Supplementary Chapters to Accompany

Finite Mathematics *(2nd. Ed.)*

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✣ Chapter L—Logic

You're the Expert—Does God Exist?

Your have been assigned the job of evaluating the attempts of mortals to prove the existence of God. And many attempts there have been. Three in particular have caught your attention: they are known as the cosmological argument, the teleological argument, and the ontological argument.

Cosmological Argument (St. Thomas Aquinas)

> No effect can cause itself, but requires another cause. If there were no first cause, there would be an infinite sequence of preceding causes. Clearly there cannot be an infinite sequence of causes, therefore there is a first cause, and this is God.

Teleological Argument (St. Thomas Aquinas)

> All things in the world act towards an end. They could not do this without there being an intelligence that directs them. This intelligence is God.

Ontological Argument (St. Anselm)

> God is a being than which none greater can be thought. A being thought of as existing is greater than one thought of as not existing. Therefore, one cannot think of God as not existing, so God must exist.

Are these arguments valid?

Introduction

Logic is the underpinning of all reasoned argument. The ancient Greeks recognized its role in mathematics and philosophy, and studied it extensively. Aristotle, in his *Organon*, wrote the first systematic treatise on logic. His work had a heavy influence on philosophy, science and religion through the Middle Ages.

But Aristotle's logic was expressed in ordinary language, so was subject to the ambiguities of ordinary language. Philosophers came to want to express logic more formally and symbolically, more like the way that mathematics is written (Leibniz, in the 17th century, was probably the first to envision and call for such a formalism). It was with the publication in 1847 of G. Boole's *The Mathematical Analysis of Logic* and A. DeMorgan's *Formal Logic* that **symbolic logic** came into being, and logic became recognized as part of mathematics. This also marked the recognition that mathematics is not just about numbers (arithmetic) and shapes (geometry), but encompasses any subject that can be expressed symbolically with precise rules of manipulation of those symbols. It is symbolic logic that we shall study in this chapter.

Since Boole and DeMorgan, logic and mathematics have been inextricably intertwined. Logic is part of mathematics, but at the same time it is the language of mathematics. In the late 19th and early 20th century it was believed that all of mathematics could be reduced to symbolic logic and made purely formal. This belief, though still held in modified form today, was shaken by K. Gödel in the 1930's, when he showed that there would always remain truths that could not be derived in any such formal system. (See some of the footnotes in this chapter.)

The study of symbolic logic is usually broken into several parts. The first and most fundamental is the **propositional calculus**. Built on top of this is the **predicate calculus**, which is the language of mathematics. We shall study the propositional calculus in the first six sections of this chapter and look at the predicate calculus briefly in the last two.

L.1 Statements and Logical Operators

In this chapter we shall study **propositional calculus**, which, contrary to what the name suggests, has nothing to do with the subject usually called "calculus." Actually, the term "calculus" is a generic name for any area of mathematics that concerns itself with *calculating*. For example, arithmetic could be called the calculus of numbers. Propositional calculus is the calculus of propositions. A **proposition**, or **statement**, is any declarative sentence which is either true (T) or false (F). We refer to T or F as the **truth value** of the statement.

Example 1 ***Propositions***

Which of the following are statements? What are their truth values?

(a) $2 + 2 = 4$
(b) $1 = 0$
(c) It will rain tomorrow.
(d) If I am Buddha, then I am not Buddha.
(e) Solve the following equation for x.
(f) The number 5.
(g) This statement is false.
(h) This statement is true.

Solution

(a) The sentence "$2 + 2 = 4$" is a statement, since it can be either true or false.[1] Since it happens to be a true statement, its truth value is T.

(b) The sentence "$1 = 0$" is also a statement, but its truth value is F.

(c) "It will rain tomorrow" is a statement. To determine its truth value we shall have to wait for tomorrow.

(d) We shall see later that the statement "If I am Buddha, then I am not Buddha" really amounts to the simpler statement "I am not Buddha." As long as the speaker is not Buddha, this is a true statement.

(e) "Solve the following equation for x" is not a statement, as it cannot be assigned any truth value whatsoever. It is an imperative, or command, rather than a declarative sentence.

(f) "The number 5" is not a statement, since it is not even a complete sentence.

(g) "This statement is false" gets us into a bind: If it were true, then, since it is declaring itself to be false, it must be false. On the other hand, if it were false, then its declaring itself false is a lie, so it is true! In other words, if it is true, then it is false, and if it is false, then it is true, and we

[1] If you doubt that "$2 + 2 = 4$" is a sentence to begin with, read it aloud: "Two plus two equals four," is a perfectly respectable English sentence.

go around in circles. We get out of this bind by refusing to call it a statement. An equivalent pseudo-statement is: "I am lying," so this sentence is known as **the liar's paradox**.

(h) "This statement is true" may *seem* like a statement, but there is no way that its truth value can be determined. It makes just as much sense to say that the sentence is true as to say that it is false. We thus refuse to call it a statement.

The last two sentences in the preceding example are called **self-referential sentences**, since they refer to themselves. Self-referential sentences are henceforth disqualified from statementhood, so this is the last time you will see them in this chapter.[1]

We shall use the letters p, q, r, s and so on to stand for propositions. Thus, for example, we might decide that p should stand for the proposition "the moon is round." We shall write

p: "the moon is round"

to express this. We read:

p is the statement "the moon is round."

We can form new propositions from old ones in several different ways. For example, starting with p: "I am an Anchovian," we can form the **negation** of p: "It is not the case that I am an Anchovian" or simply "I am not an Anchovian." We denote the negation of p by $\sim p$, read "*not p*." We mean by this that, if p is true, then $\sim p$ is false, and vice-versa. We can show the meaning of $\sim p$ in a **truth table**:

p	$\sim p$
T	F
F	T

On the left are the two possible truth values of p and on the right are the corresponding truth values of $\sim p$. The symbol ~ is our first example of a **logical operator**.

Example 2 ***Negation***

Find the negations of the following propositions.

(a) p: "$2 + 2 = 4$"

(b) q: "$1 = 0$"

(c) r: "Diamonds are a pearl's best friend."

(d) s: "All the politicians in this town are crooks."

[1] Self-referential sentences are not simply an idle indulgence; the famous logician Kurt Gödel used a mathematical formulation of the Liar's Paradox to draw very profound conclusions about the power of mathematics. For more on self-referential sentences, see *Metamagical Themas: Questing for the Essence of Mind and Pattern* by Douglas R. Hofstadter (Bantam Books, New York 1986)

Solution

(a) $\sim p$: "It is not the case that $2 + 2 = 4$," or, more simply, $\sim p$: "$2 + 2 \neq 4$."

(b) $\sim q$: "$1 \neq 0$"

(c) $\sim r$: "Diamonds are not a pearl's best friend."

(d) $\sim s$: "Not all the politicians in this town are crooks."

Before we go on… Notice that $\sim p$ is false, because p is true. However, $\sim q$ is true, because q is false. A statement of the form $\sim q$ can very well be true; it is a common mistake to think it must be false.

To say that diamonds are not a pearl's best friend is not to say that diamonds are a pearl's worst enemy. The negation is not the polar opposite, but whatever would deny the truth of the original statement. Similarly, saying that not all politicians are crooks is not the same as saying that no politicians are crooks, but is the same as saying that some politicians are not crooks. Negations of statements involving the **quantifiers** "all" or "some" are tricky. We'll study quantifiers in more depth when we discuss the predicate calculus.

Here is another way we can form a new proposition from old ones. Starting with p: "I am clever," and q: "You are strong," we can form the statement "I am clever and you are strong." We denote this new statement by $p \wedge q$, read "*p and q*." In order for $p \wedge q$ to be true, *both p and q* must be true. Thus, for example, if I am indeed clever, but you are not strong, then $p \wedge q$ is false. The symbol $\wedge$ is another logical operator. The statement $p \wedge q$ is called the **conjunction** of p and q.

Conjunction

The **conjunction** of p and q is the statement $p \wedge q$, which we read "p and q." Its truth value is defined by the following truth table.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

In the p and q columns are listed all four possible combinations of truth values for p and q, and in the $p \wedge q$ column we find the corresponding truth value for $p \wedge q$. For example, reading across the third row tells us that, if p is false and q is true, then $p \wedge q$ is false. In fact, the only way we can get a T in the $p \wedge q$ column is if both p and q are true, as the table shows.

Example 3 ***Conjunction***

If p: "This galaxy will ultimately disappear into a black hole" and q: "$2 + 2 = 4$," what is $p \wedge q$?

Solution $p \wedge q$: "This galaxy will ultimately disappear into a black hole and $2 + 2 = 4$," or the more astonishing statement: "Not only will this galaxy ultimately disappear into a black hole, but $2 + 2 = 4$!"

Before we go on… q is true, so if p is true then the whole statement $p \wedge q$ will be true. On the other hand, if p is false, then the whole statement $p \wedge q$ will be false.

Example 4

With p and q as in Example 3, what does the statement $p \wedge (\sim q)$ say?

Solution $p \wedge (\sim q)$ says: "This galaxy will ultimately disappear into a black hole and $2 + 2 \neq 4$," or "Contrary to your hopes, this galaxy is doomed to disappear into a black hole; moreover, two plus two is decidedly *not* equal to four!"

Before we go on… Since $\sim q$ is false, the whole statement $p \wedge (\sim q)$ is false (regardless of whether p is true or not).

Example 5

If p is the statement "This chapter is interesting" and q is the statement "Logic is an interesting subject," then express the statement "This chapter is not interesting even though logic is an interesting subject" in logical form.

Solution The first clause is the negation of p, so is $\sim p$. The second clause is simply q. The phrase "even though" is another way of saying that both clauses are true, and so the whole statement is $(\sim p) \wedge q$.

Example 6

Let p: "This chapter is interesting," q: "This whole book is interesting" and r: "Life is interesting." Express the statement "Not only is this chapter interesting, but this whole book is interesting, and life is interesting, too" in logical form.

Solution The statement is asserting that all three statements p, q and r are true. (Note that "but" is simply an emphatic form of "and.") Now we can combine all three in two steps: First, we can combine p and q to get $p \wedge q$, meaning "This chapter is interesting and this book is interesting." We can then conjoin this with r to get: $(p \wedge q) \wedge r$. This says: "This chapter is interesting, this book is interesting and life is interesting." On the other hand, we could equally well have done it the other way around: conjoining q and r gives "This book is interesting and life is interesting." We then conjoin p to get $p \wedge (q \wedge r)$, which again says: "This chapter is interesting, this book is interesting and life is interesting." We shall soon see that $(p \wedge q) \wedge r$ is logically the same as $p \wedge (q \wedge r)$, a fact called the **associative law** for conjunction. Thus both answers $(p\wedge q)\wedge r$ and $p\wedge(q\wedge r)$ are equally valid. This is like saying that $(1 + 2) + 3$ is the same as $1 + (2 + 3)$. As with addition, we often drop the parentheses and write $p\wedge q\wedge r$.

As we've just seen, there are many ways of expressing a conjunction in English. For example, if p: "Waner drives a fast car" and q: "Costenoble drives a slow car," the following are all ways of saying $p\wedge q$.

Waner drives a fast car and Costenoble drives a slow car.
Waner drives a fast car but Costenoble drives a slow car.
Waner drives a fast car yet Costenoble drives a slow car.
Although Waner drives a fast car, Costenoble drives a slow car.
Waner drives a fast car even though Costenoble drives a slow car.
While Waner drives a fast car, Costenoble drives a slow car.

Any sentence that says that two things are both true is a conjunction. Symbolic logic strips away any elements of surprise or judgment that are expressed in an English sentence.

Here is a third logical operator. Starting once again with p: "I am clever," and q: "You are strong," we can form the statement "I am clever or you are strong," which we write symbolically as $p\vee q$, read "*p or q*." Now in English the word "or" has several possible meanings, so we have to agree on which one we want here. Mathematicians have settled on the **inclusive or**: $p\vee q$ means p is true or q is true *or both are true*[1]. With p and q as above, $p\vee q$ stands for "I am clever or you are strong, or both." We shall sometimes include the phrase "or both" for emphasis, but if we leave it off we still interpret "or" as inclusive.

Disjunction

The **disjunction** of p and q is the statement $p\vee q$, which we read "p or q." Its truth value is defined by the following truth table.

p	q	$p\vee q$
T	T	T
T	F	T
F	T	T
F	F	F

This is the **inclusive** or, so $p\vee q$ is true when p is true or q is true *or both* are true.

Notice that the only way for $p\vee q$ to be false is for *both* p and q to be false. For this reason we can say that $p\vee q$ also means "p and q are not both false." We'll say more about this in the next section.

Example 7 ***Disjunction***

Let p: "the butler did it" and let q: "the cook did it." What does $p\vee q$ say?

[1] There is also the **exclusive or**: "p or q *but not both*." This can be expressed as $(p\vee q)\wedge\sim(p\wedge q)$. Do you see why?

Solution $p \vee q$: "either the butler or the cook did it."

Before we go on… Remember that this does not exclude the possibility that the butler and cook both did it—or that they were in fact the same person! The only way that $p \vee q$ could be false is if neither the butler nor the cook did it.

Example 8

Let p: "the butler did it," let q: "the cook did it," and let r: "the lawyer did it." What does $(p \vee q) \wedge (\sim r)$ say?

Solution $(p \vee q) \wedge (\sim r)$ says "either the butler or the cook did it, but not the lawyer."

Example 9

Let p: "55 is divisible by 5," q: "676 is divisible by 11" and r: "55 is divisible by 11." Express the following statements in symbolic form:
(a) "Either 55 is not divisible by 11 or 676 is not divisible by 11."
(b) "Either 55 is divisible by either 5 or 11, or 676 is divisible by 11."

Solution
(a) This is the disjunction of the negations of p and q, so is $(\sim p) \vee (\sim q)$.
(b) This is the disjunction of all three statements, so is $(p \vee q) \vee r$, or, equivalently, $p \vee (q \vee r)$. We often drop the parentheses and write $p \vee q \vee r$.

Before we go on… (a) is true because $\sim q$ is true. (b) is true because p is true. Notice that r is also true. If at least one of p, q, or r is true, the whole statement $p \vee q \vee r$ will be true.

We end this section with a little terminology: A **compound statement** is a statement formed from simpler statements via the use of logical operators. Examples are $\sim p$, $(\sim p) \wedge (q \vee r)$ and $p \wedge (\sim p)$. A statement that cannot be expressed as a compound statement is called an **atomic statement**[1]. For example, "I am clever" is an atomic statement. In a compound statement such as $(\sim p) \wedge (q \vee r)$, we refer to p, q and r as the **variables** of the statement. Thus, for example, $\sim p$ is a compound statement in the single variable p.

L.1 Exercises

Which of Exercises 1–14 are statements? Comment on the truth values of all the statements you encounter. If a sentence fails to be a statement, explain why.

1. All swans are white.

2. The fat cat sat on the mat.

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[1] "Atomic" comes from the Greek for "not divisible." Atoms were originally thought to be the indivisible components of matter, but the march of science proved that wrong. The name stuck, though.

3. Look in thy glass and tell whose face thou viewest.[1]

4. My glass shall not persuade me I am old.[2]

5. Father Nikolsky penned his dying confession to Patriarch Arsen III Charnoyevich of Peç in the pitch dark, somewhere in Poland, using a mixture of gunpowder and saliva, and a quick Cyrillic hand, while the innkeeper's wife scolded and cursed him through the bolted door.[3]

6. 1,000,000,000 is the largest number.

7. There is no largest number.

8. There may or may not be a largest number.

9. Intelligent life abounds in the universe.

10. This definitely is a statement.

11. He, she or it is lying.

12. This is exercise number 12.

13. This sentence no verb.[4]

14. “potato” is spelled p-o-t-a-t-o-e.

Let p: “Our mayor is trustworthy,” q: “Our mayor is a good speller,” and r = “Our mayor is a patriot.” Express each of the statements in Exercises 15–20 in logical form:

15. Although our mayor is not trustworthy, he is a good speller.

16. Either our mayor is trustworthy, or he is a good speller.

17. Our mayor is a trustworthy patriot who spells well.

18. While our mayor is both trustworthy and patriotic, he is not a good speller.

19. It may or may not be the case that our mayor is trustworthy.

20. Either our mayor is not trustworthy or not a patriot, yet he is an excellent speller.

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[1] William Shakespeare
[2] Ibid.
[3] from *Dictionary of the Khazars* by Milorad Pavic (Vintage Press).
[4] From *Metamagical Themas: Questing for the Essence of Mind and Pattern* by Douglas R. Hofstadter (Bantam Books, New York 1986)

Let p: "Willis is a good teacher," q: "Carla is a good teacher," r: "Willis' students hate math," s: "Carla's students hate math." Express the statements in Exercises 21–30 in words.

21. $p \wedge (\sim r)$

22. $(\sim p) \wedge (\sim q)$

23. $p \vee (r \wedge (\sim q))$

24. $(r \vee (\sim p)) \wedge q$

25. $q \vee (\sim q)$

26. $((\sim p) \wedge (\sim s)) \vee q$

27. $r \wedge (\sim r)$

28. $(\sim s) \vee (\sim r)$

29. $\sim (q \vee s)$

30. $\sim (p \wedge r)$

Assume that it is true that "Polly sings well," it is false that "Quentin writes well," and it is true that "Rita is good at math." Determine the truth of each of the statements in Exercises 31–40.

31. Polly sings well and Quentin writes well.

32. Polly sings well or Quentin writes well.

33. Polly sings poorly and Quentin writes well.

34. Polly sings poorly or Quentin writes poorly.

35. Either Polly sings well and Quentin writes poorly, or Rita is good at math.

36. Either Polly sings well and Quentin writes poorly, or Rita is not good at math.

37. Either Polly sings well or Quentin writes well, or Rita is good at math.

38. Either Polly sings well and Quentin writes well, or Polly sings well and Rita is good at math.

39. Polly sings well, and either Quentin writes well or Rita is good at math.

40. Polly sings poorly, or Quentin writes poorly and Rita is good at math.

Communication and Reasoning Exercises

41. The statement that either p or q is true, but not both is called the **exclusive disjunction** of p and q, which we write as $p \amalg q$. Give a formula for $p \amalg q$ in terms of the logical operators $\sim$, $\wedge$ and $\vee$.

42. The statement that either p and q are both true, or neither is true, is called the **biconditional** of p and q, and write it as $p \leftrightarrow q$. Give a formula for $p \leftrightarrow q$ in terms of the logical operators $\sim$, $\wedge$ and $\vee$.

43. Referring to Exercise 41, give an example of an everyday usage of exclusive disjunction.

44. Referring to Exercise 42, give an example of an everyday usage of exclusive conjunction.

45. Give an example of a self-referential question that is its own answer.[1]

46. Comment on the following pair of sentences:
The next statement is false.
The preceding statement is true.

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[1] Such a question was posed by Douglas Hofstadter in *Metamagical Themas: Questing for the Essence of Mind and Pattern* (Bantam Books, New York 1986)

L.2 Logical Equivalence, Tautologies, and Contradictions

We suggested in the preceding section that certain statements are equivalent. For example, we claimed that $(p\wedge q)\wedge r$ and $p\wedge(q\wedge r)$ are equivalent—a fact we called the associative law for conjunction. In this section, we use truth tables to say precisely what we mean by logical equivalence, and we also study certain statements that are either "self-evident" ("tautological"), or "evidently false" ("contradictory").

We start with some more examples of truth tables.

Example 1 *Truth Tables*

Construct the truth table for $\sim(p\wedge q)$.

Solution Whenever we encounter a complex formula like this we work from the inside out, just as we might do if we had to evaluate an algebraic expression like $-(a + b)$. Thus, we start with the p and q columns, then construct the $p\wedge q$ column, and finally, the $\sim(p\wedge q)$ column.

p	q	$p\wedge q$	$\sim(p\wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

Notice how we get the $\sim(p\wedge q)$ column from the $p\wedge q$ column: we reverse all the truth values.

Example 2

Construct the truth table for $p\vee(p\wedge q)$.

Solution Since there are two variables, p and q, we again start with the p and q columns. We then evaluate $p\wedge q$, and finally take the disjunction of the result with p.

p	q	$p\wedge q$	$p\vee(p\wedge q)$
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

How did we get the last column from the others? Since we are "or-ing" p with $p\wedge q$, we look at the values in the p and $p\wedge q$ columns and combine these according to the instructions for "or." Thus, for example, in the second row we have $T\vee F = T$ and in the third row we have $F\vee F = F$. (If you look at the second row of the truth table for "or" you will see T | F | T, and in the last row you will see F | F | F.)

Example 3

Construct the truth table for $\sim(p \wedge q) \wedge (\sim r)$.

Solution Here, there are *three* variables: p, q and r. Thus we start with three initial columns showing all eight possibilities.

