Advanced Engineering Math II
Math 144

Lecture Notes
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1. Algebra and Geometry of Complex Numbers (based on §§17.1–17.3 of Zill)

Definition 1.1 A **complex number** has the form $z = (x, y)$, where x and y are real numbers. x is referred to as the **real part** of z, and y is referred to as the **imaginary part** of z. We write

$$\text{Re}(z) = x, \text{Im}(z) = y.$$

Denote the set of complex numbers by $\mathbb{C}$. Think of the set of real numbers as a subset of $\mathbb{C}$ by writing the real number x as $(x, 0)$. The complex number $(0, 1)$ is called $\boldsymbol{i}$.

Examples
$3 = (3, 0)$, $(0, 5)$, $(-1, -\pi)$, $i = (0, 1)$.

Geometric Representation of a Complex Number- in class.

Definition 1.2 Addition and multiplication of complex numbers, and also multiplication by reals are given by:

$$(x, y) + (x', y') = ((x+x'), (y+y'))$$
$$(x, y)(x', y') = ((xx'-yy'), (xy'+x'y))$$
$$\lambda(x, y) = (\lambda x, \lambda y).$$

Geometric Representation of Addition- in class. (Multiplication later)

Examples 1.3
(a) $3+4 = (3, 0)+(4, 0) = (7, 0) = 7$ **(b)** $3\times4 = (3, 0)(4, 0) = (12-0, 0) = (12, 0) = 12$
(c) $(0, y) = y(0, 1) = yi$ (which we also write as iy).
(d) In general, $z = (x, y) = (x, 0) + (0, y) = x + iy$. $\boxed{z = x + iy}$
(e) Also, $i^2 = (0, 1)(0, 1) = (-1, 0) = -1$. $\boxed{i^2 = -1}$
(g) $4 - 3i = (4, -3)$.

Note In view of (d) above, from now on we shall write the complex number (x, y) as $x+iy$.

Definitions 1.4 The **complex conjugate**, $\bar{z}$, of the complex number $z = x+iy$ given by
$$\bar{z} = x - iy.$$
The **magnitude**, $|z|$ of $z = x+iy$ is given by
$$|z| = \sqrt{x^2+y^2}.$$

Examples and Geometric Representation of Conjugation and Magnitude - in class.

Notes

1. $z + \bar{z} = (x+iy) + (x-iy) = 2x = 2\text{Re}(z)$. Therefore, $\boxed{\text{Re}(z) = \frac{1}{2}(z+\bar{z})}$

$z - \bar{z} = (x+iy) - (x-iy) = 2iy = 2i\text{Im}(z)$. Therefore, $\boxed{\text{Im}(z) = \frac{1}{2i}(z-\bar{z})}$

2. Note that $z\bar{z} = (x+iy)(x-iy) = x^2-i^2y^2 = x^2+y^2 = |z|^2$ $\boxed{z\bar{z} = |z|^2}$

3. If $z \neq 0$, then z has a multiplicative inverse. Why? because:

$$z \cdot \frac{\bar{z}}{|z|^2} = \frac{z\bar{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1. \text{ Hence,} \qquad \boxed{z^{-1} = \frac{\bar{z}}{|z|^2}}$$

Examples

(a) $\frac{1}{i} = -i$ **(b)** $\frac{1}{3+4i} = \frac{3-4i}{25}$

(c) $\frac{1}{\frac{1}{\sqrt{2}}(1+i)} = \frac{1}{\sqrt{2}}(1-i)$ **(d)** $\frac{1}{\cos\theta + i\sin\theta} = \cos(-\theta) + i\sin(-\theta)$

4. There is also the **Triangle Inequality:**

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Proof We square both sides and compare them. Write $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$\begin{aligned} |z_1 + z_2|^2 &= (x_1+x_2)^2 + (y_1+y_2)^2 \\ &= x_1^{\ 2} + x_2^{\ 2} + 2x_1x_2 + y_1^{\ 2} + y_2^{\ 2} + 2y_1y_2. \end{aligned}$$

On the other hand,

$$\begin{aligned} (|z_1| + |z_2|)^2 &= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \\ &= x_1^{\ 2} + x_2^{\ 2} + y_1^{\ 2} + y_2^{\ 2} + 2|z_1||z_2|. \end{aligned}$$

Subtracting,

$$\begin{aligned} (|z_1| + |z_2|)^2 - |z_1 + z_2|^2 &= 2|z_1||z_2| - 2(x_1x_2 + y_1y_2) \\ &= 2\big[|(x_1,y_1)||(x_2,y_2)| - (x_1,y_1).(x_2,y_2)\big] \quad \text{(in vector form)} \\ &= 2\big[|(x_1,y_1)||(x_2,y_2)| - |(x_1,y_1)||(x_2,y_2)| \cos\alpha\big] \\ &= 2\ |(x_1,y_1)||(x_2,y_2)|\ (1 - \cos\alpha\ ) \\ &\geq 0, \end{aligned}$$

giving the result.

