average profit of $10,000 per run. (The game is strictly determined.) **25.** We list strategies as pairs (x, y), where x represents the number of regiments sent to the first location and y represents the number of regiments sent to the second location. Colonel Blotto should play (0, 4) with probability 4/9, (2, 2) with probability 1/9, and (4, 0) with probabilty 4/9. Captain Kije has a number of optimal strategies: He can play (0, 3) with probability 1/30 (1, 2) with probability 8/15, (2, 1) with probability 16/45, and (3, 0) with probability 7/90. Another optimal strategy is to reverse the order of the probabilities just given. Another is to play (0, 3) with probability 1/18 (1, 2) with probability 4/9, (2, 1) with probability 4/9, and (3, 0) with probability 1/18. The value of the game is 14/9 in favor of Colonel Blotto. **27. (a)** Keep on call a specialist in infectious diseases 27.6% of the time, a specialist in oncology 30.7% of the time, and a specialist in rheumatology 41.7% of the time. **(b)** You can expect a revenue of $1.35 million per month. **29.** The row player's optimal strategy guarantees the largest expected value for the best counter-strategy chosen by the column player. Put another way, if the row player chooses any other strategy, there will be a strategy for the column player that will make the expected value smaller. **31.** If the row player uses a pure strategy instead of the optimal mixed strategy, the column player will notice and switch from using his or her optimal strategy to a pure strategy that lowers the payoff. For example, if the row player uses the first pure strategy in Example 3, the column player will switch to his or her first pure strategy, giving a payoff of –8,000 instead of the expected value of –4,444.44. Even if the column player does not notice what the row player is doing, the row player cannot raise the expected value of the game, as long as the column player sticks to the optimal strategy. Thus, the risk of playing a nonoptimal pure strategy is not worth taking.