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

We now add columns for $p \wedge q$, $\sim(p \wedge q)$ and $\sim r$, and finally $\sim(p \wedge q) \wedge (\sim r)$, according to the instructions for these logical operators. Here is how the table would grow as we construct it.

p	q	r	$p \wedge q$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

p	q	r	$p \wedge q$	$\sim(p \wedge q)$	$\sim r$
T	T	T	T	F	F
T	T	F	T	F	T
T	F	T	F	T	F
T	F	F	F	T	T
F	T	T	F	T	F
F	T	F	F	T	T
F	F	T	F	T	F
F	F	F	F	T	T

and finally,

p	q	r	$p \wedge q$	$\sim(p \wedge q)$	$\sim r$	$\sim(p \wedge q) \wedge (\sim r)$
T	T	T	T	F	F	F
T	T	F	T	F	T	F
T	F	T	F	T	F	F
T	F	F	F	T	T	T
F	T	T	F	T	F	F
F	T	F	F	T	T	T
F	F	T	F	T	F	F
F	F	F	F	T	T	T

We say that two statements are **logically equivalent** if, for all possible truth values of the variables involved, the two statements always have the same truth values. If s and t are equivalent, we write $s \equiv t$. This is *not* another logical statement. It is simply the claim that the two statements s and t are logically equivalent. Here are some examples.

Example 4 ***Logical Equivalence***

Show that $p \equiv \sim(\sim p)$. This is called **double negation**.

Solution To demonstrate the logical equivalence of these two statements, we construct a truth table with columns for both p and $\sim(\sim p)$.

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

The p column gives the two possible truth values for p, while the $\sim p$ column gives the corresponding values for its negation. We get the values in the $\sim(\sim p)$ column from those in the $\sim p$ column by reversing the truth values: if $\sim p$ is false, then its negation, $\sim(\sim p)$, must be true, and vice-versa. Since the p and $\sim(\sim p)$ columns now contain the same truth values in all rows ("for all possible truth values of the variables involved"), they are logically equivalent.

Example 5 ***Double Negation***

Rewrite "It's not true that I'm not happy" in simpler form.

Solution Let p: "I am happy," so that the given statement is $\sim(\sim p)$. This is equivalent to p, in other words, to the statement "I am happy."

Before we go on… Unlike French ("Ceci **n**'est **pas** une pipe") and colloquial English ("This **ain't no** pipe"), a double negative in logic always means a positive statement.

Example 6 ***DeMorgan's Law***

Show that $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$. This is one of **DeMorgan's Laws**.

Solution We construct a truth table showing both $\sim(p \wedge q)$ and $(\sim p) \vee (\sim q)$.

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$(\sim p) \vee (\sim q)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Since the $\sim(p \wedge q)$ column and $(\sim p) \vee (\sim q)$ column agree, we conclude that they are equivalent.

Before we go on… The statement $\sim(p \wedge q)$ can be read as "It is not the case that both p and q are true" or "p and q are not both true." We have just shown that this is equivalent to "Either p is false or q is false."

Example 7 *DeMorgan's Law*

Let p: "the President is a Democrat," and q: "the President is a Republican." Interpret $\sim(p \wedge q)$ and the equivalent statement given by DeMorgan's Law.

Solution $\sim(p \wedge q)$: "the President is not both a Democrat and a Republican." This is the same as saying: "either the President is not a Democrat, or he is not a Republican, or he is neither," which is $(\sim p) \vee (\sim q)$.

Before we go on… This is *not* the same as "the President is a Republican or a Democrat," which would be $q \vee p$. The statement $\sim(p \wedge q)$ would be true if the President were from a third party, while $q \vee p$ would not.

Here are the two equivalences known as DeMorgan's Laws.

DeMorgan's Laws

If p and q are statements, then

$$\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$$
$$\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$$

Mechanically speaking, this means that, when we distribute a negation sign, it reverses $\wedge$ and $\vee$, and the negation applies to both parts.

A compound statement is a **tautology** if its truth value is always T, regardless of the truth values of its variables. It is a **contradiction** if its truth value is always F, regardless of the truth values of its variables. Notice that these are properties of a *single* statement, while logical equivalence relates two statements.

Example 8 *Tautologies*

Show that the statement $p \vee (\sim p)$ is a tautology.

Solution We look at its truth table.

p	$\sim p$	$p\vee(\sim p)$
T	F	T
F	T	T

all T's.

Since there are only Ts in the $p\vee(\sim p)$ column, we conclude that $p\vee(\sim p)$ is a tautology. We can think of this as saying that the truth value of the statement $p\vee(\sim p)$ is *independent* of the value of the "input" variable p.

Before we go on…

"You are sad," the Knight said in an anxious tone: "let me sing you a song to comfort you. . . Everybody that hears me sing it—either it brings the *tears* into their eyes, or else—"

"Or else what?" said Alice, for the Knight had made a sudden pause.

"Or else it doesn't, you know.[1]"

Example 9

Show that $(p\vee q)\vee[(\sim p)\wedge(\sim q)]$ is a tautology.

Solution Its truth table is the following.

p	q	$\sim p$	$\sim q$	$p\vee q$	$(\sim p)\wedge(\sim q)$	$(p\vee q)\vee[(\sim p)\wedge(\sim q)]$
T	T	F	F	T	F	T
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	F	T	T

Again, since the last column contains only Ts, the statement is a tautology.

When a statement is a tautology, we also say that the statement is **tautological**. In common usage this sometimes means simply that the statement is convincing. In logic it means something stronger: that the statement is always true, under all circumstances. In contrast, a contradiction, or **contradictory** statement, is *never* true, under any circumstances.

Example 10 ***Contradictions***

Show that the statement $(p\vee q)\wedge[(\sim p)\wedge(\sim q)]$ is a contradiction.

Solution Its truth table is the following.

[1] From *Through the Looking-Glass*, by Lewis Carroll. Lewis Carroll was the pen name of the Rev. Charles Lutwidge Dodgson (1832–1898), a logician who taught at Christ Church College, Oxford.

p	q	$\sim p$	$\sim q$	$p\vee q$	$(\sim p)\wedge(\sim q)$	$(p\vee q)\wedge[(\sim p)\wedge(\sim q)]$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	F

Since the last column contains only Fs, we conclude that $(p\vee q)\wedge[(\sim p)\wedge(\sim q)]$ is a contradiction.

Before we go on… In common usage we sometimes say that *two* statements are contradictory. By this we mean that their conjunction is a contradiction: they cannot both be true. For example, the statements $p\vee q$ and $(\sim p)\wedge(\sim q)$ are contradictory, since we've just shown that their conjunction is a contradiction. In other words, no matter what the truth values of p and q, it is never true that both $p\vee q$ and $(\sim p)\wedge(\sim q)$ are true at the same time. (Can you see why this is so from the meaning of $p\vee q$?)

Most statements are neither tautologies nor contradictions. The first three examples in this section were of statements that were sometimes true and sometimes false.

Here is a list of some important logical equivalences, most of which we have already encountered. (The verifications of some of these appear as exercises.) We shall add to this list as we go along.

Important Logical Equivalences: First List

$\sim(\sim p) \equiv p$	the Double Negative Law
$p\wedge q \equiv q\wedge p$	the Commutative Law for conjunction.
$p\vee q \equiv q\vee p$	the Commutative Law for disjunction.
$(p\wedge q)\wedge r \equiv p\wedge(q\wedge r)$	the Associative Law for conjunction.
$(p\vee q)\vee r \equiv p\vee(q\vee r)$	the Associative Law for disjunction.
$\sim(p\vee q) \equiv (\sim p)\wedge(\sim q)$	DeMorgan's Laws
$\sim(p\wedge q) \equiv (\sim p)\vee(\sim q)$	
$p\wedge(q\vee r) \equiv (p\wedge q)\vee(p\wedge r)$	the Distributive Laws
$p\vee(q\wedge r) \equiv (p\vee q)\wedge(p\vee r)$	
$p\wedge p \equiv p$	Absorption Laws
$p\vee p \equiv p$	

Note that these logical equivalences apply to *any* statements. The *p*s, *q*s and *r*s can stand for atomic statements or compound statements.

Example 11 *Simplifying*

Simplify the statement $\sim([p\wedge(\sim q)]\wedge r)$.

Solution By "simplify" we mean "find a simpler equivalent statement." We can analyze this statement from the outside in. It is first of all a negation, but further it is the negation $\sim(A\wedge B)$, where A is $(p\wedge(\sim q))$ and B is r. To see that the statement has this structure, look for the "principal connective," the *last* connective ("and" or "or") you would evaluate in forming the truth table. Now one of DeMorgan's Laws is

$$\sim(A \wedge B) \equiv (\sim A) \vee (\sim B).$$

Applying this equivalence gives

$$\sim([p \wedge (\sim q)] \wedge r) \equiv (\sim[p \wedge (\sim q)]) \vee (\sim r).$$

We can apply DeMorgan's Law again, this time to the statement $\sim(p \wedge (\sim q))$. Doing so gives

$$\sim[p \wedge (\sim q)] \equiv (\sim p) \vee \sim(\sim q) \equiv (\sim p) \vee q.$$

Notice that we've also used the Double Negative law. Putting these equivalences together gives

$$\sim([p \wedge (\sim q)] \wedge r) \equiv (\sim[p \wedge (\sim q)]) \vee (\sim r) \equiv ((\sim p) \vee q) \vee (\sim r),$$

which we can write as

$$(\sim p) \vee q \vee (\sim r),$$

since the Associative Law tells us that it does not matter which two expressions we "or" first.

Example 12

Consider: "You will get an A if either you are clever and the sun shines, or you are clever and it rains." Rephrase the condition more simply.

Solution The condition is "you are clever and the sun shines, or you are clever and it rains." Let's analyze this symbolically: Let p: "you are clever," q: "the sun shines," and r: "it rains." The condition is then $(p \wedge q) \vee (p \wedge r)$. We can "factor out" the p using one of the distributive laws in reverse, getting

$$(p \wedge q) \vee (p \wedge r) \quad \equiv \quad p \wedge (q \vee r).$$

We are taking advantage of the fact that the logical equivalences we listed can be read from right to left as well as from left to right. Putting $p \wedge (q \vee r)$ back into English, we can rephrase the sentence as "You will get an A if you are clever and either the sun shines or it rains."

L.2 Exercises

Construct the truth tables for expressions in Exercises 1–10.

1. $p \wedge (\sim q)$

2. $p \vee (\sim q)$

3. $\sim(\sim p)\vee p$

4. $p\wedge(\sim p)$

5. $(\sim p)\wedge(\sim q)$

6. $(\sim p)\vee(\sim q)$

7. $(p\wedge q)\wedge r$

8. $p\wedge(q\wedge r)$

9. $p\wedge(q\vee r)$

10. $(p\wedge q)\vee(p\wedge r)$

Use truth tables to verify the logical equivalences given in Exercises 11–20.

11. $p\wedge p \equiv p$

12. $p\vee p \equiv p$

13. $p\vee q \equiv q\vee p$. . . the Commutative Law for disjunction.

14. $p\wedge q \equiv q\wedge p$. . . the Commutative Law for conjunction.

15. $\sim(p\vee q) \equiv (\sim p)\wedge(\sim q)$

16. $\sim(p\wedge(\sim q)) \equiv (\sim p)\vee q$

17. $(p\wedge q)\wedge r \equiv p\wedge(q\wedge r)$ . . . the Associative Law for conjunction.

18. $(p\vee q)\vee r \equiv p\vee(q\vee r)$. . . the Associative Law for disjunction.

19. $p\vee(q\wedge(\sim q)) \equiv p$

20. $p\wedge(\sim p) \equiv q\wedge(\sim q)$

Use truth tables to check whether each statement in Exercises 21–26 is a tautology, contradiction, or neither.

21. $p\wedge(\sim p)$

22. $p\wedge p$

23. $p\wedge\sim(p\vee q)$

24. $p\vee\sim(p\vee q)$

25. $p\vee\sim(p\wedge q)$

26. $q\vee\sim(p\wedge(\sim p))$

Apply the stated logical equivalence to each of the statements in Exercises 27–34.

27. $p\vee(\sim p)$; the Commutative law

28. $p\wedge(\sim q)$; the Commutative law

29. $\sim(p\wedge(\sim q))$; DeMorgan's Law

30. $\sim(q\vee(\sim q))$; DeMorgan's Law

31. $p\vee\sim(p\wedge q)$; DeMorgan's Law

32. $q\vee\sim(p\wedge(\sim p))$; DeMorgan's Law

33. $p \vee ((\sim p) \wedge q)$; the Distributive Law **34.** $(\sim q) \wedge ((\sim p) \vee q)$; the Distributive Law.

Use logical equivalences to rewrite each of the sentences in Exercises 35–42. If possible, rewrite more simply.

35. It is not true that both I am Julius Caesar and you are a fool.

36. It is not true that either I am Julius Caesar or you are a fool.

37. Either it's raining and I have forgotten my umbrella, or it's raining and I have forgotten my hat.

38. I forgot my hat or my umbrella, and I forgot my hat or my glasses.

39. My computer crashes when it has been on a long time, and when it's not the case that either the air is dry or the moon is not full.

40. The study determined that the market crashed because interest rates rose, or because it was not the case that both earnings rose and the moon was not full.

41. The warning light will come on if the pressure drops while the temperature is high, or if the pressure drops while not both the emergency override and the manual controls are activated.

42. The alarm will sound if the door is opened and the override button is not pushed while the alarm is activated, or if there is motion and it is not the case that either the override button is pushed or the alarm is not activated.

Communication and Reasoning Exercises

43. If two propositions are logically equivalent, what can be said about their truth tables?

44. If a proposition is neither a tautology nor a contradiction, what can be said about its truth table?

45. Can an atomic statement be a tautology or a contradiction? Explain.

46. Can a statement with a single variable p be a tautology or a contradiction? Explain.

47. If A and B are two (possibly compound statements) such that $A \vee B$ is a contradiction, what can you say about A and B?

48. If A and B are two (possibly compound statements) such that $A \wedge B$ is a tautology, what can you say about A and B?

49. Your friend thinks that all tautologies are logically equivalent to one another. Is he correct? Explain.

50. Another friend thinks that, if two statements are logically equivalent to each other, then they must either be tautologies or contradictions. Is she correct? Explain.

L.3 The Conditional and the Biconditional

Consider the following statement: "If you earn an A in logic, then I'll buy you a new car." It seems to be made up out of two simpler statements,

p: "you earn an A in logic," and
q: "I will buy you a new car."

The original statement says: *if p is true, then q is true*, or, more simply, **if** p, **then** q. We can also phrase this as p **implies** q, and we write the statement symbolically as $p \longrightarrow q$.

Now let us suppose for the sake of argument that the original statement: "If you earn an A in logic, then I'll buy you a new car," is true. This does *not* mean that you *will* earn an A in logic. All it says is that *if* you do so, then I will buy you that car. Thinking of this as a promise, the only way that it can be broken is if you *do* earn an A and I do *not* buy you a new car. With this in mind we define the logical statement $p \longrightarrow q$ as follows.

Conditional

The **conditional** $p \longrightarrow q$, which we read "if p, then q" or "p implies q," is defined by the following truth table.

p	q	$p \longrightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The arrow "$\longrightarrow$" is the **conditional** operator, and in $p \longrightarrow q$ the statement p is called the **antecedent**, or **hypothesis**, and q is called the **consequent**, or **conclusion**.

Note

(1) The only way that $p \longrightarrow q$ can be false is if p is true and q is false—this is the case of the "broken promise."

(2) If you look at the truth table again, you see that we say that "$p \longrightarrow q$" is true when p is false, *no matter what the truth value of q*. Think again about the promise—if you don't get that A, then whether or not I buy you a new car, I have not broken my promise. However, this part of the truth table seems strange if you think of "if p then q" as saying that p *causes* q. The problem is that there are really many ways in which the English phrase "if . . . then . . ." is used. Mathematicians have simply agreed that the meaning given by the truth table above is the most useful for mathematics, and so that is the meaning we shall always use. Shortly we'll list some other English phrases that we interpret as conditional statements.

Here are some examples that will help to explain each line in the truth table.

Example 1 ***True Implies True***

Is the following statement true or false? "If 1 + 1 = 2 then the sun rises in the east."

Solution

Yes, since both "1 + 1 = 2" and "the sun rises in the east" are true, and the first line in the truth table of the conditional yields a true statement. In general,

If p and q are both true, then $p \longrightarrow q$ *is true.*

Before we go on... Notice that the statements *p* and *q* need not have anything to do with one another. We are not saying that the sun rises in the east *because* 1 + 1 = 2, simply that the whole statement is logically true.

Example 2 ***True Can't Imply False***

Is the following statement true or false? "When it rains, I need to water my lawn."

Solution

No. We can rephrase this statement as "If it rains then I need to water my lawn," which is clearly false: if it truly does rain, then it is clearly false that I need to water my lawn. The second line of the truth table for the conditional yields a false statement. In general,

If p is true and q is false, then $p \longrightarrow q$ *is false.*

Before we go on... Notice that we interpreted "When *p*, *q*" as "If *p* then *q*."

Example 3 ***False Implies Anything***

Is the following statement true or false? "If the moon is made of green cheese, then I am a professor of mathematics."

Solution

True. While the first part of the statement is false, the second part could be true or false, depending on the speaker. But, the third and fourth lines of the truth table for the conditional both yield true statements. In general,

If p is false, then $p \longrightarrow q$ *is true, no matter whether q is true or not.*

Before we go on... "If I had a million dollars I'd be on Easy Street." "Yeah, and if my grandmother had wheels she'd be a bus." The point of the retort is that anything follows from a false hypothesis.

Example 4 ***The "Switcheroo" Law***

Show that $p \longrightarrow q \equiv (\sim p) \vee q$.

Solution

We show the equivalence using a truth table.

p	q	$p \longrightarrow q$	$\sim p$	$(\sim p) \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

same

Before we go on... In other words, $p \longrightarrow q$ is true if either p is false or q is true. By DeMorgan's Law these statements are also equivalent to $\sim(p \wedge (\sim q))$. The only way the conditional can be false is the case of the *broken promise*: when p is true and q is false.

For lack of a better name, we shall call the equivalence $p \longrightarrow q \equiv (\sim p) \vee q$ the "Switcheroo" law.[1]

The fact that we can convert implication to disjunction should surprise you. In fact, behind this is a very powerful technique. It is not too hard (using truth tables) to convert *any* logical statement into a disjunction of conjunctions of atoms or their negations. This is called *disjunctive normal form*, and is essential in the design of the logical circuitry making up digital computers.

We have already seen how colorful language can be. Not surprisingly, it turns out that there are a great variety of different ways of saying that p implies q. Here are some of the most common:

Some Phrasings of the Conditional

We interpret each of the following as equivalent to the conditional $p \longrightarrow q$.

If p then q.	p implies q.
q follows from p.	Not p unless q.
q if p.	p only if q.
Whenever p, q.	q whenever p.
p is sufficient for q.	q is necessary for p.
p is a sufficient condition for q.	q is a necessary condition for p.

Notice the difference between "if" and "only if." We say that "p only if q" means $p \longrightarrow q$ since, assuming that $p \longrightarrow q$ is true, p can be true only if q is also. In other words, the only line of the truth table that has $p \longrightarrow q$ true and p true also has q true. The phrasing "p is a sufficient

[1] This name was used by Douglas R. Hofstadter in his book *Gödel, Escher, Bach: An Eternal Golden Braid* (Basic Books 1979).

condition for q" says that it suffices to know that p is true to be able to conclude that q is true. For example, it is sufficient that you get an A in logic for me to buy you a new car. Other things might induce me to buy you the car, but an A in logic would suffice. The phrasing "q is necessary for p" says that, for p to be true q must be true (just as we said for "p only if q").

Example 5 *Rephrasing a Conditional*

Rephrase the sentence "If it's Tuesday, this must be Belgium."

Solution

Here are various ways of rephrasing the sentence:

"Its being Tuesday implies that this is Belgium."
"This is Belgium if it's Tuesday."
"It's Tuesday only if this is Belgium."
"It can't be Tuesday unless this is Belgium."
"Its being Tuesday is sufficient for this to be Belgium."
"That this is Belgium is a necessary condition for its being Tuesday."

In the exercises for §2, we saw that the commutative law holds for both conjunction and disjunction: $p \wedge q \equiv q \wedge p$, and $p \vee q \equiv q \vee p$.

Question Does the commutative law hold for the conditional. In other words, is $p \longrightarrow q$ equivalent to $q \longrightarrow p$?

Answer No, as we can see in the following truth table.

p	q	$p \longrightarrow q$	$q \longrightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

not the same

Converse

The statement $q \longrightarrow p$ is called the **converse** of the statement $p \longrightarrow q$. A conditional and its converse are *not* equivalent.

The fact that a conditional can easily be confused with its converse is often used in advertising. For example, the slogan "Drink Boors, the official beverage of the US Olympic Team" suggests that all US Olympic athletes drink Boors (i.e., if you are a US Olympic athlete, then you drink Boors). What it is *trying* to insinuate at the same time is the converse: that all drinkers of Boors become US Olympic athletes (if you drink Boors then you are a US Olympic athlete, or: it is sufficient to drink Boors to become a US Olympic athlete).