Note The triangle inequality can also be seen by drawing a picture of $z_1 + z_2$.

5. We now consider the **polar form** of these things: If $z = x+iy$, we can write $x = r\cos\theta$ and $y = r\sin\theta$, getting $z = r\cos\theta + ir\sin\theta$, so

This is called the polar form of z. It is important to draw pictures in order to feel comfortable with the polar representation. Here r is the magnitude of z, $r = |z|$, and ø is called the **argument of** z, denoted arg(z). To calculate θ, we can use the fact that $\tan\theta =$

y/x. Thus θ is not arctan(y/x) as claimed in the book, but by:

since the *arctan* function takes values between $-\pi/2$ and $\pi/2$. The **principal value** of arg(z) is the unique choice of θ such that $-\pi < \theta \leq \pi$. We write this as **Arg(z)**

$\boxed{-\pi < \text{Arg}(z) \leq \pi}$ $-\pi \leq$

Examples

(a) Express $z = 1+i$ in polar form, using the principal value

(b) Same for $\sqrt{3} + 3\sqrt{3}\, i$

(c) $6 = 6(\cos 0 + i \sin 0)$

6. Multiplication in Polar Coordinates

If $z_1 = r_1(\cos\theta_1 + i\ \sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\ \sin\theta_2)$, then

$$z_1 z_2 = r_1 r_2(\cos\theta_1 + i\ \sin\theta_1)(\cos\theta_2 + i\ \sin\theta_2)$$
$$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i\ (\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2).$$

Thus $$\boxed{z_1 z_2 = r_1 r_2[[\cos(\theta_1+\theta_2) + i\ \sin(\theta_1+\theta_2)]}$$

That is, we *multiply the magnitudes and add the arguments*.

Examples In class.

7. Multiplicative Inverses in Polar Coordinates

Once we know how to do multiplication, division follows formally: Let $z = r(cos\theta + isin\theta)$ be given. We want to find z^{-1}. So let $z^{-1} = s(\cos\phi + i\sin\phi)$. Then, since $zz^{-1} = 1$, we have

$$rs(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = 1$$

ie., $$rs(\cos(\theta+\phi) + i\sin(\theta+\phi)) = 1 = 1(\cos 0 + i\sin 0).$$

Thus, we can take $s = 1/r$ and $\phi = -\theta$. In other words, $\boxed{z^{-1} = r^{-1}(\cos(-\theta) + i\ \sin(-\theta))}$

Examples In class.

8. Division in Polar Coordinates

Finally, since $\dfrac{z_1}{z_2} = z_1 z_2^{\ -1}$, we have: $$\boxed{\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\ \sin(\theta_1-\theta_2)\right]}$$

That is, we *divide the magnitudes and subtract the arguments*.

Examples

(a) $z_1 = -2 + 2i,\ z_2 = 3i$

(b) Formula for z^n **De Moivre's formula** $\boxed{z^n = r^n(\cos n\theta + i\ \sin n\theta)}$

In words, to take the nth power, *we take the nth power of the magnitude and multiply the argument by n*.

Examples Powers of unit complex numbers.

9. *n*th Roots of Complex Numbers

Write

$$z = r(\cos(\theta+2k\pi) + i\ \sin(\theta+2k\pi)),$$

even though different values of k give the same answer.

Then

$$\boxed{z^{1/n} = r^{1/n}[\ \cos(\theta/n+2k\pi/n) + i\ \sin(\theta/n+2k\pi/n)]}$$

Note that we get *different* answers for $k = 0, 1, 2, \ldots, n-1$. Thus there are ***n*** **distinct** ***nth*** **roots of** z.

Examples

(a) $\sqrt{i}$ **(b)** $\sqrt{4i}$ **(c)** Solve $z^2 - (5+i)z + 8 + i = 0$

(d) ***n*****th roots of unity:** Since $1 = \cos 0 + i\sin 0$, the distinct nth roots of unity are:

$$\boxed{\gamma_k = \cos(2k\pi/n) + i\ \sin(2k\pi/n), \quad (k = 0, 1, 2, \ldots, n-1)}$$

More examples In class.