Although the conditional $p \longrightarrow q$ is not the same as its converse, it *is* the same as its so-called **contrapositive**, $(\sim q) \longrightarrow (\sim p)$. While this could easily be shown with a truth table (which you will be asked to do in an exercise) we can show this equivalence by using the equivalences we already know:

$p \longrightarrow q \equiv (\sim p) \vee q$	(Switcheroo)
$\equiv q \vee (\sim p)$	(Commutativity of $\vee$)
$\equiv \sim(\sim q) \vee (\sim p)$	(Double Negative)
$\equiv (\sim q) \longrightarrow (\sim p)$	(Switcheroo)[1]

Contrapositive
The statement $(\sim q) \longrightarrow (\sim p)$ is called the **contrapositive** of the statement $p \longrightarrow q$. A conditional and its contrapositive are equivalent.

Example 6 ***Converse and Contrapositive***

Give the converse and contrapositive of the statement "If you earn an A in logic, then I'll buy you a new car."

Solution
As we noted earlier, this statement has the form $p \longrightarrow q$ where p is the statement "you earn an A" and q is the statement "I'll buy you a new car." The converse is $q \longrightarrow p$. In words, this is "If I buy you a new car then you earned an A in logic."

The contrapositive is $(\sim q) \longrightarrow (\sim p)$. In words, this is "If I don't buy you a new car, then you didn't earn an A in logic."

Before we go on… Assuming that the original statement is true, notice that the converse is not necessarily true. There is nothing in the original promise that prevents me from buying you a new car anyway if you do not earn the A. On the other hand, the contrapositive is true. If I don't buy you a new car, it must be that you didn't earn an A, otherwise I would be breaking my promise.

It sometimes happens that we do want both a conditional and its converse to be true. The conjunction of a conditional and its converse is called a **biconditional**.

[1] Note that the Switcheroo law applies to any pair of statements, and says that $\sim A \vee B \equiv A \longrightarrow B$, no matter what A and B are. In the last step, we had $A = (\sim q)$ and $B = (\sim p)$.

Biconditional

The **biconditional**, written $p \leftrightarrow q$, is defined to be the statement $(p \rightarrow q) \wedge (q \rightarrow p)$. Its truth table is the following.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Looking at the truth table, we can see that $p \leftrightarrow q$ is true when p and q have the same truth values and it is false when they have different truth values. Here are some common phrasings of the biconditional.

Phrasings of the Biconditional

We interpret each of the following as equivalent to $p \leftrightarrow q$.

p if and only if q.
p is necessary and sufficient for q.
p is equivalent to q.

For the phrasing "p if and only if q," remember that "p if q" means $q \rightarrow p$ while "p only if q" means $p \rightarrow q$. For the phrasing "p is equivalent to q," the statements A and B are logically equivalent if and only if the statement $A \leftrightarrow B$ is a tautology (why?). We'll return to that in the next section. Notice that $p \leftrightarrow q$ is logically equivalent to $q \leftrightarrow p$ (you are asked to show this as an exercise), so we can reverse p and q in the phrasings above.

Example 7 ***Rephrasing a Biconditional***

Rephrase the statement "I teach math if and only if I am paid a large sum of money."

Solution

Here are some possible rephrasings:

I am paid a large sum of money if and only if I teach math.
My teaching math is necessary and sufficient for me to be paid a large sum of money.
For me to teach math it is necessary and sufficient that I be paid a large sum of money.

Before we go on... Another possibility is: "I will not be paid a large sum of money if and only if I do not teach math." Why is this equivalent to the others?

L.3 Exercises

Find the truth value of each of the statements in Exercises 1–28.

1. "If 1=1, then 2=2."

2. "If 1=1, then 2=3."

3. "If 1=0, then 1=1."

4. "If 1≠0, then 2≠2."

5. "If 1=1 and 1=2, then 1=2."

6. "If 1=3 or 1=2 then 1=1."

7. "If everything you say is false, then everything you say is true."

8. "If everything you say is false, then 1=2."

9. "A sufficient condition for 1 to equal 2 is 1=3."

10. "1=1 is a sufficient condition for 1 to equal 2."

11. "1=0 is a necessary condition for 1 to equal 2."

12. "1=1 is a necessary condition for 1 to equal 2."

13. "1=2 is a necessary condition for 1 to be unequal to 2."

14. "1≠2 is a necessary condition for 1 to be unequal to 3."

15. "If I pay homage to the great Den, then the sun will rise in the east."

16. "If I fail to pay homage to the great Den, then the sun will still rise in the east."

17. "In order for the sun to rise in the east, it is necessary that it sets in the west."

18. "In order for the sun to rise in the east, it is sufficient that it sets in the west."

19. "The sun rises in the west only if it sets in the west."

20. "The sun rises in the east only if it sets in the east."

21. "The Milky Way Galaxy will fall into a great black hole if everything I say is false."

22. "The Milky Way Galaxy will not fall into a great black hole only if 1=1."

23. "1=2 is a necessary and sufficient condition for 1 to be unequal to 2."

24. "1≠2 is a necessary and sufficient condition for 1 to be unequal to 3."

25. "The sun will rise in the east if and only if it sets in the west."

26. "The sun will rise in the east if and only if it does not set in the west."

27. "In order for the sun to rise in the west, it is necessary and sufficient that it sets in the east."

28. "In order for the sun to rise in the east, it is necessary and sufficient that it sets in the west."

Construct the truth table for each of the statements in Exercises 29–40, and indicate which (if any) are tautologies or contradictions.

29. $p \longrightarrow (q \vee p)$

30. $(p \vee q) \longrightarrow \sim p$

31. $(p \wedge q) \longrightarrow \sim p$

32. $(p \longrightarrow \sim p) \longrightarrow \sim p$

33. $(p \longrightarrow \sim p) \longrightarrow p$

34. $p \wedge (p \longrightarrow \sim p)$

35. $(p \wedge \sim p) \longrightarrow q$

36. $\sim((p \wedge \sim p) \longrightarrow q)$

37. $p \leftrightarrow (p \vee q)$

38. $(p \wedge q) \leftrightarrow \sim p$

39. $(p \wedge \sim p) \leftrightarrow (q \wedge \sim q)$

40. $(p \vee \sim p) \leftrightarrow (q \vee \sim q)$

Use truth tables to demonstrate the equivalences in Exercises 41–46.

41. $p \longrightarrow q \equiv (\sim q) \longrightarrow (\sim p)$

42. $\sim(p \longrightarrow q) \equiv p \wedge (\sim q)$

43. $p \longrightarrow q \equiv (\sim p) \vee q$

44. $(p \longrightarrow \sim p) \equiv \sim p$

45. $(p \leftrightarrow \sim p) \equiv (q \leftrightarrow \sim q)$

46. $(p \leftrightarrow \sim q) \equiv (q \leftrightarrow \sim p)$

Give the contrapositive and converse of each of the statements in Exercises 47–54, phrasing your answers in words.

47. "If I think, then I am."

48. "If I do not think, then I do not exist."

49. "If I do not think, then I am Buddha."

50. "If I am Buddha, then I think."

51. "These birds are of a feather only if they flock together."

52. “These birds flock together only if they are of a feather.”

53. “In order to worship Den, it is necessary to sacrifice beasts of burden.”

54. “In order to read the Tarot, it is necessary to consult the Oracle.”

Express each of the statements in Exercises 55–60 in equivalent disjunctive form.

55. “I am if I think.”

56. “I think if I am.”

57. “Symphony orchestras will cease to exist without government subsidy.”

58. “The education system will collapse without continued taxpayer support.”

59. “Research in the pure sciences will continue if our society wishes it.”

60. “Nuclear physicists would be out of work if their accomplishments were measured purely by the generation of profit.”

Translate the statements in Exercises 61–70 into compound statements utilizing either the conditional or the biconditional, and using p for the statement "I am Julius Caesar" and q for the statement "You are Brutus"

61. “If I am Julius Caesar then you are not Brutus.”

62. “It is not the case that if I am Julius Caesar then you are Brutus.”

63. “I am Julius Caesar only if you are not Brutus.”

64. “You are Brutus only if I am not Julius Caesar.”

65. “I am Julius Caesar if and only if you are not Brutus.”

66. “You are not Brutus if and only if I am not Julius Caesar.”

67. “Either you are Brutus, or I am Julius Caesar.”

68. “Either I am not Julius Caesar, or you are Brutus.”

69. “In order for you to be Brutus, it is necessary and sufficient that I am not Julius Caesar.”

70. “In order for you to not be Brutus, it is necessary and sufficient that I am not Julius Caesar.”

Communication and Reasoning Exercises

71. Give an example of an instance where $p \longrightarrow q$ means that q causes p.

72. Complete the following. If $p \longrightarrow q$, then its converse, ___ , is the statement that ___ and (is/is not) logically equivalent to $p \longrightarrow q$.

73. Complete the following sentence. If both $p \longrightarrow q$ and its ____ are true, then the biconditional, ___ , is ____ .

74. If B is a tautology, why is $A \longrightarrow B$ also a tautology, regardless of A?

75. If A is a contradiction, why is $A \longrightarrow B$ a tautology, regardless of B?

76. If A is a tautology and B is a contradiction, what can you say about $A \longrightarrow B$?

77. If A and B are both contradictions, what can you say about $A \leftrightarrow B$?

78. Give an instance of a biconditional $p \leftrightarrow q$ where neither p nor q causes the other.

L.4 Tautological Implications and Tautological Equivalences

In this section we enlarge our list of "standard" tautologies by adding ones involving the conditional and the biconditional. From now on, we use small letters like p and q to denote atomic statements only, and uppercase letters like A and B to denote statements of all types, compound or atomic.

We first look at some **tautological implications**, tautologies of the form $A \longrightarrow B$. You should check the truth table of each of the statements we give to see that they are, indeed, tautologies.

Modus Ponens or Direct Reasoning

$$[(p \longrightarrow q) \wedge p] \longrightarrow q$$

In words: If an implication and its premise are both true, then so is its conclusion.

For example, if p: "I love math" and q: "I will pass this course," then we have the following tautology:

> *If my loving math implies that I will pass this course, and if I do love math, then I will pass this course.*

We can write a statement like this in **argument form**[1] as follows:

If I love math, then I will pass this course.
I love math.
———————————————
Therefore, I will pass this course.

In symbol form again, we write the following.

$$\begin{array}{l} p \longrightarrow q \\ p \\ \hline \therefore\ q \end{array}$$

What appears above the line in an argument is what is "given;" what appears below is the conclusion we can draw.

Modus ponens is the most direct form of everyday reasoning, hence its alternate name "direct reasoning." When we know that p implies q and we know that p is true, we can conclude that q is also true. This is sometimes known as **affirming the hypothesis**. You should not confuse this with a fallacious argument like: "If I were an Olympic athlete then I would drink Boors. I do drink Boors, therefore I am an Olympic athlete." (Do you see why

[1] We shall define arguments precisely in Section 6, but we shall start using them informally now.

this is nonsense? See the preceding section.) This is known as the fallacy of *affirming the consequent*. There is, however, a correct argument in which we *deny* the consequent:

Modus Tollens or Indirect Reasoning

$[(p \longrightarrow q) \wedge (\sim q)] \longrightarrow (\sim p)$

In words: If an implication is true but its conclusion is false, then its premise is false.

In argument form, this is:

$$\begin{array}{l} p \longrightarrow q \\ \underline{\sim q \qquad\qquad} \\ \therefore\ \sim p \end{array}$$

For example:

If I love math, then I will pass this course.
I will not pass this course.
____
Therefore, I must not love math.

This argument is not quite so direct as before; it contains a little twist: "If I loved math I would pass this course. However, I will not pass this course. Therefore, it must be that I don't love math (else I *would* pass this course)." Hence the name "indirect reasoning."

Note that, again, there is a similar, but fallacious argument to avoid: "If I were an Olympic athlete then I would drink Boors. However, I am not an Olympic athlete. Therefore, I won't drink Boors." This is a mistake Boors certainly hopes you do not make!

Simplification

$(p \wedge q) \longrightarrow p$

and

$(p \wedge q) \longrightarrow q$

In words, the first says: If both p and q are true, then, in particular, p is true.

Quick Example

If the sky is blue and the moon is round, then (in particular) the sky is blue.

Addition

$p \longrightarrow (p \vee q)$

and

$q \longrightarrow (p \vee q)$

In words, the first says: If p is true, then we know that either p or q is true.

Quick Example

If the sky is blue, then either the sky is blue of some ducks are kangaroos.

Note that it doesn't matter in this example whether q is true or not. As long as we know that p is true then $p \vee q$ must also be true.

Warning The following are *not* tautologies:

$(p \vee q) \longrightarrow p$

$p \longrightarrow (p \wedge q)$

In the exercise set you will be asked to check that these are, indeed, not tautologies.

Disjunctive Syllogism or One-or-the-Other

$[(p \vee q) \wedge (\sim p)] \longrightarrow q$

and

$[(p \vee q) \wedge (\sim q)] \longrightarrow p$

In words: If either p or q is true, and one is known to be false, then the other must be true.

Quick Example

If either the cook or the butler did it, but we know that the cook didn't do it, then the butler did it.

Transitivity

$[(p \longrightarrow q) \wedge (q \longrightarrow r)] \longrightarrow (p \longleftarrow r)$

In words: If q is implied by p and r is implied by q, then r is implied by p.

Quick Example

When it rains the ground gets muddy and when the ground is muddy my shoes get dirty. So, when it rains my shoes get dirty.

We sometimes think of transitivity as a "chain rule," allowing us to chain arrows together. In other words, follow the arrows: An arrow from p to q and an arrow from q to r give us an arrow all the way from p to r.

Also important are the **tautological equivalences**, tautologies of the form $A \leftrightarrow B$. Recall that the statement $A \leftrightarrow B$ is true exactly when A and B have the same truth value. When A and B are compound statements, this must be true for all truth values of the atomic statements used in A and B. This means that A and B are logically equivalent statements.

Logical Equivalence and Tautological Equivalence
To say that $A \equiv B$ is the same as saying that $A \leftrightarrow B$ is a tautology.

So, every logical equivalence we already know gives us a tautological equivalence. Here is an example. We give lots more in the table at the end of the section.

Double Negation

$$p \leftrightarrow \sim(\sim p)$$

Since the biconditional can be read either way, we get two argument forms from each tautological equivalence. In this case, they are:

$$\begin{array}{l} p \\ \hline \therefore\ \sim(\sim p) \end{array}$$

and

$$\begin{array}{l} \sim(\sim p) \\ \hline \therefore\ p \end{array}$$

We conclude this section with a list of useful tautologies.

A. Tautological Implications

Symbolic Form	Argument Form		Name
$[(p \longrightarrow q) \wedge p] \longrightarrow q$	$p \longrightarrow q$ p —————— $\therefore\ q$		Modus Ponens (Direct Reasoning)
$[(p \longrightarrow q) \wedge \sim q] \longrightarrow \sim p$	$p \longrightarrow q$ $\sim q$ —————— $\therefore\ \sim p$		Modus Tollens (Indirect Reasoning)
$(p \wedge q) \longrightarrow p$ $(p \wedge q) \longrightarrow q$	$p \wedge q$ ———— $\therefore\ p$	$p \wedge q$ ———— $\therefore\ q$	Simplification
$p \longrightarrow (p \vee q)$	p ——————— $\therefore\ p \vee q$		Addition
$[(p \vee q) \wedge (\sim p)] \longrightarrow q$ $[(p \vee q) \wedge (\sim q)] \longrightarrow p$	$p \vee q$ $\sim p$ ———— $\therefore\ q$	$p \vee q$ $\sim q$ ———— $\therefore\ p$	Disjunctive Syllogism (One-or-the-Other)
$[(p \longrightarrow q) \wedge (q \longrightarrow r)] \longrightarrow (p \longrightarrow r)$	$p \longrightarrow q$ $q \longrightarrow r$ ——————— $\therefore\ p \longrightarrow r$		Transitivity

B. Tautological Equivalences

Symbolic Form	Argument Forms		Name
$p \leftrightarrow \sim(\sim p)$	p ——————— $\therefore\ \sim(\sim p)$	$\sim(\sim p)$ ——————— $\therefore\ p$	Double Negative
$p \wedge q \leftrightarrow q \wedge p$ $p \vee q \leftrightarrow q \vee p$	$p \wedge q$ ——————— $\therefore\ q \wedge p$	$p \vee q$ ——————— $\therefore\ q \vee p$	Commutative Laws
$(p \wedge q) \wedge r \leftrightarrow p \wedge (q \wedge r)$ $(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$	$(p \wedge q) \wedge r$ ——————— $\therefore\ p \wedge (q \wedge r)$ $(p \vee q) \vee r$ ——————— $\therefore\ p \vee (q \vee r)$	$p \wedge (q \wedge r)$ ———————————— $\therefore\ (p \wedge q) \wedge r$ $p \vee (q \vee r)$ ———————————— $\therefore\ (p \vee q) \vee r$	Associative Laws
$\sim(p \vee q) \leftrightarrow (\sim p) \wedge (\sim q)$ $\sim(p \wedge q) \leftrightarrow (\sim p) \vee (\sim q)$	$\sim(p \vee q)$ ——————— $\therefore\ (\sim p) \wedge (\sim q)$ $\sim(p \wedge q)$ ——————— $\therefore\ (\sim p) \vee (\sim q)$	$(\sim p) \wedge (\sim q)$ ———————————— $\therefore\ \sim(p \vee q)$ $(\sim p) \vee (\sim q)$ ———————————— $\therefore\ \sim(p \wedge q)$	DeMorgan's Laws

$p\wedge(q\vee r)\leftrightarrow(p\wedge q)\vee(p\wedge r)$ $p\vee(q\wedge r)\leftrightarrow(p\vee q)\wedge(p\vee r)$	$p\wedge(q\vee r)$ / $\therefore(p\wedge q)\vee(p\wedge r)$ $p\vee(q\wedge r)$ / $\therefore(p\vee q)\wedge(p\vee r)$	$(p\wedge q)\vee(p\wedge r)$ / $\therefore p\wedge(q\vee r)$ $(p\vee q)\wedge(p\vee r)$ / $\therefore p\vee(q\wedge r)$	Distributive Laws
$p\wedge p\leftrightarrow p$ $p\vee p\leftrightarrow p$	$p\wedge p$ / $\therefore\ p$ $p\vee p$ / $\therefore\ p$	p / $\therefore\ p\wedge p$ p / $\therefore\ p\vee p$	Idempotent Laws
$(p\longrightarrow q)\leftrightarrow(\sim p\vee q)$	$p\longrightarrow q$ / $\therefore\ \sim p\vee q$	$\sim p\vee q$ / $\therefore\ p\longrightarrow q$	Switcheroo
$(p\longrightarrow q)\leftrightarrow(\sim q\longrightarrow\sim p)$	$p\longrightarrow q$ / $\therefore\ \sim q\longrightarrow\sim p$	$\sim q\longrightarrow\sim p$ / $\therefore\ p\longrightarrow q$	Contrapositive
$(p\leftrightarrow q)\leftrightarrow((p\longrightarrow q)\wedge(q\longrightarrow p))$	$p\leftrightarrow q$ / $\therefore(p\rightarrow q)\wedge(q\rightarrow p)$	$(p\rightarrow q)\wedge(q\rightarrow p)$ / $\therefore\ p\leftrightarrow q$	Meaning of $\leftrightarrow$

L.4 Exercises

Use truth tables to check the tautologies in Exercises 1–10 (these refer to the lists at the end of the section).

1. Modus Tollens

2. Simplification: $p \wedge q \longrightarrow p$

3. Addition

4. Disjunctive Syllogism: $[(p \vee q) \wedge (\sim p)] \longrightarrow q$

5. Transitivity

6. Double Negative

7. Commutative Law: $(p \wedge q) \leftrightarrow (q \wedge p)$

8. Commutative Law: $(p \vee q) \leftrightarrow (q \vee p)$

9. Switcheroo

10. Contrapositive.

Show that the statements in Exercises 11–16 are *not* tautologies by giving examples of statements p and q for which these implications are false.

11. $(p \vee q) \longrightarrow p$

12. $p \longrightarrow (p \wedge q)$

13. $(p \longrightarrow q) \longrightarrow (q \longrightarrow p)$

14. $\sim(p \wedge q) \longrightarrow [(\sim p) \wedge (\sim q)]$

15. $(p \longrightarrow q) \longrightarrow (\sim p \longrightarrow \sim q)$

16. $((p \longrightarrow q) \wedge (p \longrightarrow r)) \longrightarrow (q \longrightarrow r)$

Write each of the statements in Exercises 17–32 in symbolic form, and then decide whether it is a tautology or not.

17. If I am hungry and thirsty, then I am hungry.

18. If I am hungry or thirsty, then I am hungry.

19. If it's not true that roses are red and violets are blue, then roses are not red and violets are not blue.

20. If roses are not red or violets are not blue, then it's not true that roses are red and violets are blue.

21. For me to bring my umbrella it's necessary that it rain. Therefore if it does not rain I will not bring my umbrella.

22. For me to bring my umbrella it's sufficient that it rain. Therefore if it does not rain I will not bring my umbrella.

23. For me to bring my umbrella it's necessary and sufficient that it rain. Therefore if it does not rain I will not bring my umbrella.

24. For me to bring my umbrella it's necessary and sufficient that it rain. Therefore if I do not bring my umbrella it will not rain.

25. For me to pass math it is sufficient that I have a good teacher. Therefore, I will either have a good teacher or I will not pass math.

26. For me to pass math it is necessary that I have a good teacher. Therefore, I will either have a good teacher or I will not pass math.

27. I am either tired or hungry, but I am not tired, so I must be hungry.

28. I am either smart or athletic, and I am athletic, so I must not be smart.

29. To get good grades it is necessary to study, and if you get good grades you will get a good job. Therefore, it is sufficient to study to get a good job.

30. To get good grades it is sufficient to study, and to get a good job it is necessary to get good grades. Therefore, if you study you will get a good job.

31. To get good grades it is necessary to study, but John did not get good grades. Therefore John did not study.

32. To get good grades it is necessary and sufficient to study, but Jill did not study. Therefore Jill will not get good grades.

Communication and Reasoning Exercises

33. How would you convert a tautology of the form $A \vee B$ into a tautological implication?

34. How would you convert a tautology of the form $(A \longrightarrow B) \wedge (C \longrightarrow D)$ into *two* tautological implications?

35. Complete the following sentence. A tautological equivalence can be expressed as ___ tautological implications.

36. Complete the following sentence. If $A \longrightarrow (B \longrightarrow C)$ is a tautological implication, then, given ___ and ___, we can always deduce ___ .

L.5 Rules of Inference

In the preceding section, we introduced the "argument form" of a tautological implication. For example, we wrote Modus Ponens in the following way:

$$\begin{array}{l} p \longrightarrow q \\ p \\ \hline \therefore\ q \end{array}$$

We think of the statements above the line, the **premises**, as statements given to us as true, and the statement below the line, the **conclusion**, as a statement that must then also be true.

Our convention has been that small letters like p stand for atomic statements. But, there is no reason to restrict Modus Ponens to such statements. For example, we would like to be able to make the following argument:

If roses are red and violets are blue, then sugar is sweet and so are you.
Roses are red and violets are blue.
Therefore, sugar is sweet and so are you.

In symbols, this is

$$\begin{array}{l} (p \wedge q) \longrightarrow (r \wedge s) \\ p \wedge q \\ \hline \therefore\ r \wedge s \end{array}$$

So, we really should write Modus Ponens in the following more general and hence usable form:

$$\begin{array}{l} A \longrightarrow B \\ A \\ \hline \therefore\ B \end{array}$$

where, as our convention has it, A and B can be any statements, atomic or compound.

In this form, Modus Ponens is our first **rule of inference**. We shall use rules of inference to assemble lists of true statements, called **proofs**. A proof is a way of showing how a conclusion follows from a collection of premises. Modus Ponens, in particular, allows us to say that, if $A \longrightarrow B$ and A both appear as statements in a proof, then we are justified in adding B as another statement in the proof. (We shall say more about proofs in the next section.)

Example 1 ***Applying Modus Ponens***

Apply Modus Ponens to statements 1 and 3 in the following list.

1. $(p \vee q) \longrightarrow (r \wedge \sim s)$
2. $\sim r \longrightarrow s$
3. $p \vee q$

Solution
All three statements in this list are compound statements. We can rewrite them in the following way.

1. $A \longrightarrow B$
2. C
3. A

Recall that Modus Ponens tells us that, if the two statements $A \longrightarrow B$ and A appear in a list, we can write down B as well. Applying Modus Ponens, then, to lines 1 and 3 (line 2 doesn't get involved here), we can lengthen our list to get the following.

1. $(p \vee q) \longrightarrow (r \wedge \sim s)$	Premise
2. $\sim r \longrightarrow s$	Premise
3. $p \vee q$	Premise
4. $r \wedge \sim s$	1, 3 Modus Ponens

On the right we have recorded the justification for each line. Lines 1, 2, and 3 were given to us as premises, but we got line 4 by applying Modus Ponens to lines 1 and 3.

Before we go on... The above list of four statements consitutes a *proof* that Statement 4 follows from the premises 1–3, and we refer to it as a **proof of the argument**

$(p \vee q) \longrightarrow (r \wedge \sim s)$	Premise
$\sim r \longrightarrow s$	Premise
$p \vee q$	Premise
$\therefore\ r \wedge \sim s$	Conclusion

In general, a **rule of inference** is an instruction for obtaining new true statements from a list of statements we already know or assume to be true. If you were studying logic as a mathematics or philosophy major, you might use the smallest collection of rules of inference you could get away with. We'll be more generous and give you more tools to work with. For example, *any* of the tautologies listed in the preceding section gives us a rule of inference.

Rule of Inference T1

Any tautology that appears in the lists at the end of the preceding section may be used as a rule of inference.

Example 2 ***Applying Modus Tollens***

Apply Modus Tollens to the following premises.

1. $(p \lor q) \longrightarrow (r \land \sim s)$
2. $\sim(r \land \sim s)$
3. $(p \lor q) \longrightarrow p$

Solution

The given list can be written like this:

1. $A \longrightarrow B$
2. $\sim B$
3. $A \longrightarrow C$

As a rule of inference, Modus Tollens has the following form.

$$\begin{array}{l} A \longrightarrow B \\ \sim B \\ \hline \therefore\ \sim A \end{array}$$

This matches the first two premises, so we can apply Modus Tollens to get the following.

1. $(p \lor q) \longrightarrow (r \land \sim s)$	Premise
2. $\sim(r \land \sim s)$	Premise
3. $(p \lor q) \longrightarrow p$	Premise
4. $\sim(p \lor q)$	1, 2 Modus Tollens

Before we go on… We used $A \longrightarrow C$ to represent the third premise when thinking about how to use Modus Tollens. It didn't really matter how we represented it; we could just as easily have written D. Since we're not using this statement at all, it doesn't matter how we represent it. On the other hand, we needed to write the first two statements as $A \longrightarrow B$ and $\sim B$ in order to see that we could apply Modus Tollens to them. Part of learning to apply the rules of inference is learning how to analyze the structure of statements at the right level of detail.

Recall that a tautology is a statement that is always true. As such, we should be allowed to add it to a list of true statements. This give us our next rule of inference.

Rule of Inference T2

Any tautology that appears in the lists at the end of the preceding section may be added as a new line in a proof.

Example 3 *Using T2*

Justify each step in the following.

1. $p \longrightarrow \sim(\sim p)$
2. $\sim(\sim p) \longrightarrow p$
3. $p \longrightarrow p$

Solution

Each of the first two steps is an application of rule of inference T2. Recall that $p \leftrightarrow \sim(\sim p)$ is a tautology, called Double Negation. We permit ourselves to break a tautological equivalence into its two tautological implications and write down either one. In this case, we wrote down both. The third step is an application of rule of inference T1, using Transitivity. Thus, we can write our justifications like this:

1. $p \longrightarrow \sim(\sim p)$	Double Negative
2. $\sim(\sim p) \longrightarrow p$	Double Negative
3. $p \longrightarrow p$	1, 2 Transitivity

Before we go on… What we have just written down is a proof of the following argument, in which there are no premises:

$$\therefore\ p \longrightarrow p$$

Rules T1 and T2 are the two we shall use most often. The next two are used less often, but are sometimes necessary.

Rule of Inference S (Substitution)

We can replace any part of a compound statement with a tautologically equivalent statement.

As with T2, we rely on our list at the end of the preceding section to decide what statements are tautologically equivalent. Notice that Rule S is the same as the mathematical rule of substitution: in any equation, if part of an expression is equal to something else, then we can replace it by that something else.

Example 4 *Substitution*

Justify the third and fourth steps in the following proof.

1. $(\sim(\sim p))\longrightarrow q$ Premise
2. p Premise
3. $p\longrightarrow q$
4. q

Solution

The third line resembles the first except that $\sim(\sim p)$ has been replaced by p. But, that replacement can be justified by the substitution rule because the Double Negation tautology tells us that p is tautologically equivalent to $\sim(\sim p)$. To get the fourth line we simply apply Modus Ponens to the second and third lines. Thus, we can fill in the following justifications.

1. $(\sim(\sim p))\longrightarrow q$	Premise
2. p	Premise
3. $p\longrightarrow q$	1, Substitution
4. q	2, 3 Modus Ponens

Rule of Inference C (Conjunction)

If A and B are any two lines in a proof, then we can add the line $A\wedge B$ to the proof.

This is just the obvious fact that, if we already know A and B to be true, then we know that $A\wedge B$ is true.

Question Are we done yet?

Answer Not quite. Recall that we will generally be given premises that we are to assume true. We get to write them down as steps in our proof as well, so we may as well record one last rule of inference.

Rule of Inference P (Premise)

We can write down a premise as a line in a proof.

Of course, we cannot make up premises as we go along, they will be given to us at the start. It is traditional, but not necessary, to write down all of the premises as the first lines of a proof. On the other hand, some people like to write them down only as they are needed.

In the following rather tricky proof, we start with two premises, and shall manage to use every single rule of inference except for T2:

Example 5 ***The Rules of Inference***

Fill in the justifications in the following proof.

1. $a \longrightarrow q$	Premise
2. $b \longrightarrow q$	Premise
3. $\sim a \vee q$	
4. $\sim b \vee q$	
5. $(\sim a \vee q) \wedge (\sim b \vee q)$	
6. $(\sim a \wedge \sim b) \vee q$	
7. $\sim(a \vee b) \vee q$	
8. $(a \vee b) \longrightarrow q$	

Solution

Here are the justifications, with the type of rule of inference noted after each:

1. $a \longrightarrow q$	Premise (P)
2. $b \longrightarrow q$	Premise (P)
3. $\sim a \vee q$	1, Switcheroo (T1)
4. $\sim b \vee q$	2, Switcheroo (T1)
5. $(\sim a \vee q) \wedge (\sim b \vee q)$	3, 4 Conjunction (C)
6. $(\sim a \wedge \sim b) \vee q$	5, Distributive Law[1] (T1)
7. $\sim(a \vee b) \vee q$	6, DeMorgan (S)
8. $(a \vee b) \longrightarrow q$	7, Switcheroo (T1)

Before we go on… This proves shows the validity of the following argument:

$$\begin{array}{l} a \longrightarrow q \\ b \longrightarrow q \\ \hline \therefore \quad (a \vee b) \longrightarrow q \end{array}$$

In other words, if each of a and b implies q, then if either one is true then so is q. This should be obvious if you think about it a bit, but its proof is not!

[1] Note that we have used the Distributive law "backwards." That is, we have used it in the following form (as listed among our tautological equivalences):

$$\begin{array}{l} (A \vee C) \wedge (B \vee C) \\ \hline \therefore (A \wedge B) \vee C \end{array}$$

L.5 Exercises

In each of Exercises 1–34, supply the missing statement or reason, as the case may be. (To make life simpler, we shall allow you to write $\sim(\sim p)$ as just p whenever it occurs. This saves an extra step in practice.)

1.

Statement	Reason
1. $p \longrightarrow \sim q$	Premise
2. p	Premise
3. - - - -	1,2 Modus Ponens

2.

Statement	Reason
1. $\sim p \longrightarrow q$	Premise
2. $\sim p$	Premise
3. - - - -	1,2 Modus Ponens

3.

Statement	Reason
1. $(\sim p \vee q) \longrightarrow \sim(q \wedge r)$	Premise
2. $\sim p \vee q$	Premise
3. - - - -	1,2 Modus Ponens

4.

Statement	Reason
1. $(\sim p \wedge q) \longrightarrow (q \wedge \sim r)$	Premise
2. $\sim p \wedge q$	Premise
3. - - - -	1,2 Modus Ponens

5.

Statement	Reason
1. $(\sim p \vee q) \longrightarrow \sim(q \wedge r)$	Premise
2. $q \wedge r$	Premise
3. - - - -	1,2 Modus Tollens

6.

Statement	Reason
1. $(\sim p \wedge q) \longrightarrow (q \wedge \sim r)$	Premise
2. $\sim(q \wedge \sim r)$	Premise
3. - - - -	1,2 Modus Tollens

7.

Statement	Reason
1. $\sim(\sim p \vee q)$	Premise
2. - - - -	1, DeMorgan

8.

Statement	Reason
1. $\sim(p \wedge \sim q)$	Premise
2. - - - -	1, DeMorgan

9.

Statement	Reason
1. $(p \wedge r) \longrightarrow \sim q$	Premise
2. $\sim q \longrightarrow r$	Premise
3. - - - -	1,2 Transitive Law

10.

Statement	Reason
1. $(\sim p \wedge q) \longrightarrow (q \wedge \sim r)$	Premise
2. $(q \wedge \sim r) \longrightarrow s$	Premise
3. - - - -	1,2 Transitive Law

11.

Statement	Reason
1. $(p \wedge r) \longrightarrow \sim q$	Premise
2. $\sim q \longrightarrow r$	Premise
3. $\sim r$	Premise
4. - - - -	1,2 Transitive Law
5. - - - -	3,4 Modus Tollens

12.

Statement	Reason
1. $(\sim p \wedge q) \longrightarrow (q \wedge \sim r)$	Premise
2. $(q \wedge \sim r) \longrightarrow s$	Premise
3. $\sim s$	Premise
4. - - - -	1,2 Transitive Law
5. - - - -	3,4 Modus Tollens

13.

Statement	Reason
1. $(p \longrightarrow q) \vee r$	Premise
2. $\sim r$	Premise
3. - - - -	1,2 Disjunctive Syllogism

14.

Statement	Reason
1. $\sim(p \wedge q) \vee s$	Premise
2. $\sim s$	Premise
3. - - - -	1,2 Disjunctive Syllogism

15.

Statement	Reason
1. $p \longrightarrow (r \wedge q)$	Premise
2. $\sim r$	Premise
3. - - - -	2, Addition of $\sim q$
4. - - - -	3, DeMorgan
5. - - - -	1,4 Modus Tollens

16.

Statement	Reason
1. $(p \vee q) \longrightarrow r$	Premise
2. $\sim r$	Premise
3. - - - -	1,2 Modus Tollens
4. - - - -	3, DeMorgan
5. - - - -	Simplification[1]

[1] Use simplification in the following form:

$$\frac{A \wedge B}{\quad}$$

17.

1. $(p \wedge q) \longrightarrow r$	Premise
2. q	Premise
3. p	Premise
4. - - - -	3,2 Rule C
5. - - - -	1,4 Modus Ponens

18.

1. $p \longrightarrow r$	Premise
2. p	Premise
3. s	Premise
4. - - - -	1,2 Modus Ponens
5. - - - -	3,4 Rule C

19. 1. $\sim(\sim p \vee q) \longrightarrow p \wedge \sim q$ - - - -

20. 1. $[(\sim p \longrightarrow q) \wedge \sim q] \longrightarrow p$ - - - -

21.

1. $p \longrightarrow \sim(q \wedge r)$	Premise
2. $q \wedge r$	Premise
3. $\sim p$	- - - -

22.

1. $(s \wedge t) \longrightarrow (q \wedge \sim r)$	Premise
2. $(s \wedge t)$	Premise
3. $q \wedge \sim r$	- - - -

23.

1. $\sim p \vee (r \longrightarrow s)$	Premise
2. $p \longrightarrow (r \longrightarrow s)$	- - - -

24.

1. $\sim p \longrightarrow q$	Premise
2. $\sim p \vee \sim q$	- - - -

25.

1. $\sim[p \longrightarrow \sim(q \wedge r)]$	Premise
2. $\sim[\sim p \vee \sim(q \wedge r)]$	- - - -
3. $p \wedge (q \wedge r)$	- - - -

26.

1. $(\sim p \wedge \sim q) \longrightarrow p$	Premise
2. $\sim(\sim p \wedge \sim q) \vee p$	- - - -
3. $(p \vee q) \vee p$	- - - -

27.

1. $(p \vee q) \longrightarrow (r \wedge s)$	Premise
2. p	Premise
3. $p \vee q$	- - - -
4. $r \wedge s$	- - - -
5. r	- - - -

28.

1. $(p \vee q) \vee \sim r$	Premise
2. $\sim p \wedge \sim q$	Premise
3. $\sim(p \vee q)$	- - - -
4. $\sim r$	- - - -
5. $\sim r \vee s$	- - - -

29.

1. $p \longrightarrow \sim q$	Premise
2. $\sim q \longrightarrow \sim r$	Premise
3. $(r \longrightarrow \sim p) \longrightarrow t$	Premise
4. $p \longrightarrow \sim r$	- - - -
5. $r \longrightarrow \sim p$	- - - -
6. t	- - - -

30.

1. $(p \vee q) \longrightarrow (r \vee \sim s)$	Premise
2. $\sim r \wedge s$	Premise
3. $\sim(r \vee \sim s)$	- - - -
4. $\sim(p \vee q)$	- - - -
5. $\sim p \wedge \sim q$	- - - -
6. $\sim p$	- - - -

$\therefore B$

31.[*]

1.	$p\wedge{\sim}p$	Premise
2.	p	- - - -
3.	${\sim}p$	- - - -
4.	${\sim}p\vee q$	- - - -
5.	$p\longrightarrow q$	- - - -
6.	q	- - - -

32.

1.	${\sim}p$	Premise
2.	p	Premise
3.	${\sim}p\vee{\sim}p$	- - - -
4.	$p\longrightarrow{\sim}p$	- - - -
5.	$p\vee p$	- - - -
6.	${\sim}p\longrightarrow p$	- - - -

33.

1.	$p\longrightarrow{\sim}({\sim}p)$	- - - -
2.	$p\longrightarrow p$	- - - -
3.	${\sim}p\vee p$	- - - -
4.	$({\sim}p\vee p)\vee{\sim}q$	- - - -
5.	${\sim}p\vee(p\vee{\sim}q)$	- - - -
6.	${\sim}p\vee({\sim}q\vee p)$	- - - -
7.	$p\longrightarrow({\sim}q\vee p)$	- - - -
8.	$p\longrightarrow(q\longrightarrow p)$	- - - -

34.

1.	${\sim}p\vee p$	Premise
2.	t	Premise
3.	$t\vee{\sim}p$	- - - -
4.	${\sim}p\vee t$	- - - -
5.	$({\sim}p\vee p)\wedge({\sim}p\vee t)$	- - - -
6.	${\sim}p\vee(p\wedge t)$	- - - -

Convert each of Exercises 35–40 into a symbolic proof, and supply the justifications for each step.

35. For me to carry my umbrella it is necessary that it rain. When it rains I always wear my hat. Today I did not wear my hat. Therefore, it must not be raining, and so I am not carrying my umbrella.

36. For me to take my umbrella it is sufficient that it rain. For me to wear my hat it is necessary that it rain. I am wearing my hat today. Therefore, it must be raining, and so I must have taken my umbrella.

37. You cannot be both happy and rich. Therefore, you are either not happy, or not rich. Now you do appear to be happy. Therefore, you must not be rich.

38. If I were smart or good-looking, I would be happy and rich. But I am not rich. So it's true that either I'm not happy or I'm not rich. In other words, I am not both happy and rich. Therefore I am not smart or good-looking. In other words I am not smart and neither am I good-looking. In particular, I am not smart.

39. If interest rates fall, then the stock market will rise. If interest rates do not fall, then housing starts and consumer spending will fall. Now, consumer spending is not falling. So, it's true that housing starts are not falling or consumer spending is not falling, that is, it is false that housing

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[*] This is a proof that, if we assume the contradiction $p\wedge({\sim}p)$ true, then q follows, no matter what q is. In argument form:

$$\frac{p\wedge({\sim}p)}{\therefore\, q}$$

In other words, if you permit a contradiction in an argument, then everything is true! This proof is discussed again in the next section.

starts and consumer spending are both falling. This means that interest rates are falling, so the stock market will rise.

40. If interest rates or the bond market fall, then the stock market will rise. If interest rates do not fall, then housing starts will fall. Housing starts are rising, so interest rates must be falling. Therefore, it is true that interest rates or the bond market are falling, and so the stock market will rise.

Communication and Reasoning Exercises

41. Complete the following sentence. The Modus Tollens rule of inference says that, if both ___ and ___ appear on a list of statements known to be true, then we can add ___ .

42. Complete the following sentence. The Modus Ponens rule of inference says that, if both ___ and ___ appear on a list of statements known to be true, then we can add ___ .

43. Modify Example 5 to produce a proof that uses every type of inference rule we have discussed. (Try replacing q by b and referring to Example 3.)

44. Explain why the following is not a reasonable candidate for a new rule of inference:

$$\frac{A}{\therefore\ A \wedge B}$$

L.6 Arguments and Proofs

Let us now make precise the notions of argument and proof. In Example 5 in the preceding section we saw the following argument.

$$\begin{array}{l} a \longrightarrow q \\ b \longrightarrow q \\ \hline \therefore\ (a \vee b) \longrightarrow q \end{array}$$

Precisely, an **argument** is a list of statements called **premises** followed by a statement called the **conclusion**. (We allow the list of premises to be *empty*, as in Example 3 in the preceding section.) We say that an argument is **valid** if the conjunction of its premises implies its conclusion. In other words, validity means that *if all the premises are true, then so is the conclusion*. Validity of an argument does not guarantee the truth of its premises, so does not guarantee the truth of its conclusion. It only guarantees that the conclusion will be true if the premises are.

Arguments and Validity

An **argument** is a list of statements called **premises** followed by a statement called the **conclusion**.

$$\begin{array}{l} P_1 \\ P_2 \\ \dots \\ P_n \\ \hline \therefore\ C \end{array}$$

If, as above, the premises are P_1 through P_n and the conclusion is C, then the argument is said to be **valid** if the statement

$$(P_1 \wedge P_2 \wedge \ . \ . \ . \ \wedge P_n) \longrightarrow C$$

is a tautology.

Question To show the validity of an argument like

$$\begin{array}{l} a \longrightarrow q \\ b \longrightarrow q \\ \hline \therefore\ (a \vee b) \longrightarrow q \end{array}$$

what we need to do is check that the statement $[(a \longrightarrow q) \wedge (b \longrightarrow q)] \longrightarrow [(a \vee b) \longrightarrow q]$ is a tautology. So to show that an argument is valid we need to construct a truth table, right?

Answer Well, that would work, but there are a couple of problems. First, the truth table can get quite large. The truth table for $[(a \longrightarrow q) \wedge (b \longrightarrow q)] \longrightarrow [(a \vee b) \longrightarrow q]$ has eight rows and nine columns. It gets worse quickly, since each extra variable doubles the number of rows.

Second, checking the validity of an argument mechanically by constructing a truth table is almost completely unenlightening; it gives you no good idea *why* an argument is valid. We'll concentrate on an alternative way of showing that an argument is valid, called a **proof**, that is far more interesting and tells you much more about what is going on in the argument.

Lastly, while truth tables suffice to check the validity of statements in the propositional calculus, they do not work for the predicate calculus we will begin to discuss in the following section. Hence, they do not work for real mathematical arguments. One of our ulterior motives is to show you what mathematicians really do: They create proofs.

Question OK, so what is a proof?

Answer Informally, a proof is a way of convincing you that the conclusion follows from the premises, or that the conclusion must be true if the premises are. Formally, a proof is a list of statements, usually beginning with the premises, in which each statement that is not a premise must be true if the statements preceding it are true. In particular, the truth of the last statement, the conclusion, must follow from the truth of the first statements, the premises. How do we know that each statement follows from the preceding ones? We cite a rule of inference that guarantees that it is so.

Proofs

A **proof** of an argument is a list of statements, each of which is obtained from the preceding statements using one of the rules of inference T1, T2, S, C, or P. The last statement in the proof must be the conclusion of the argument.

As an example, we have the following proof of the argument given above, which we considered in the preceding section:

1. $a \longrightarrow q$	Premise
2. $b \longrightarrow q$	Premise
3. $\sim a \vee q$	1, Switcheroo
4. $\sim b \vee q$	2, Switcheroo
5. $(\sim a \vee q) \wedge (\sim b \vee q)$	3, 4 Conjunction
6. $(\sim a \wedge \sim b) \vee q$	5, Distributive Law
7. $\sim (a \vee b) \vee q$	6, DeMorgan
8. $(a \vee b) \longrightarrow q$	7, Switcheroo

It is reassuring, but not at all obvious, that every valid argument in the propositional calculus has a proof[1]. Equally reassuring is the fact that no *invalid* argument has a proof. The only way to learn to find proofs is by looking at lots of examples and doing lots of practice. In the following examples we'll try to give you some tips as we go along.

Example 1 *Modus Ponens*

Proof the following valid argument (which we saw at the beginning of the preceding section).

$$\begin{array}{l} (p\wedge q)\longrightarrow(r\wedge s) \\ p\wedge q \\ \hline \therefore\ r\wedge s \end{array}$$

Solution

As we mentioned earlier, part of finding a proof is recognizing the form of what you have to work with. In this case, the argument we are given has the following form.

$$\begin{array}{l} A\longrightarrow B \\ A \\ \hline \therefore\ B \end{array}$$

But this is nothing more than Modus Ponens. Thus, we get the following one-step proof.

1. $(p\wedge q)\longrightarrow(r\wedge s)$	Premise
2. $p\wedge q$	Premise
3. $r\wedge s$	1, 2 Modus Ponens

Before we go on… Here is a case in which a proof is much shorter than a truth table. Since there are four variables, the truth table would have 16 rows. Also, the proof shows you that the argument is just an elaborate version of Modus Ponens.

Modus Ponens and Modus Tollens are, perhaps, the most commonly used rules of inference. You should get used to looking for places you can apply them.

[1] This does not apply to the predicate calculus, and in particular it does not apply to the arguments used in real mathematics. The logician Kurt Gödel shook the mathematical world in 1931 when he showed that *there are valid mathematical arguments that have no proofs!* This result is known as Gôdel's Incompleteness Theorem.

Example 2 ***Modus Tollens***

Prove the following valid argument.

$p \wedge q$
$r \longrightarrow (\sim p)$

$\therefore \sim r$

Solution
Looking at the second premise and the conclusion, it looks like we should use Modus Tollens. However, to do that we would need to know that p is true, since Modus Tollens tells us that p and $r \longrightarrow (\sim p)$ together give us $\sim r$. How do we get p by itself? Since we are given $p \wedge q$, we can use Simplification. Thus, we get the following proof.

1. $p \wedge q$	Premise
2. $r \longrightarrow (\sim p)$	Premise
3. p	1, Simplification
4. $\sim r$	2, 3 Modus Tollens

Before we go on… Notice that, when thinking about how to do the proof, we worked backwards from what we wanted. This is an enormously useful technique. Sometimes you need to work forward from what you are given and also backwards from what you want, until the two meet in the middle.

Rule C plays an important role in the next proof.

Example 3 ***Rule C Invoked***

Prove the following valid argument.

$p \longrightarrow a$
$p \longrightarrow b$
p

$\therefore a \wedge b$

Solution We can get both a and b individually using Modus Ponens. To get their conjunction, all we need do is invoke Rule C.

1. $p \longrightarrow a$	Premise
2. $p \longrightarrow b$	Premise
3. p	Premise
4. a	1, 3 Modus Ponens
5. b	2, 3 Modus Ponens
6. $a \wedge b$	4, 5 Rule C

Example 4 ***Strategy***

Prove the following valid argument

$$\begin{array}{l} p \longrightarrow (q \vee r) \\ p \\ \sim r \\ \hline \therefore \quad q \end{array}$$

Solution

Let us think of a strategy for finding a proof. We first examine what we need.

1. We need q. The only place q appears in the premises is in the first, as part of the consequent, $q \vee r$. We can pull out the consequent using the first two premises and Modus Ponens.

2. To get q alone from $q \vee r$ we need to exclude r. But the third premise says that r is false, so we can use Disjunctive Syllogism to complete the proof.

Thus, we get the following proof.

1. $p \longrightarrow (q \vee r)$	Premise
2. p	Premise
3. $\sim r$	Premise
4. $q \vee r$	1, 2 Modus Ponens
5. q	3, 4 Disjunctive Syllogism

Before we go on… Again, notice the back-and-forth thought process. We started to work backwards from q. We noticed that, working forwards, we could get $q \vee r$. Working backwards from q again, we noticed that Disjunctive Syllogism would make the ends meet.

Example 5 ***More Strategy***

Prove the following valid argument

$$\begin{array}{l} (p \vee r) \longrightarrow (s \wedge t) \\ p \\ \hline \therefore \quad t \end{array}$$

Solution Here is a strategy:
1. We need t. This occurs in the consequent of the first premise. We could get this by Modus Ponens if we knew that $p \vee r$ were true.
2. All we know is that p is true from the second premise. But the Addition rule will give us $p \vee r$.
3. Combining (1) and (2) will give us the consequent $s \wedge t$. To get t on its own, we can then use simplification:

1. $(p \vee r) \longrightarrow (s \wedge t)$	Premise
2. p	Premise
3. $p \vee r$	2, Addition
4. $s \wedge t$	1, 3 Modus Ponens
5. t	4, Simplification

Example 6 *Working Backwards*

Prove the following valid argument.

$$a \longrightarrow (b \wedge c)$$
$$\sim b$$
$$\therefore \;\; \sim a$$

Solution
1. We need $\sim a$, which occurs as the *negation* of the antecedent in the first premise. We could get this using Modus Tollens, *if* we knew that $b \wedge c$ was false.
2. Thus, we have a new goal: Show $\sim(b \wedge c)$. What we have is $\sim b$. We can make these look closer by applying DeMorgan's Law to rewrite $\sim(b \wedge c)$ as $(\sim b) \vee (\sim c)$.
3. Now we recognize that we can use Addition to get $(\sim b) \vee (\sim c)$ from $\sim b$.
This gives us the following proof.

1. $a \longrightarrow (b \wedge c)$	Premise
2. $\sim b$	Premise
3. $\sim b \vee \sim c$	2, Addition
4. $\sim(b \wedge c)$	3, DeMorgan
5. $\sim a$	1, 4 Modus Tollens

Before we go on… This time we worked almost entirely backwards. However, we must write the proof forwards. This is a common complaint when students first start to do proofs in symbolic logic or in mathematics. The proof does not follow the train of thought that went into finding it. Often, the thought process is exactly the reverse of what the proof suggests.

Another point to keep in mind is that there are often many different proofs of a given valid argument. Here is another proof of the argument above:

1. $a \longrightarrow (b \wedge c)$	Premise
2. $\sim b$	Premise
3. $\sim(b \wedge c) \longrightarrow (\sim a)$	1, Contrapositive
4. $\sim b \vee \sim c$	2, Addition
5. $\sim(b \wedge c)$	4, DeMorgan
6. $\sim a$	3, 5 Modus Ponens

Constructing a proof is a little like playing a game of chess. You need to pick the right moves, out of all that are possible, to get you to your goal.

Example 7 ***Working Forwards***

Prove the following valid argument.

$$s \longrightarrow r$$
$$(p \vee q) \longrightarrow (\sim r)$$
$$(\sim s) \longrightarrow (\sim q \longrightarrow r)$$
$$p$$
$$\therefore \quad q$$

Solution

1. We need to end up with q alone. Now, q appears in both the second and third premises, and it is not clear at this point which to focus on. Perhaps we should look at what we have and see what we can get working forwards.

2. The last premise, p, is the simplest and so should be the easiest to do something with. It looks like we should be able to combine it with the second using Modus Ponens. In fact, we can use Addition to get $p \vee q$ from p and then use Modus Ponens to get $\sim r$ from the second premise.

3. Now we can combine $\sim r$ with the first premise to get $\sim s$ by Modus Tollens.

4. Things are moving along nicely. We can combine $\sim s$ with the third premise to get $\sim q \longrightarrow r$, using Modus Ponens.

5. Remembering that we still have $\sim r$, we can use Modus Tollens now to get q, which is what we wanted!

This gives us the following proof.

1. $s \longrightarrow r$	Premise
2. $(p \vee q) \longrightarrow \sim r$	Premise
3. $\sim s \longrightarrow (\sim q \longrightarrow r)$	Premise
4. p	Premise
5. $p \vee q$	4, Addition
6. $\sim r$	4, 2 Modus Ponens
7. $\sim s$	1, 6 Modus Tollens
8. $\sim q \longrightarrow r$	3, 7 Modus Ponens
9. q	6, 8 Modus Tollens

As the preceding example shows, not all proofs are easy to find. Sometimes you have to fiddle a bit to get one. If the line of argument you're trying doesn't pan out, experiment with something else. Here are some things to try that often help:

1. Replace an implication with its contrapositive.
2. Use DeMorgan's Law to rewrite a negation of a conjunction or disjunction.
3. Use any of the other tautological equivalences to rewrite a statement.
4. Take a coffee break to clear your head.

Above all, be persistent. After you take that coffee break, get back to work!

The next argument basically asserts that if we permit a single contradiction in an argument, then anything is possible. (A proof appeared in the exercise set at the end of the last section, but it is interesting enough to warrant further inspection.)

Example 8 ***Slippery Argument***

Prove and comment on the argument

$$\frac{p\wedge(\sim p)}{\therefore\ q}$$

Solution Notice that the premise $p\wedge(\sim p)$ is a contradiction. If you write out the truth table for $[p\wedge(\sim p)]\longrightarrow q$, you can see why this is a valid argument. But let us try to come up with a proof.
1. The easiest way to begin is to use simplification to "break up" the statement $p\wedge(\sim p)$ into the two separate statements p and $\sim p$.
2. Notice that q does not occur anywhere among the premises. One way we can get it out of thin air is to use Addition, so let's try adding it to p to get $p\vee q$.
3. Now the statement $\sim p$ that we got from (1) tells us that p is false, so that this, combined with $p\vee q$, gives us q, by Disjunctive Syllogism.

1. $p\wedge(\sim p)$	Premise
2. p	1, Simplification
3. $\sim p$	1, Simplification
4. $p\vee q$	2, Addition
5. q	3, 4 Disjunctive Syllogism

Before we go on... Notice that this proof is one step shorter than the one you saw in the exercises. This illustrates again the fact that there may be several different proofs of the same argument. The simplest proof (which often means the shortest one) is considered the most **elegant**.

We were also asked to comment on the argument. Look at the premise: it is making the contradictory claim that both p and $\sim p$ are true. What the argument says is that, once you allow a contradiction into an argument, *anything* is true. Notice that the conclusion, q, has nothing to do with the premise.

In general, a false antecedent implies any consequent, true or not. Here is an example which illustrates how to make this claim precise: "If 0 = 1, then I am the King of England" is a true statement no matter who says it. To write this as an argument, let us take p to be the statement "0 = 1" and q to be the statement "I am the King of England." Then our statement is equivalent to the argument

$$\begin{array}{l} p \\ \hline \therefore\ q \end{array}$$

But that's not all! As mathematicians, we happen to know that the statement p is false, so that $\sim p$ is a true statement. Thus we are really arguing that

$$\begin{array}{l} p \\ \sim p \\ \hline \therefore\ q \end{array}$$

By Rule C, this is really the same as

$$\begin{array}{l} p \wedge (\sim p) \\ \hline \therefore\ q \end{array}$$

which we know is valid.*

Example 9 ***An Invalid Argument***

Show that the following argument is not valid.

$$\begin{array}{l} p \rightarrow q \\ q \\ \hline \therefore\ p \end{array}$$

Solution

Proofs can only be used to show that an argument is valid. If you try to prove this argument, you'll get nowhere. It looks sort of like Modus Ponens, except that it's backwards. It looks sort of like Modus Tollens, but the negations are missing. It just looks wrong, and it is.

To show that an argument is invalid we need to find a **counterexample**. This is an assignment of truth values to the variables that makes the premises true but the conclusion false, thus showing that the conclusion does not follow from the premises.

* When mathematician and philosopher Bertrand Russell claimed to a colleague that, from a false statement he could prove anything, he was challenged as follows:
"Prove that, if 0 = 1, then you are the King of England."
To this he replied, "Simple. If 0 = 1, then, adding 1 to each side, 1 = 2. Since the King and I are two, it follows that the King and I are one, and I am the King of England!"

In this case, for the conclusion to be false we need p to be F. For the premises to be true we certainly need q to be T. All we need to do is check that both premises are then true: The first premise is $p \longrightarrow q$, which is true when p is F and q is T. This is our counterexample.

A counterexample is more vivid if we illustrate it with concrete statements. For p, which must be F, let us take the statement "0 = 1." For q, which must be T, let us take the statement "The earth is round." Our argument then has the following, patently ridiculous, form:

If 0 = 1, then the earth is round.	True
The earth is round.	True
Therefore, 0 = 1.	False.

Before we go on… This particular argument is a common **fallacy** know as the **fallacy of affirming the consequent**. It is also know as the **fallacy of the converse** since it seems to come from a confusion of $p \longrightarrow q$ with its converse $q \longrightarrow p$. (If the first premise were $q \longrightarrow p$, then the argument would be an example of the valid Modus Ponens.)

Example 10 *Valid or Invalid?*

Decide whether the following arguments are valid or not. If an argument is valid, give a proof; if it is not, give a counterexample:

(a) Every irreversible chemical reaction dissipates heat. Therefore, if a chemical reaction is reversible, it dissipates no heat.
(b) The moon is made of blue cheese. If the moon is made of blue cheese, it must be gorgonzola. Therefore, the moon is made of gorgonzola.

Solution

(a) To analyze any argument given in words we first translate it into symbolic form. The first sentence discusses two aspects of a chemical reaction, whether it is irreversible and whether is dissipates heat. Let p: "this chemical reaction is irreversible" and q: "this chemical reaction dissipates heat." Then the first statement is $p \longrightarrow q$. The conclusion is the implication $(\sim p) \longrightarrow (\sim q)$. Therefore, the argument is, in symbolic form, the following.

$$\begin{array}{l} p \longrightarrow q \\ \hline \therefore\ (\sim p) \longrightarrow (\sim q) \end{array}$$

This argument may remind us of the Contrapositive rule. However, the conclusion is backwards, since the contrapositive of $p \longrightarrow q$ is $(\sim q) \longrightarrow (\sim p)$. This suggests that the argument is invalid, so let us try to find a counterexample.

Our counterexample should make the premise true but the conclusion false. The only way to make an implication false is for the antecedent to be true and the consequent false, so we must have $\sim p$ true and $\sim q$ false. In other words, p should be false and q true. Fortunately, this makes the premise true, so we have found our counterexample. In terms of the meanings we assigned to p and q, a counterexample would be given by a chemical reaction that was reversible but dissipated heat.

If we want to express the counterexample in a more immediately understandable way, we could let p: "this creature is a horse" and q: "this creature is a mammal." A counterexample would then be given by any creature who was not a horse but was a mammal, say a dog.

(b) Let us take p: "The moon is made of blue cheese." q: "The moon is made of gorgonzola." Then the argument takes the following form.

$$\begin{array}{l} p \longrightarrow q \\ p \\ \hline \therefore\ q \end{array}$$

We see that this is just an application of modus ponens, so the argument is valid (even though the conclusion is false!)

Before we go on… In (a), the conclusion of the original argument is in fact true: Reversible chemical reactions dissipate no heat. However, the *argument* used to arrive at this conclusion was invalid. In part (b), a premise and the conclusion are false (one of the premises happens to be true. Do you see which one?) and yet the argument used to arrive at the false conclusion is valid. This points up the difference between *truth* and *validity*. The validity of an argument depends solely on its form. Validity assures you that *if* the premises happen to be true for some interpretation of the variables *then* the conclusion will also be true. Validity tells you nothing about whether the premises are true, nor does it tell you what happens when a premise is false. Likewise, if an argument is invalid it does not necessarily mean that the conclusion is false, just that its truth does not follow from the truth of the premises.

The following example is reminiscent of the kind of question that often appears in aptitude test such as the LSAT.

Example 11 *Logical Reasoning*

Decide whether the following argument is valid or not. If it is, give a proof; if it is not, give a counterexample:

> When Alexis attends math class, her friends Guppy and Desmorelda also attend. Since Desmorelda is in love with Luke, Luke's attendance at class is a sufficient condition for her to attend as well. On the other hand, for Desmorelda to attend class it is necessary that Alexis also be there (as she needs someone to talk to during the boring portions of the class). Therefore, Luke won't attend class unless Guppy also attends.

Solution

The only way to make heads or tails out of all this is to translate into symbols. To make life easier, let us choose the first letter of a person's name to symbolize their attendance at math class. Thus, a: "Alexis attends math class," and so on. Our argument now has the following form.

$a \longrightarrow (g \wedge d)$
$l \longrightarrow d$
$d \longrightarrow a$
$\therefore\ l \longrightarrow g$

Let us try to prove this. Looking at the premises we can see the following string of implications:

$$l \longrightarrow d \longrightarrow a \longrightarrow (g \wedge d).$$

Thus, the Transitivity rule will give us $l \longrightarrow (g \wedge d)$ pretty easily. It does appear that l implies g, so the argument is valid. To get from $l \longrightarrow (g \wedge d)$ to $l \longrightarrow g$ we would like to use Simplification, but we can't do it directly. The way around this is to use Switcheroo, a useful tool for manipulating implications. Here's the proof:

1. $a \longrightarrow (g \wedge d)$	Premise
2. $l \longrightarrow d$	Premise
3. $d \longrightarrow a$	Premise
4. $l \longrightarrow a$	2, 3 Transitivity
5. $l \longrightarrow (g \wedge d)$	1, 4 Transitivity
6. $\sim l \vee (g \wedge d)$	5, Switcheroo
7. $(\sim l \vee g) \wedge (\sim l \vee d)$	6, Distributive Law
8. $(\sim l \vee g)$	7, Simplification
9. $l \longrightarrow g$	8, Switcheroo

L.6 Exercises

Prove each of the valid arguments in Exercises 1–22.

1. $(p \vee r) \longrightarrow \sim q$
$p \vee r$
$\therefore\ \sim q$

2. $\sim p \longrightarrow (q \longrightarrow s)$
$\sim p$
$\therefore\ q \longrightarrow s$

3. $\sim p \longrightarrow (r \longrightarrow \sim t)$
$\sim (r \longrightarrow \sim t)$
$\therefore\ p$

4. $(\sim p \vee r) \longrightarrow \sim q$
q
$\therefore\ \sim (\sim p \vee r)$

5. $\sim p \longrightarrow (q \wedge r)$
$\sim p \wedge s$
$\therefore\ r$

6. $p \longrightarrow (r \wedge s)$
$\sim r$
$\therefore\ \sim p$

7. $p \longrightarrow q$

8. $p \longrightarrow (q \wedge r)$

$\sim(q \vee r)$

$\therefore \sim p$

$r \longrightarrow s$
p

$\therefore s$

9. $(p \vee r) \longrightarrow q$
$s \longrightarrow p$
s

$\therefore q$

10. $(p \vee q) \longrightarrow r$
$\sim r$
$t \longrightarrow q$

$\therefore \sim t$

11. $(p \vee \sim q) \longrightarrow r$
$s \longrightarrow (t \wedge u)$
$s \wedge p$

$\therefore r \wedge u$

12. $(p \wedge \sim q) \longrightarrow r$
$s \longrightarrow p$
$q \longrightarrow \sim u$
$u \wedge s$

$\therefore r$

13. $(p \longrightarrow q) \longrightarrow r$
$\sim(q \vee r)$

$\therefore p$

14. $(p \longrightarrow q) \longrightarrow (p \longrightarrow r)$
$q \wedge p$

$\therefore r$

15. $p \longrightarrow (q \longrightarrow r)$
q

$\therefore p \longrightarrow r$

16. $p \longrightarrow (q \longrightarrow r)$
$\sim r$

$\therefore p \longrightarrow (\sim q)$

17. -

$\therefore (p \wedge q) \longrightarrow (p \vee q)$

18. -

$\therefore p \longrightarrow \sim(q \wedge \sim p)$

19. -

$\therefore \sim(p \wedge \sim p)$

20. -

$\therefore (p \wedge \sim p) \longrightarrow q$

21. -

$\therefore (p \longrightarrow \sim p) \longrightarrow \sim p$

22. -

$\therefore \sim p \longrightarrow (p \longrightarrow \sim p)$

Give counterexamples to each of the fallacies in Exercises 23–26 by finding truth values for the variables making the premises true and the conclusion false.

23. $p \longrightarrow q$

24. $p \longrightarrow q$

$\sim p$

$\therefore \sim q$

$p \longrightarrow r$

$\therefore q \longrightarrow r$

25. $(p \wedge q) \longrightarrow r$
p

$\therefore r$

26. $(p \wedge q) \longrightarrow r$
$\sim r$

$\therefore \sim p$

Decide whether each of the arguments in Exercises 27–30 is a valid argument. If it is valid, give a proof. If it is invalid, give a counterexample. In any case, supply verbal statements that make all the premises true. If the argument is invalid make sure that they also make the conclusion false.

27. $p \longrightarrow (q \vee r)$
$\sim q$

$\therefore \sim p$

28. $p \longrightarrow (q \wedge r)$
$\sim q$

$\therefore \sim p$

29. $p \longrightarrow r$
$\sim q \longrightarrow \sim r$

$\therefore p \longrightarrow q$

30. $\sim (p \wedge q)$
p

$\therefore q$

In Exercises 31–44, convert each argument into symbolic form and then decide whether or not it is valid. If invalid, give a counterexample by supplying truth values for the various atomic statements.

31. If I were a cow, then I would eat grass. However, I am not a cow, so I don't eat grass.

32. If we were meant to fly, we would have wings. Since we do indeed fly, and since we would not fly unless we were meant to fly, it follows that we have wings.

33. If factories in the west pollute the air, then acid rain will lead to damage in the east. Indeed, acid rain is leading to considerable damage in the east, so the factories in the west must be polluting the air.

34. If factories in the west pollute the air, then acid rain will lead to damage in the east. The factories in the west do not pollute the air, so acid rain will not be a problem in the east.

35. If interest rates go down, inflation will rise. If interest rates go down, the stock market will also rise. Therefore, if inflation rises so will the stock market.

36. If interest rates go down, inflation will rise. If inflation rises, so will the bond market. Therefore, if interest rates go down, the bond market will rise.

37. If mortgage rates go down, or prices fall, then housing starts will rise. Mortgage rates are falling, therefore housing starts will rise.

38. If mortgage rates go down or prices fall, then housing starts will rise. Prices are rising, therefore housing starts will not rise.

39. When it rains on the great plains of the Nile, Sagittarius is in the shadow of Jupiter, and as you know, it always rains if Mercury is ascending. On the other hand, Sagittarius falls in Jupiter's shadow only when either the moon is full or Mercury is ascending. I have noticed a disturbing pattern to the weather predictions of Desmorelda, so-called "Mistress of the Zodiac." It seems that she always predicts that it will rain on the plains of the Nile when Sagittarius ventures into the Shadow of Jupiter and the moon is full. Should I replace her as royal meteorologist?

40. My stereo system is faulty: there is no sound coming out of the left speaker. Switching the speaker leads will not bring sound to the left speaker if and only if the left speaker is faulty. If switching the speaker leads causes the right speaker to fail, then there is a fault with either the amplifier or the CD player. Switching the leads from the CD player has no effect if and only if there is no problem with the CD player. I discovered the following: switching the leads to the speakers resulted in both channels failing, and switching the leads from the CD player reversed the problem from the left to the right speaker. Therefore replacing the CD player and the left speaker will solve the problem.

41. You are chairing an important committee at the UN, and are faced with the following predicament. Upper Volta refuses to sign your new peace accord unless both Costa Rica and Bosnia sign as well. Since Bosnia has a lucrative trade agreement with Iraq, Iraq's signing the peace accord is a sufficient condition for Bosnia to sign the accord. On the other hand, Bosnia, fearful of Upper Volta's recent military buildup, refuses to sign the accord unless Upper Volta also signs. You conclude that Iraq won't sign unless Costa Rica also signs.

42. *Continuation of Exercise 41* Just as you are about to arrange details for the signing ceremony, Bosnia's representative informs your office that due to a recent scandal involving highly placed Costa Rican officials, the Bosnians refuse to sign any accord with Costa Rica. In retaliation, the Costa Rican government announces a hard-line position: they will not sign the accord unless Iraq also signs. After thinking things over, you come to the depressing conclusion that it will be impossible to have *anyone* sign the accord. [To make your analysis less time consuming, you may assume the following tautology (which we proved earlier): $(p \longrightarrow \sim p) \longrightarrow \sim p$. You may also feel free to quote the result of Exercise 42.]

43. *If no violets are red and some roses are blue,*
Then nobody loves you; that is certainly true.
But Lucille does love you,
And Susan and Joy.
Yet some roses are deep blue
Or you're just a boy.

You assure me you're grown up
And that I believe;
The brew in your cup
Is so strong, I perceive.
Now this surely implies
Violets red as your eyes.

44. (In Memory of Dr. Seuss)
If you glue, then Wu glues
And Golly glues too.
If Golly glues, Molly glues
And Solly, you too!
But Holly, not Solly glues
With green gooey glue.
Thus Dolly or Holly glues
But not you nor Wu!

Communication and Reasoning Exercises

45. Can anything be proved without any premises being given? Explain.

46. Your friend James claims that every argument can be proved in more than one way. Is he correct? Explain.

47. Your friend Jane claims to have come up with a proof of the following argument

$$\begin{array}{l} p \\ \hline \therefore\ q \end{array}$$

Comment on her claim.

48. Your other friend Janet claims that the simplification rule can be deduced from the addition rule, and is therefore not necessary. Comment on her claim.

L.7 Predicate Calculus

One of the most famous arguments in logic goes as follows.

All men are mortal.
Socrates is a man.
Therefore, Socrates is mortal.

There is really no good way to express this argument using propositional calculus. The problem is, how do we express symbolically the statement "All men are mortal?" And how do we do so in such a way that we can relate it to the statement "Socrates is a man?" We need to go beyond the propositional calculus to the **predicate calculus**, which allows us to manipulate statements about all or some things.

We begin by rewording "All men are mortal" in something closer to a statement of propositional calculus:

"For all x, if x is a man then x is mortal."

The sentence "x is a man" is not a statement in the sense we've discussed so far, since it involve an unknown thing x and we can't assign a truth value without knowing what x we're talking about. This sentence can be broken down into its subject, x, and a **predicate**, "is a man." We say that the sentence is a **statement form**, since it becomes a statement once we fill in x. Here is how we shall write it symbolically: The subject is already represented by the symbol x, called a **term** here, and we use the symbol P for the predicate "is a man." We then write Px for the statement form. (It is traditional to write the predicate before the term; this is related to the convention of writing function names before variables in other parts of mathematics.)

Similarly, if we use Q to represent the predicate "is mortal" then Qx stands for "x is mortal." We can then write the statement "If x is a man then x is mortal" as $Px \longrightarrow Qx$.

To write our whole statement, "For all x, if x is a man then x is mortal" symbolically, we need symbols for "For all x." We use the symbol "$\forall$" to stand for the words "for all" or "for every." Thus, we can write our complete statement as

$$\forall x[Px \longrightarrow Qx].$$

The symbol "$\forall$" is called a **quantifier** because it describes the number of things we are talking about: all of them[1]. Specifically, it is the **universal quantifier** because it makes a claim that something happens universally.

[1] There were arguments in the late 19th century as to the "existential import" of the quantifier $\forall$. The question was, if you say "all men are mortals," are you at the same time saying that there are in fact some men? Eventually it was decided that the most useful choice would be to say that, no, you are making no such claim. For example, the statement "all moon men are green" would actually be considered a true statement, since there is no example of a moon man who is not green to serve as a counterexample. We say then that the statement is **vacuously** true. This is related to the meaning of implication: If "M" = "is a moon man" and "G" = "is green," then our statement is $\forall x[Mx \longrightarrow Gx]$. Since, for every x, Mx is false, the statement $Mx \longrightarrow Gx$ is always true. Thus, $\forall x[Mx \longrightarrow Gx]$ is a true statement.

Question What are those square brackets doing around $Px \longrightarrow Qx$?

Answer They define what is called the **scope** of the quantifier $\forall x$. That is, they surround what it is we are claiming is true for all x.

Example 1 ***A Syllogism***

Express the argument above, about Socrates, in symbolic form.

Solution
We've done most of the work. The statement "Socrates is mortal" uses the predicate P to make a statement about a particular man, Socrates. Let us use the letter s to stand for Socrates. (We shall always use small letters for terms and big letters for predicates.) Then Ps is the statement "Socrates is a man." Similarly, Qs is the statement "Socrates is mortal." The argument now looks like this:

$$\begin{array}{l} \forall x[Px \longrightarrow Qx] \\ Ps \\ \hline \therefore \quad Qs \end{array}$$

Before we go on... This is an example of a classical form of argument known as a **syllogism**. Roughly, a syllogism is an argument in the predicate calculus with two premises sharing a common term (in this case, the predicate P, "is a man"). In the following section we shall see how to prove that such an argument is valid.

Mathematics is expressed in the language of the predicate calculus. Here's an example of a mathematical statement expressed symbolically.

Example 2 ***A Mathematical Statement***

Write the following statement symbolically: "If a number is greater than 1 then it is greater than 0."

Solution
Since this is a statement meant to be true of every number, we need to rephrase it to make the universal quantifier obvious: "For all x, if x is a number and x is greater than 1, then x is greater than 0." Let us write N for the predicate "is a number" and use the standard notation ">" for "is greater than." Our statement is then:

$$\forall x[(Nx \wedge (x>1)) \longrightarrow (x>0)].$$

Notice that we put the phrases "$x>1$" and "$x>0$" in parentheses to make the meaning clearer.

Before we go on… Mathematicians being as lazy as they are, they often don't bother to specify that x is a number, regarding it as understood if they write something like $x>1$. So, a mathematician might write

$$\forall x[(x>1)\longrightarrow(x>0)].$$

In fact, we're being a little sloppy even in our original solution. We can run into logical paradoxes if we allow ourselves to let x range over "everything" possible. We should, instead, agree beforehand what **universe** of things the quantifier "$\forall x$" really refers to. In this example, we might agree that the universe is the set of all real numbers. There is no need to allow x to also refer to, say, an elephant.

Example 3 ***Another Mathematical Statement***

Now write the following statement symbolically: "Given any two numbers, the square of their sum is never negative."

Solution

Again, we are making a statement about all numbers, in fact, about all pairs of numbers. We can rephrase the statement as follows: "For all x and all y, if x and y are numbers then the square of their sum is not negative." Since "the square of their sum" is $(x+y)^2$, our statement can be written like this:

$$\forall x[\forall y[(Nx\wedge Ny)\longrightarrow\sim\{(x+y)^2<0\}]].$$

Rather than write $\forall x[\forall y[\ .\ .\ .\ ]]$ we often write

$$\forall x,y[(Nx\wedge Ny)\longrightarrow\sim\{(x+y)^2<0\}].$$

If we prefer not to have the negation, we could write

$$\forall x,y[(Nx\wedge Ny)\longrightarrow\{(x+y)^2\geq 0\}].$$

Once more, we could be lazy and write

$$\forall x,y[(x+y)^2\geq 0].$$

There are times when, rather than claim that something is true about *all* things, we only want to claim that it is true about *at least one* thing. For example, we might want to make the claim that "some politicians are honest," but we would probably not want to claim this universally. A way that mathematicians often phrase this is "*there exists* a politician who is honest." Our abbreviation for "there exists" is "$\exists$", which is called the **existential quantifier** because it claims the existence of something. If we use P for the predicate "is a politician" and H for the predicate "is honest," we can write "some politicians are honest" as

$\exists x[Px \wedge Hx]$.

Example 4 ***Mixing Quantifiers***

Write the following statement symbolically: "Every person is better off than someone else."

Solution

Let's rephrase this statement to get closer to our logical symbolism: "For every person x, there is a person y such that x is better off than y." Now we can see two quantifiers, a universal one and an existential one. We also need some predicates, including P for "is a person," and one more predicate $B(x,y)$ to stand for "x is better off than y." This is a new kind of predicate, taking two terms. Since it relates its two terms, such a **2-place predicate** is often called a **relation**.

Now we can write our statement symbolically, using lots of brackets to make the meaning clear:

$$\forall x[Px \longrightarrow (\exists y[Py \wedge B(x,y)])].$$

Many mathematical definitions are made in terms of quantifiers. An interesting example is the notion of "divisible by." To say that a number x is divisible by 2, for example, is to say that x is 2 times some integer, or that there exists some integer n such that $x = 2n$. Generalizing a bit and writing symbolically, we can make the following definition.

Divisible By

If x and y are integers, we say that ***x* is divisible by *y*** if

$$\exists n[In \wedge (x=yn)].$$

Here, I is the predicate "is an integer."

Note: If we agree to restrict our variable to the universe of integers, we don't have to use the predicate I and we get the following simpler version:

$$\exists n[x = yn].$$

Example 5 ***Divisibility***

Write the following statement symbolically: "If a number is divisible by 6 then it is divisible by 3 and by 2."

Solution

To simplify the notation, let us agree that our universe is the set of integers and all variables are therefore integers.

First notice that our statement is a universal one about integers: "For every integer n, if n is divisible by 6 then it is divisible by 3 and by 2." Now, when we want to write "n is divisible by 6" we have to watch out for the fact that we've already used the variable n and can't reuse it as in the definition of "divisible by" above. What we do is pick another letter, say m, and write $\exists m[n = 6m]$ for "n is divisible by 6." In general, the variable being quantified (the one immediately to the right of the quantifier) is a **dummy variable**; its name does not matter, as long as the same name is used consistently throughout the statement.

Doing the same with divisibility by 3 and 2, we can write our statement as follows:

$$\forall n[(\exists m[n = 6m]) \longrightarrow (\exists m[n = 3m]) \wedge (\exists m[n = 2m])].$$

Before we go on… We used m as the dummy variable in all three parts of this statement, but it stands for different numbers in each case. If we wanted to emphasize that the three numbers are different, we could use three different letters, like this:

$$\forall n[(\exists m[n = 6m]) \longrightarrow (\exists i[n = 3i]) \wedge (\exists j[n = 2j])].$$

This leads to an interesting question: For a given n, how are i and j related to m? Pondering this question leads to the mathematical proof of the statement.

In this last example we've started to see how mathematics can be translated into symbolic form. It was the hope of mathematicians at the end of the nineteenth century that all of mathematics could be made purely formal and symbolic in this way. The most serious attempt to do this was in Whitehead and Russell's *Principia Mathematica* (1910), which translated a large part of mathematics into symbolic language. The hope then was that there could be developed a purely formal procedure for checking the truth of mathematical statements and producing proofs. This project was cut short by Gödel's incompleteness theorem (1931), which effectively showed the impossibility of any such procedure. Nonetheless, mathematicians still feel that anything that they do should be expressible in symbolic logic, and the language that they actually use in writing down their work is a somewhat less formal version of the predicate calculus.

L.7 Exercises

Translate each of the sentences in Exercises 1–26 into a statement in the predicate calculus. (Underlined letters are to be used for the relevant predicates or terms where appropriate.)

1. Every good girl deserves fruit.

2. Good boys deserve fruit always.

3. All cows eat grass.

4. No cows eat grass.

5. Some cows eat grass.

6. Some birds are fishes.

7. Some cows are not birds and some are.

8. Some cows are birds but no cows are fishes.

9. Although some city drivers are insane, Dorothy is a very sane city driver.

10. Even though all mathematicians are nerds, Waner and Costenoble are not nerds.

11. If one or more lives are lost, then all lives are lost.

12. If every creature evolved from lower forms, then you and I did as well.

13. Some numbers are larger than two; others are not.

14. Every number smaller than 6 is also smaller than 600.

In Exercises 15–26, you can use the convention that the letters i through n represent positive integers.

15. 12 is divisible by 6.

16. 13 is not divisible by 6.

17. For any positive integer m, if 12 is divisible by m, then so is 24.

18. If 13 is not divisible by m, then neither is 17.

19. 15 is divisible by some positive integer.

20. 15 is divisible by a positive integer other than 15 or 1.

21. 17 is prime (that is, not divisible by any positive integer except itself and 1).

22. 15 is not prime. (See (21).)

23. There is no smallest positive real number. (Use the convention that the letters x through z represent real numbers.)

24. There is no largest positive integer.

25. If 1 has property P, and if $(n+1)$ has property P whenever n does, then every positive integer has property P. (This statement is called the *Principle of Mathematical Induction*.)

26. If 2 has property P, and if $(n+2)$ has property P whenever n does, then every even positive integer has property P.

Translate the statements in Exercises 27–34 into words.

27. $\forall x[Rx \longrightarrow Sx]$; R = "is a raindrop," S = "makes a splash."

28. $\forall y[Cy \longrightarrow My]$; C = "is a cowboy," M = "is macho."

29. $\exists z[Dz \wedge Wz]$; D = "is a dog," W = "whimpers."

30. $\exists z[Dz \wedge \sim Wz]$; D = "is a dog," W = "whimpers."

31. $\forall x[Dx \longrightarrow \sim Wx]$; D = "is a dog," W = "whimpers."

32. $\sim\forall x[Dx \longrightarrow Wx]$; D = "is a dog," W = "whimpers."

33. $\exists z,y[Cz \wedge Cy \wedge Wz \wedge \sim Wy]$; C = "is a cat," W = "whimpers"

34. $\forall x[Px \longrightarrow \exists y[Py \wedge L(x,y)]]$, P = "is a person," $L(x,y)$ = "y is older than x."

Communication and Reasoning Exercises

35. The claim that every athlete drinks ThirstPro is false. In other words, no athletes drink ThirstPro, right?

36. Give one advantage that predicate calculus has over propositional calculus.

37. Your friend claims that the quantifiers $\forall$ and $\exists$ are insufficient for her purposes; she requires new quantifiers to express the phrases "for some" and "there does not exist". How would you respond?

38. Consider a new quantifier, "∇" meaning "for no" (as in "for no x can x be larger than itself"). Express ∇ in terms of the quantifiers you already have.

L.8 Arguments and Proofs in the Predicate Calculus

In Example 1 in the preceding section we discussed the classical syllogism that goes:

All men are mortal.
Socrates is a man.
Therefore, Socrates is mortal.

We represent this argument symbolically as follows:

$$\begin{array}{l} \forall x[Px \longrightarrow Qx] \\ Ps \\ \hline \therefore \;\; Qs \end{array}$$

Here, again, P is the predicate "is a man" while Q is the predicate "is mortal." What we now need to discuss are the rules of inference that allow us to prove that this and other arguments in the predicate calculus are valid.

Let's first consider the statement $\forall x[Px \longrightarrow Qx]$, or, "For all x, if x is a man then x is mortal." From this general statement about men we should be able to *specialize* to any particular man, like Socrates. What is true of all things should be true of any one of them. This gives us the following rule of inference.

Specialization (or Substitution or Dropping a Universal Quantifier)

If s stands for a particular thing, then the following is a tautology for any predicate P:

$$\forall x[Px] \longrightarrow Ps.$$

Stated as a rule of inference, this is:

$$\begin{array}{l} \forall x[Px] \\ \hline \therefore \; Ps \end{array}$$

Example 1 ***Proving a Syllogism***

Prove that the argument discussed above is valid.

Solution

The proof is a simple application of Specialization and Modus Ponens.

1. $\forall x[Px \longrightarrow Qx]$	Premise
2. Ps	Premise
3. $Ps \longrightarrow Qs$	1, Specialization

4. Qs 2, 3 Modus Ponens

Before we go on… How did we know to specialize to $Ps \longrightarrow Qs$ and not, say to $Pc \longrightarrow Qc$ where c stands for Costenoble? Either is a legitimate deduction, but the conclusion and one of the premises of our original argument mention Socrates, not Costenoble, so the specialization to Socrates is probably more useful.

Here is another argument.

0 < 1.
For any x, if $x \geq 2$ then $x \geq 1$.
Therefore, there exists a number less than 2.

We can express this argument symbolically as follows (using the convention that our universe is the set of real numbers).

$$0 < 1$$
$$\forall x[(x \geq 2) \longrightarrow (x \geq 1)]$$
$$\therefore \exists x[x < 2]$$

Here is an informal proof of the argument: Substitute 0 for x in the second premise to get the statement $(0 \geq 2) \longrightarrow (0 \geq 1)$. Using the first premise and Modus Tollens we conclude that $0 < 2$. But now we have an example of a number less than 2, so certainly we have shown that there exists such a number, and we can conclude that $\exists x[x<2]$.

The last step in that informal argument is the next rule of inference we need to write down.

Existential Generalization (or Adding an Existential Quantifier)

If s stands for a particular thing, then the following is a tautology for any predicate P:

$$Ps \longrightarrow \exists x[Px].$$

Stated as a rule of inference, this is:

$$Ps$$
$$\therefore \exists x[Px]$$

Example 2 ***Proving a Generalization***

Prove that the argument discussed above is valid.

Solution

We can now turn our informal proof into a formal one.

1. $0 < 1$	Premise
2. $\forall x[(x\geq 2) \longrightarrow (x\geq 1)]$	Premise
3. $(0\geq 2) \longrightarrow (0\geq 1)$	2, Specialization
4. $0 < 2$	1, 3 Modus Tollens
5. $\exists x[x<2]$	4, Existential Generalization

The final rules of inference we'll discuss have to do with negations of quantified statements. Suppose that you have a universal statement $\forall x[Px]$, which claims that, for all x, Px is true. What does it mean to say that its negation $\sim(\forall x[Px])$ is true, instead? If it is not true that Px is always true, then there must be some example where Px is false. In other words, there must exist an x for which Px is false. This gives us the following tautology.

Negation of a Universal Quantifier

For any predicate P the following is a tautology:

$$\sim(\forall x[Px]) \leftrightarrow \exists x[\sim Px].$$

Mechanically, this says that we can pass a negation past "$\forall$" if we then change the quantifier to "$\exists$".

Example 3 ***Negating a Universal Quantifier***

What is the negation of "Every swan is white?"

Solution

Let S be the predicate "is a swan" and W be the predicate "is white." Our statement, before negation, is $\forall x[Sx\longrightarrow Wx]$. After negation it becomes the following.

$$\sim\forall x[Sx\longrightarrow Wx] \equiv \exists x[\sim(Sx\longrightarrow Wx)].$$

We can simplify this further if we use Switcheroo to rewrite the implication inside.

$$\begin{aligned}\sim\forall x[Sx\longrightarrow Wx] &\equiv \exists x[\sim(Sx\longrightarrow Wx)]\\ &\equiv \exists x[\sim((\sim Sx)\vee Wx)]\\ &\equiv \exists x[(Sx)\wedge(\sim Wx)].\end{aligned}$$

Translating back into words this is: "There exists a swan that is not white." A moment's reflection shows that this is, indeed, the exact negation of the statement that all swans are white.

Before we go on… Note that the negation is *not* "No swans are white." To disprove a universal statement it suffices to produce *one* **counterexample**. In this case, it suffices to say that there is (at least) one swan that is not white. To say that no swans are white would be to make a much stronger statement than to say that not all swans are white.

For the existential quantifier we have a very similar rule.

Negation of an Existential Quantifier

For any predicate P the following is a tautology:

$$\sim(\exists x[Px]) \leftrightarrow \forall x[\sim Px].$$

Example 4 ***Negating an Existential Quantifier***

What is the negation of "There exists a man who is immortal?"

Solution

Let P be the predicate "is a man" and let R be the predicate "is immortal." We are looking for the negation of the statement $\exists x[Px \wedge Rx]$. The negation is:

$$\begin{aligned} \sim(\exists x[Px \wedge Rx]) &\equiv \forall x[\sim(Px \wedge Rx)] \\ &\equiv \forall x[(\sim Px) \vee (\sim Rx)] \\ &\equiv \forall x[Px \longrightarrow (\sim Rx)]. \end{aligned}$$

In words, the negation of "There exists a man who is immortal" is "All men are not immortal" or, "All men are mortal."

Before we go on… The reasoning behind the negation of $\exists x[Px]$ being $\forall x[\sim Px]$ is this: If there is no example of an x for which Px is true, then it must be false for all x. For example, if there is no man who is immortal, then all men must be mortal.

Here's an interesting logical equivalence:

$$\exists x[Qx] \equiv \sim\forall x[\sim Qx].$$

This tells us that existential statements could be rewritten as (the negations of) universal statements. If we wished, we could do without the existential quantifier entirely. However, allowing ourselves to use it often produces shorter and more readable logical statements.

Example 5 ***A Lengthy Proof***

Proof that the following argument is valid:

Every student can swim.
Everyone can either swim or surf.

Betty cannot swim.
Therefore, not everyone who can surf is a student.

Solution

Let S = "is a student," W = "can swim," R = "can surf," and b = Betty. Symbolically, our argument is this:

$$\forall x[Sx \longrightarrow Wx]$$
$$\forall x[Wx \vee Rx]$$
$$\sim Wb$$
$$\therefore \sim\forall x[Rx \longrightarrow Sx]$$

Consider how we might prove the conclusion. It is always hard to prove a negative, so rewrite it as $\exists x[\sim(Rx \longrightarrow Sx)]$. We can prove this existential statement by finding an example of someone for whom $\sim(Rx \longrightarrow Sx)$ is true. Again, let us pull the negation further inside the statement: $\sim(Rx \longrightarrow Sx)$ is equivalent to $Rx \wedge (\sim Sx)$. Now, the only person we know anything about specifically is Betty, so perhaps we can prove that $Rb \wedge (\sim Sb)$. Working from the beginning, we have general statements that we should probably specialize to our subject, Betty. Putting these ideas together we get the following proof.

1. $\forall x[Sx \longrightarrow Wx]$	Premise
2. $\forall x[Wx \vee Rx]$	Premise
3. $\sim Wb$	Premise
4. $Sb \longrightarrow Wb$	1, Specialization
5. $Wb \vee Rb$	2, Specialization
6. $\sim Sb$	3, 4 Modus Tollens
7. Rb	3, 5 One-or-the-other
8. $Rb \wedge (\sim Sb)$	6, 7 Conjunction
9. $\sim((\sim Rb) \vee Sb)$	8, DeMorgan
10. $\sim(Rb \longrightarrow Sb)$	9, Switcheroo
11. $\exists x[\sim(Rb \longrightarrow Sb)]$	10, Generalization
12. $\sim\forall x[Rx \longrightarrow Sx]$	11, Negation of a quantifier

We have only scratched the surface of the list of rules of inference necessary to do proofs in the predicate calculus. Although we would very much like to continue this discussion for a few more sections, we would be straying rather far beyond the scope of this text. Instead, we recommend for further reading any of the many books on symbolic logic that exist, including:

I. M. Copi, *Symbolic Logic*, 5th Ed., Prentice Hall, 1979.

J. E. Rubin, *Mathematical Logic: Applications and Theory*, Holt, Rinehart, and Winston, 1997.

P. Suppes, *Introduction to Logic*, Dover Publications, 1999.

L.8 Exercises

Write the negation of each of the statements in Exercises 1–26. (No, your answer may not start with "~".)

1. $\forall x[Px \longrightarrow \sim Qx]$

2. $\forall x[(\sim Px) \longrightarrow Qx]$

3. $\forall x[\sim(Px \longrightarrow Qx)]$

4. $\forall x[Px \longleftrightarrow Qx]$

5. $\exists x[Px \wedge Qx]$

6. $\exists x]Px \vee Qx]$

7. $\exists x[Px \longleftrightarrow Qx]$

8. $\exists x[Px \longrightarrow (Qx \vee Rx)]$

9. $\forall x[\exists y[P(x,y)]]$

10. $\forall x[\exists y[Px \longrightarrow Qy]]$

11. $\forall x[Px \longrightarrow \exists y[Q(x,y)]]$

12. $\forall x[Px \longrightarrow \exists y[Qx \longrightarrow Ry]]$

13. $\forall x[\exists y[\forall z[P(x,y,z)]]]$

14. $\forall x[\exists y[\forall z[Px \wedge Qz \longrightarrow Ry]]]$

15. All men are mortal.

16. All birds can fly.

17. All pigs with wings can fly.

18. All pigs either have no wings or can fly.

19. Some men are mortal.

20. Some birds can fly.

21. Some pigs can fly.

22. Some pigs have wings and can fly.

23. For every positive number there is a smaller positive number.

24. For every number x there is a number y such that $y^2 = x$.

25. There is a number smaller than every positive number.

26. There is a person older than all other people.

Prove each of the arguments in Exercises 27–42.

27. $\forall x[Px \longrightarrow Qx]$
$\sim Qb$

28. $\forall x[Px \longrightarrow Qx]$
$\forall x[Qx \longrightarrow Rx]$

$\therefore \sim Pb$

Pb

$\therefore Rb$

29. $\sim\exists x[Px \wedge Qx]$
Pb

$\therefore \sim Qb$

30. $\sim\exists x[Px \longrightarrow Qx]$

$\therefore Pb$

31. $\forall x[Px \longrightarrow \sim Qx]$
$\sim\exists x[(Rx \vee Sx) \wedge \sim Qx]$
Rb

$\therefore \sim Pb$

32. $\forall x[Px \longrightarrow (Qx \vee Rx)]$
$\sim\exists x[(Qx \vee Rx) \wedge \sim Sx]$
$\sim Sb$

$\therefore \sim Pb$

33. $\forall x[Px \longrightarrow Qx]$
Pb

$\therefore \exists x[Qx]$

34. $\forall x[Px \longrightarrow Qx]$
$\sim Qb$

$\therefore \sim\forall x[Px]$

35. $\sim\exists x[Px \wedge Qx]$
Pb

$\therefore \sim\forall x[Qx]$

36. $\sim\exists x[Px \wedge \sim Qx]$
$\forall x[Qx \longrightarrow Rx]$
Pb

$\therefore \exists x[Rx]$

37. All men are mortal.
Bach is immortal.
∴ Bach was not a man.

38. All men are mortal
All mortals require food.
Aristotle was a man.
∴ Aristotle required food.

39. No pig can fly.
Spot is a pig.
∴ Spot cannot fly.

40. No mathematicians are fools.
No one, who is not a fool, is an administrator.
Rita is a mathematician.
∴ Rita is not an administrator.

41.[1] No children are patient.
No impatient person can sit still.
Alex is a child.
∴ Alex cannot sit still.

42. Babies are illogical
Nobody is despised who can manage a crocodile.
Illogical persons are despised.

[1] This and the following exercise, after Lewis Carroll's *Symbolic Logic*.

Andrew is a baby.
∴ Andrew cannot manage a crocodile.

Communication and Reasoning Exercises

43. In view of the tautologies about negation of quantifiers, we could do away with the universal quantifier completely, right?

44. In view of the tautologies about negation of quantifiers, we could do away with the existential quantifier completely, right?

45. Comment on the following "generalization rule":

$$Px \longrightarrow \forall x[Px]$$

46. Comment on the following "specialization rule':

$$\exists x[Px] \longrightarrow Px$$

47. Can one switch the order of universal and existential quantifiers? In other words, is $\exists x[\forall y[P(x,y)]]$ equivalent to $\forall y[\exists x[P(x,y)]]$?

48. Is $\exists x[\exists y[P(x,y)]]$ equivalent to $\exists y\,[\exists x[P(x,y)]]$?

You're the Expert—Does God Exist?

Faith
Faith is an island in the setting sun
But proof, yes
Proof is the bottom line for everyone

Paul Simon, *Proof*, from *The Rhythm of the Saints*, Warner Bros. Records, 1990

Lord, what fools these mortals be.

William Shakespeare, *A Midsummer Night's Dream*

As head of your university's Department of Logic, you have been asked to evaluate several proofs of the existence of God. And many attempts there have been. Three in particular have caught your attention: they are known as the cosmological argument, the teleological argument, and the ontological argument.

The first argument that comes across your desk is the cosmological argument, put forward by St. Thomas Aquinas (1226–1274), the philosopher who introduced Aristotle's philosophy and logic into Christian theology. As its name suggests, this argument is based on a so-called "cosmological" consideration—that of the origin of the universe. It goes as follows.

> No effect can cause itself, but requires another cause. If there were no first cause, there would be an infinite sequence of preceding causes. Clearly there cannot be an infinite sequence of causes, therefore there is a first cause, and this is God.

You make the following quick analysis. Let F: "there is a first cause," and let I: "there is an infinite sequence of causes." Then the argument has the overall form

$$\begin{array}{l} \sim F \longrightarrow I \\ \sim I \\ \hline \therefore\ F \end{array}$$

This you quickly recognize as a correct application of Modus Tollens, together with the Double Negation law. The question then is, are the premises obvious, or do they require further justification? The first seems reasonable, although it looks like a job for the Department of Mathematics, so you send it down the hall for their opinion. You also make a note to ask the theologians whether they really mean to say that God is simply the first cause.

The next argument to cross your desk is another one from Aquinas, known as the teleological[1] argument, which goes like this:

> All things in the world act towards an end. They could not do this without there being an intelligence that directs them. This intelligence is God.

[1] *Teleology* is the study of evidence of *design* or *purpose* in nature.

Again, you give this a quick analysis. You let *A*: "all things act towards an end" and you let *D*: "there is an intelligent director." The argument is then

$$A$$
$$\sim D \longrightarrow \sim A$$
$$\overline{\therefore\ D}$$

Again, you recognize a correct application of Modus Tollens. But again you could question the premises. Do all things really act toward some end? If they do, then must it be the case that there is an intelligent director? Perhaps this should be passed on to the Department of Philosophy for further consideration. . .

The third argument you consider is the ontological[1] argument, put forward by St. Anselm (1033–1103), Archbishop of Canterbury. It goes like this.

> God is a being than which none greater can be thought. A being thought of as existing is greater than one thought of as not existing. Therefore, one cannot think of God as not existing, so God must exist.

Now this is a little more complicated than Aquinas' arguments. You look at the conclusion first. This is certainly a logical statement, so we take *GE* to be the statement "God exists." Now look at the next-to-last phrase: "one cannot think of God as not existing." Let us take *GNE* to represent the statement "God can be thought of as not existing."

Now, in the last sentence is really a hidden premise: "if one cannot think of God as not existing, then God does exist." It is actually a debatable point: if one cannot think of some entity as not existing, does that necessarily imply that it does exist? You decide to give St. Anselm the benefit of the doubt, and include the premise $\sim GNE \longrightarrow GE$.

You now look at the first sentence: "God is a being than which none greater can be thought." This is simple enough. In more modern language it says simply that "one cannot think of a being greater than God." You let *BGG* stand for the statement "one can think of a being greater than God," and so the first premise will be $\sim BGG$.

The second sentence says "A being thought of as existing is greater than one thought of as not existing." In particular, if one can think of a being existing but can think of God as not existing, than one has thought of a being greater than God. Letting *BE* stand for the statement "one can think of a being existing," this is the premise $(BE \wedge GNE) \longrightarrow BGG$.

These are the premises you have to work with. After fiddling around with them and getting nowhere, you realize that an additional premise is being assumed: that one can think of a being existing: *BE*.

You now assemble the following version of the argument.

[1] *Ontology* is the study of the nature of being or existence.

$\sim BGG$
$(BE \wedge GNE) \longrightarrow BGG$
BE
$\sim GNE \longrightarrow GE$

$\therefore\ GE$

You find that this is a valid argument, since it has the following proof.

1. $\sim BGG$	Premise
2. $(BE \wedge GNE) \longrightarrow BGG$	Premise
3. BE	Premise
4. $\sim GNE \longrightarrow GE$	Premise
5. $\sim(BE \wedge GNE)$	1, 2 Modus Tollens
6. $\sim BE \vee \sim GNE$	5, DeMorgan
7. $\sim GNE$	3, 6 One-or-the-other
8. GE	4, 7 Modus Ponens

So, you conclude that anyone accepting the premises must accept the conclusion, and so be convinced of the existence of God. But are these premises obviously true, or do they themselves require justification?

Exercises

1. Analyze the following version of the cosmological argument, due to the philosopher Richard Taylor:[1] Everything has a sufficient reason for its existence—either it was caused by something

[1] See: Richard Taylor, *Metaphysics*, Prentice-Hall, Inc., 1963.

else (its existence is **contingent**) or it must exist by its very nature (its existence is **necessary**). Neither the existence of the world, nor anything in it, is necessary, and therefore the existence of the world was caused by something else. If a thing's existence was not ultimately caused by something whose existence is necessary, there would be no sufficient reason for that thing's existence. Now the world does in fact exist. Therefore, the existence of the world must ultimately be caused by something whose existence is necessary. This ultimate cause (which Taylor calls a **necessary being**) cannot be the world itself or anything in it, and this is God.

2. Find other arguments for the existence of God and analyze them as logical arguments (arguments from your own religion, those of your friends, etc.).

3. Find examples of arguments in politics, advertisements, newspapers, etc., and analyze them.

Answers to Odd-Numbered Exercises

L.1

1. False statement **3.** Not a statement, since it is not a declarative sentence **5.** False statement; Father Nikolsky is a fictitious character, and thus he didn't exist. **7.** True statement **9.** True (we hope!) statement **11.** Statement whose truth value depends on what he, she, or it has uttered. **13.** Not a statement, since it is self-referential. **15.** $(\sim p)\wedge q$ **17.** $(p\wedge r)\wedge q$ **19.** $p\vee(\sim p)$ **21.** Willis is a good teacher and his students do not hate math. **23.** Either Willis is a good teacher, or his students hate math and Carla is not a good teacher. **25.** Either Carla is a good teacher, or she is not. **27.** Willis' students both hate and do not hate math. **29.** It is not true that either Carla is a good teacher or her students hate math. **31.** F **33.** F **35.** T **37.** T **39.** T **41.** $p \amalg q = (p\vee q)\wedge\sim(p\wedge q)$ **43.** I shall either buy a new calculus book or a used one. **45.** Here is one possible answer: "What does this question ask?"

L.2

1.

p	q	$\sim q$	$p\wedge\sim q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

3.

p	$\sim p$	$\sim(\sim p)$	$\sim(\sim p)\vee p$
T	F	T	T
F	T	F	F

5.

p	q	$\sim p$	$\sim q$	$(\sim p)\wedge(\sim q)$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

7.

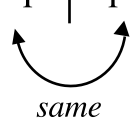
p	q	r	$p\wedge q$	$(p\wedge q)\wedge$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	F
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

9.

p	q	r	$q\vee r$	$p\wedge(q\vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

11.

p	$p\wedge p$
T	T
F	F



13.

p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

same

15.

p	q	$p \vee q$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$(\sim p) \wedge (\sim q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

same

17.

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$	$q \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	F	T	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

same

19.

p	q	$\sim q$	$q \wedge \sim q$	$p \vee (q \wedge \sim q$
T	T	F	F	T
T	F	T	F	T
F	T	F	F	F
F	F	T	F	F

same

21. Contradiction

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

23. Contradiction

p	q	$p \vee q$	$\sim(p \vee q)$	$p \wedge \sim(p \vee q)$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	F

25. Tautology

p	q	$p \wedge q$	$\sim(p \wedge q)$	$p \vee \sim(p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

27. $(\sim p) \vee p$ **29.** $(\sim p) \vee \sim(\sim q)$ **31.** $p \vee ((\sim p) \vee (\sim q))$ **33.** $(p \vee (\sim p)) \wedge (p \vee q)$ **35.** Either I am not Julius Caesar or you are not a fool. **37.** It's raining and I have forgotten either my umbrella or my hat. **39.** My computer crashes when it has been on a long time, the air is not dry, and the moon is full. **41.** The warning light will come on if the pressure drops and either the temperature is high, the emergency override is not activated, or the manual controls are not activated.

L.3

1. T **3.** T **5.** T **7.** F **9.** T **11.** T **13.** F **15.** T **17.** T **19.** T **21.** T **23.** F **25.** T **27.** T

29.

p	q	$q\vee p$	$p\longrightarrow(q\vee p)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

31.

p	q	$p\wedge q$	$\sim p$	$(p\wedge q)\longrightarrow\sim p$
T	T	T	F	F
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

33.

p	$\sim p$	$p\longrightarrow(\sim p)$	$(p\longrightarrow\sim p)\longrightarrow p$
T	F	F	T
F	T	T	F

35. Tautology

p	q	$\sim p$	$p\wedge\sim p$	$(p\wedge\sim p)\longrightarrow q$
T	T	F	F	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

37.

p	q	$p\vee q$	$p\longleftrightarrow(p\vee q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

39. Tautology

p	q	$\sim p$	$\sim q$	$p\wedge\sim p$	$q\wedge\sim q$	$(p\wedge\sim p)\longleftrightarrow(q\wedge\sim q)$
T	T	F	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	F	T
F	F	T	T	F	F	T

41.

p	q	$p\longrightarrow q$	$\sim p$	$\sim q$	$(\sim q)\longrightarrow(\sim p)$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

same

43.

p	q	$p\longrightarrow q$	$\sim p$	$(\sim p)\vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

same

45.

p	q	$\sim p$	$\sim q$	$p\longleftrightarrow\sim p$	$q\longleftrightarrow\sim q$
T	T	F	F	F	F
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	F	F

same

47. Contrapositive: "If I do not exist, then I do not think." Converse: "If I am, then I think." **49.** Contrapositive: "If I am not Buddha, then I think." Converse: "If I am Buddha, then I do not think." **51.** Contrapositive: "These birds do not flock together only if they are not of a feather." Converse: "These birds flock together only if they are of a feather." **53.** Contrapositive: "In order not to sacrifice beasts of burden, it is necessary not to worship Den." Converse: "In order to sacrifice beasts of burden, it is necessary to worship Den." **55.** "Either I am, or I do not think." **57.** "Either symphony orchestras are subsidized by the government, or they will cease to exist." **59.** "Either our society wishes research in the pure sciences to cease, or it will continue." **61.** $p \longrightarrow \sim q$ **63.** $p \longrightarrow \sim q$ **65.** $p \leftrightarrow \sim q$ **67.** $\sim q \longrightarrow p$ **69.** $q \leftrightarrow \sim p$

L.4

1.

p	q	$p \longrightarrow q$	$\sim q$	$[(p \longrightarrow q) \wedge \sim q] \longrightarrow \sim p$
T	T	T	F	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

3.

p	q	$p \vee q$	$p \longrightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

5.

p	q	r	$p \longrightarrow q$	$q \longrightarrow r$	$(p \longrightarrow q) \wedge (q \longrightarrow r)$	$p \longrightarrow r$	$[(p \longrightarrow q) \wedge (q \longrightarrow r)] \longrightarrow (p \longrightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

7.

p	q	$p \wedge q$	$q \wedge p$	$(p \wedge q) \longleftrightarrow (q \wedge p)$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	F	F	T

9.

p	q	$p \longrightarrow q$	$\sim q$	$(\sim p) \vee q$	$(p \longrightarrow q) \longleftrightarrow [(\sim p) \vee q]$
T	T	T	F	T	T
T	F	F	T	F	T
F	T	T	F	T	T
F	F	T	T	T	T

11. p = "Some cows are chickens"; q = "Some chickens lay eggs." Then $p \vee q$ is true, whereas p is false. Thus, $(p \vee q) \longrightarrow q$ is false. **13.** p = "All swans are white"; q = "Some swans are white." Then $p \longrightarrow q$ is true (since the statement p is false; not *all* swans are in fact white). On the other hand, $q \longrightarrow p$ says that if some swans are white, then all swans are white. But *some* swans *are* white, so q is true; whereas p is false. Thus, $q \longrightarrow p$ is false. Therefore, $(p \longrightarrow q) \longrightarrow (q \longrightarrow p)$ is false. **15.** Use the same example as in Exercise 13. **17.** $(h \wedge t) \longrightarrow h$; tautology **19.** $\sim(r \wedge v) \longrightarrow ((\sim r) \wedge (\sim v))$; not a tautology **21.** $(u \longrightarrow r) \longrightarrow (\sim r \longrightarrow \sim u)$; tautology **23.** $(u \leftrightarrow r) \longrightarrow (\sim r \longrightarrow \sim u)$; tautology **25.** $(g \longrightarrow p) \longrightarrow (g \vee \sim p)$; not a tautology **27.** $((t \vee h) \wedge \sim t) \longrightarrow h$; tautology **29.** $((g \longrightarrow s) \wedge (g \longrightarrow j)) \longrightarrow (s \longrightarrow j)$; not a tautology **31.** $((g \longrightarrow s) \wedge \sim g) \longrightarrow \sim s$; not a tautology

L.5

1. $\sim q$ **3.** $\sim(q\wedge r)$ **5.** $\sim(\sim p\vee q)$ **7.** $p\wedge\sim q$ **9.** $(p\wedge r)\longrightarrow r$ **11.** 4. $(p\wedge r)\longrightarrow r$; 5. $\sim(p\wedge r)$ **13.** $p\longrightarrow q$ **15.** 3. $(\sim r)\vee(\sim q)$; 4. $\sim(r\wedge q)$; 5. $\sim p$ **17.** 4. $p\wedge q$; 5. r **19.** DeMorgan **21.** 1,2 Modus Tollens **23.** 1, Switcheroo **25.** 1, Switcheroo; 2, DeMorgan **27.** 2, Addition; 1,3 Modus Ponens; 4, Simplification **29.** 1,2 Transitive Law; 4, Contrapositive; 3,5 Modus Ponens **31.** 1, Simplification; 1, Simplification; 3, Addition, 4, Switcheroo; 2,5 Modus Ponens **33.** Double Negative; Double Negative (S); 2, Switcheroo; 3, Addition; 4, Associative Law; 5, Commutative Law; 6, Switcheroo; 7, Switcheroo

35.

1. $u\longrightarrow r$	Premise	
2. $r\longrightarrow h$	Premise	
3. $\sim h$	Premise	
4. $\sim r$	2,3	Modus Tollens
5. $\sim u$	1,4	Modus Tollens

37.

1. $\sim(h\wedge r)$	Premise
2. $\sim h\vee\sim r$	1, DeMorgan
3. h	Premise
4. $\sim r$	2,3 Disjunctive Syllogism

39.

1. $i\longrightarrow s$	Premise
2. $\sim i\longrightarrow(h\wedge c)$	Premise
3. $\sim c$	Premise
4. $\sim h\vee\sim c$	3, Addition
5. $\sim(h\wedge c)$	4, DeMorgan
6. i	2,5 Modus Tollens
7. s	1,6 Modus Ponens

L.6

1.

1. $(p\vee r)\longrightarrow\sim q$	Premise
2. $p\vee r$	Premise
3. $\sim q$	1, 2 Modus Ponens

3.

1. $\sim p\longrightarrow(r\longrightarrow\sim t)$	Premise
2. $\sim(r\longrightarrow\sim t)$	Premise
3. p	1, 2 modus tollens

5.

1. $\sim p\longrightarrow(q\wedge r)$	Premise
2. $\sim p\wedge s$	Premise
3. $\sim p$	2, simplification
4. $q\wedge r$	1, 3 modus ponens
5. r	4, simplification

7.

1. $p\longrightarrow q$	Premise
2. $\sim(q\vee r)$	Premise
3. $\sim q\wedge\sim r$	2, DeMorgan
4. $\sim q$	3, simplification
5. $\sim p$	1, 4 modus tollens

9.

1. $(p\vee r)\longrightarrow q$	Premise
2. $s\longrightarrow p$	Premise
3. s	Premise
4. p	2, 3 modus ponens
5. $p\vee r$	4, addition
6. q	1, 5 modus ponens

11.

1. $(p\vee\sim q)\longrightarrow r$	Premise
2. $s\longrightarrow(t\wedge u)$	Premise
3. $s\wedge p$	Premise
4. s	3, simplification
5. $t\wedge u$	2, 4 modus ponens
6. u	5, simplification
7. p	3, simplification
8. $p\vee\sim q$	7, addition
9. r	1, 8 modus ponens
10. $r\wedge u$	6, 9 rule C

13.

1. $(p\longrightarrow q)\longrightarrow r$	Premise

15.

1. $p\longrightarrow(q\longrightarrow r)$	Premise

2. $\sim(q\vee r)$	Premise
3. $\sim q\wedge\sim r$	2, DeMorgan
4. $\sim r$	3, simplification
5. $\sim(p\longrightarrow q)$	1, 4 modus tollens
6. $\sim(\sim p\vee q)$	5, switcheroo
7. $p\wedge\sim q$	6, DeMorgan
8. p	7, simplification

2. q	Premise
3. $\sim p\vee(q\longrightarrow r)$	1, switcheroo
4. $\sim p\vee(\sim q\vee r)$	3, switcheroo
5. $\sim p\vee(r\vee\sim q)$	4, commutativit y
6. $(\sim p\vee r)\vee\sim q$	5, associativity
7. $\sim p\vee r$	2,6 disjunctive syllogism
8. $p\longrightarrow r$	7, switcheroo

17.

1. $(p\wedge q)\longrightarrow p$	simplification
2. $p\longrightarrow(p\vee q)$	addition
3. $(p\wedge q)\longrightarrow(p\vee q)$	1, 2 transitivity

19.

1. $p\longrightarrow\sim(\sim p)$	double negative
2. $p\longrightarrow p$	double negative
3. $\sim p\vee p$	2, switcheroo
4. $\sim(p\wedge\sim p)$	3, DeMorgan

21.

1. $p\longrightarrow\sim(\sim p)$	double negative
2. $p\longrightarrow p$	double negative
3. $\sim p\vee p$	2, switcheroo
4. $p\vee\sim p$	3, commutative law
5. $(p\vee\sim p)\wedge(p\vee\sim p)$	4, rule C
6. $(p\wedge p)\vee\sim p$	5, distributive law
7. $\sim(\sim p\vee\sim p)\vee\sim p$	6, DeMorgan
8. $\sim(p\longrightarrow\sim p)\vee\sim p$	7, switcheroo
9. $(p\longrightarrow\sim p)\longrightarrow p$	8, switcheroo

23. p false, q true **25.** p true, q false, r false **27.** Invalid. For a counterexample, we can take p = "1 = 1"; q = "The moon is made of green cheese", and r = "2+2 = 4."

29. Valid.

1. $p\longrightarrow r$	Premise
2. $\sim q\longrightarrow\sim r$	Premise
3. $r\longrightarrow q$	2, contrapositive
4. $p\longrightarrow q$	1,3 transitive law

31. Invalid. This is the same argument as the one in (23). Note that, although the reasoning is incorrect, the conclusion is true. I don't eat grass.

33. Invalid:

$p\longrightarrow d$ Let p be false and d true.
d
______
$\therefore\ p$

35. Invalid:

$d\longrightarrow r$ Let d be false, r true, and s false.
$d\longrightarrow s$
______
$\therefore\ r\longrightarrow s$

37. Valid:

$(m\vee p)\longrightarrow h$
m
____

$\therefore\ h$

Proof:

1. $(m\vee p)\longrightarrow h$	Premise
2. m	Premise
3. $m\vee p$	2, addition
4. h	1,4 modus ponens

39. Replace her; her prediction is invalid. Using r = "it rains on the Nile," s = "Sagittarius falls in Jupiter's shadow," m = "Mercury is ascending," f = "The moon is full," the argument becomes:

$r\longrightarrow s$
$m\longrightarrow r$
$s\longrightarrow(f\vee m)$

$\therefore\ (s\wedge f)\longrightarrow r$

Intuitively, the argument seems invalid because, according to the premises, to be assured of r we need to have m, whereas all s guarantees is f *or* m. To get a counterexample, we try to make the conclusion false, which means s and f must be true, but r must be false. What about m? Since $m\longrightarrow r$ is one of the premises and r is false, so must m be false for $m\longrightarrow r$ to be true. Thus we have: s and f true; r and m false. Let us take: s = "1+1 = 2," f = "the moon is round," r = "the moon is square," and m = "the moon is a balloon." Then the argument becomes:

If the moon is square, then 1+1 = 2	- true
If the moon is a balloon, then it is square	- true
If 1+1 = 2, then the moon is either round or a balloon	- true
$\therefore$ If 1 + 1 = 2 and the moon is round, then the moon is square	- false

41. A correct deduction; if we use the first letter in a country's name to represent the statement that it signs the accord, then the argument is:

$u\longrightarrow(c\wedge b)$
$i\longrightarrow b$
$b\longrightarrow u$

$\therefore\ i\longrightarrow c$

To prove it, first obtain $i\longrightarrow(c\wedge b)$ using transitivity, then use switcheroo and simplification to obtain $\sim i\vee c$, and finally, use switcheroo.

43. Valid: Let v = "some violets are red," b = "some roses are blue," l = "somebody loves you," and g = "you're grown up."

$(\sim v\wedge b)\longrightarrow\sim l$
l
$b\vee(\sim g)$
g

$\therefore\ v$

Proof:

1. $(\sim v\wedge b)\longrightarrow\sim l$	Premise
2. l	Premise

3. $b\vee(\sim g)$	Premise
4. g	Premise
5. $\sim(\sim v\wedge b)$	1, 2 modus tollens
6. $v\vee\sim b$	5, DeMorgan
7. b	3,4 disjunctive syllogism
8. v	6,7 disjunctive syllogism

45. Yes; rule of Inference T2 permits us to add any tautology in our list of tautologies in Section L.4 as a premise. Thus, we can start with no premises and add tautologies from our list. **47.** The argument is invalid, and so there is not proof.

L.7

1. $\forall x[Gx\longrightarrow Dx]$ **3.** $\forall x[Cx\longrightarrow Ex]$ **5.** $\exists x[Cx\wedge Ex]$ **7.** $\exists x[Cx\wedge\sim Bx]\wedge\exists x[Cx\wedge Bx]$, or $\exists x,y[Cx\wedge Cy\wedge Bx\wedge\sim By]$ **9.** $\exists x[Cx\wedge Ix]\wedge(Cd\wedge\sim Id)$ **11.** $\exists x[Lx\wedge Sx]\longrightarrow\forall x[Lx\longrightarrow Sx]$ **13.** $\exists x[Nx\wedge(x>2)]\wedge\exists x[Nx\wedge(x\leq 2)]$, or $\exists x,y[Nx\wedge Ny\wedge(x>2)\wedge(y\leq 2)]$ **15.** $\exists n[12=6n]$ **17.** $\forall m[\exists n[12=mn]\longrightarrow\exists n[24=mn]]$ **19.** $\exists m[\exists n[15=mn]]$, or $\exists m,n[15=mn]$ **21.** $\sim\exists m,n[(m\neq 1)\wedge(m\neq 17)\wedge(17=mn)]$, or $\forall m,n[(17=mn)\longrightarrow(m=1)\vee(m=17)]$ **23.** $\forall x[(x>0)\longrightarrow\exists y[(y>0)\wedge(y<x)]]$ **25.** $\{P1\wedge\forall n[Pn\longrightarrow P(n+1)]\}\longrightarrow\forall n[Pn]$ **27.** Every raindrop makes a splash. **29.** Some dogs whimper. **31.** No dogs whimper. **33.** Some cats whimper and some cats won't. **35.** Wrong; saying that not all athletes drink ThirstPro amounts to saying that *some* athletes do not drink it. **37.** "For some" can be expressed as $\exists$ and "there does not exist" as $\sim\exists$. Hence no new quantifiers are necessary.

L.8

1. $\exists x[Px\wedge Qx]$ **3.** $\exists x[Px\longrightarrow Qx]$ **5.** $\forall x[Px\longrightarrow\sim Qx]$ **7.** $\forall x[\sim(Px\longleftrightarrow Qx)]$ **9.** $\exists x[\forall y[\sim P(x,y)]]$ **11.** $\exists x[Px\wedge\forall y[\sim Q(x,y)]]$ **13.** $\exists x[\forall y[\exists z[\sim P(x,y,z)]]]$ **15.** Some men are immortal. **17.** Some pigs with wings cannot fly. **19.** All men are immortal. **21.** No pigs can fly. **23.** There is a positive number for which there is no smaller. **25.** For every number there is a positive number that is no larger.

27.

1. $\forall x[Px\longrightarrow Qx]$	Premise
2. $\sim Qb$	Premise
3. $Pb\longrightarrow Qb$	1, Specialization
4. $\sim Pb$	2,3 Modus Tollens

29.

1. $\sim\exists x[Px\wedge Qx]$	Premise
2. Pb	Premise
3. $\forall x[\sim(Px\wedge Qx)]$	1, Negation
4. $\forall x[(\sim Px)\vee(\sim Qx)]$	3, DeMorgan
5. $(\sim Pb)\vee(\sim Qb)$	4, Specialization
6. $\sim Qb$	2, 5 Disjunctive Syllogism

31.

1. $\forall x[Px\longrightarrow\sim Qx]$	Premise
2. $\sim\exists x[(Rx\vee Sx)\wedge\sim Qx]$	Premise
3. Rb	Premise
4. $\forall x[\sim((Rx\vee Sx)\wedge\sim Qx)]$	2, Negation
5. $\forall x[\sim(Rx\vee Sx)\vee Qx)]$	4, DeMorgan
6. $\sim(Rb\vee Sb)\vee Qb$	5, Specialization

7. $Rb \vee Sb$	3, Addition
8. Qb	6,7 Disjunctive Syllogism
9. $Pb \longrightarrow \sim Qb$	1, Specialization
10. Pb	8,9 Modus Ponens

33.

1. $\forall x[Px \longrightarrow Qx]$	Premise
2. Pb	Premise
3. $Pb \longrightarrow Qb$	1, Specialization
4. Qb	2,3 Modus Ponens
5. $\exists x[Qx]$	4, Generalization

35.

1. $\sim\exists x[Px \wedge Qx]$	Premise
2. Pb	Premise
3. $\forall x[\sim(Px \wedge Qx)]$	1, Negation
4. $\sim(Pb \wedge Qb)$	3, Specialization
5. $(\sim Pb) \vee (\sim Qb)$	4, DeMorgan
6. $\sim Qb$	2,5 Disjunctive Syllogism
7. $\exists x[\sim Qx]$	6, Generalization
8. $\sim\forall x[Qx]$	7, Negation

37.

1. $\forall x[Mx \longrightarrow Rx]$	Premise
2. $\sim Rb$	Premise
3. $Mb \longrightarrow Rb$	1, Specialization
4. $\sim Mb$	2,3 Modus Tollens

39.

1. $\sim\exists x[Px \wedge Fx]$	Premise
2. Ps	Premise
3. $\forall x[\sim(Px \wedge Fx)]$	1, Negation
4. $\sim(Ps \wedge Fs)$	3, Specialization
5. $\sim Ps \vee \sim Fs$	4, DeMorgan
6. $\sim Fs$	2,5 Disjunctive Syllogism

41.

1. $\sim\exists x[Cx \wedge Px]$	Premise
2. $\sim\exists x[\sim Px \wedge Sx]$	Premise
3. Ca	Premise
4. $\forall x[\sim(Cx \wedge Px)]$	1, Negation
5. $\sim(Ca \wedge Pa)$	4, Specialization
6. $\sim Ca \vee \sim Pa$	5, DeMorgan
7. $\sim Pa$	3,6 Disjunctive Syllogism
8. $\forall x[\sim(\sim Px \wedge Sx)]$	2, Negation
9. $\sim(\sim Pa \wedge Sa)$	8, Specialization
10. $Pa \vee \sim Sa$	9, DeMorgan
11. $\sim Sa$	7,10 Disjunctive Syllogism

43. Right: one can replace $\forall x[Px]$ by $\sim\exists x[\sim Px]$. **45.** It is invalid; for instance, the fact that 3 is a positive number does ot permit us to conclude that all numbers are positive. **47.** No; For let $P(x,y)$ stand for $x > y$. Then the statement on the right is "For all (numbers) y there exists a (number) x with $x > y$"—a true statement about numbers, whereas the statement on the left is "There is a (number) x which is bigger than every number y"— a false statement.