10. Exponential Notation

We know what e raised to a real number is. We now define what e raised to an *imaginary* number is:

Definition: $e^{i\theta} = \cos\theta + i\sin\theta$.

Thus, the typical complex number is **Exponential Form of a Complex Number**

$$\boxed{re^{i\theta} = r[\cos\theta + i\sin\theta]}$$

De Moivre's Theorem now implies that $e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$, so that the exponential rule for addition works, and the inverse rule shows that $1/e^{i\phi} = e^{-i\theta}$, so that the inverse exponent law also works. Similarly, the other laws also work. Duly emboldened, we now define

$$e^{x+iy} = e^x e^{iy} = e^x[\cos y + i\sin y] \qquad \boxed{e^{x+iy} = e^x[\cos y + i\sin y]}$$

Examples in class

Exercise Set 1

p.793 #1–17 odd, 27, 29, 37, 39

p. 797. 1–15 odd, 21, 23, 25, 27, 29, 31, 33

p. 800 #1, 5, 7, 11, 15, 23, 26

Hand In

1 (a) One of the quantum mechanics wave functions of a particle of unit mass trapped in an infinite potential square well of width 1 unit is given by

$$\Psi(x,t) = \sin(\pi x)\, e^{-i(\pi^2\hbar/2)\,t} + \sin(2\pi x)e^{-i(4\pi^2\hbar/2)\,t},$$

where $\hbar$ is a certain constant. Show that

$|\Psi(x,t)|^2 = \sin^2\pi x + \sin^2 2\pi x + 4\sin^2\pi x\cos\pi x\cos 3\theta$,

where $\theta = -(\pi^2\hbar/2)t$.

($|\Psi(x,t)|^2$ is the probability density function for the position of the particle at time t.)

(b) The expected position of the particle referred to in part (a) is given by

$$\langle x\rangle = \int_0^1 x\,|\Psi(x,t)|^2\,dx \qquad .$$

Calculate $\langle x\rangle$ and compute its amplitude of oscillation.

2. Functions of a Complex Variable: Analytic Functions and the Cauchy-Riemann Equations)

(§§17.4, 17.5 in Zill)

Definition 2.1 Let $S \subset \mathbb{C}$ A **complex valued function on S** is a function

$$f: S \longrightarrow \mathbb{C}.$$

S is called the **domain** of f.

Examples 2.2

(a) Define $f: \mathbb{C} \to \mathbb{C}$ by $f(z) = z^2$;

(b) Define $g: \mathbb{C} - \{0\} \to \mathbb{C}$ by $g(z) = -\dfrac{1}{z} + \bar{z}$. Find $g(1+i)$.

(c) Define h: $\mathbb{C} \to \mathbb{C}$ by $h(x+iy) = x + i(xy)$.

Notes

(a) In general, a complex valued function is completely specified by its real and imaginary parts. For example, in **(a)** above,

$$f(x+iy) = (x+iy)^2 = (x^2-y^2) + i(2xy).$$

Write this as $u(x,y) + iv(x,y)$,

where $u(x,y)$ and $v(x,y)$ are a pair of *real−valued* functions.

(b) An important way to picture a function f: $S \longrightarrow \mathbb{C}$ is as a "mapping" − picture in class.

Examples 2.3

(a) Look at the action of the functions $z + z_0$ and αz for fixed $z_0 \in \mathbb{C}$ and α real.

(b) Let S be the unit circle; $S = S^1 = \{z : |z| = 1\}$. Then the functions

$$f: S \to S; \; f(z) = z^n$$

are "winding" maps.

(c) The function f: $\mathbb{C} \to \mathbb{C}$ given by $f(z) = 1/z = z^{-1}$ is a special case of (a) above, and "winds" the unit circle backwards. It maps the circle of radius r backwards around the circle of radius $-r$.

(d) The function f: $\mathbb{C} \to \mathbb{C}$ given by $f(z) = \bar{z}$ agrees with $1/z$ on the unit circle, but not elsewhere.

Limits and Derivatives of Complex-valued Functions

Definition 2.4 If $D \subset \mathbb{C}$ then a point z_0 not necessarily in D is called a **limit point of D** if every neighborhood of z_0 contains points in D *other than itself*.

Illustrations in class

Definition 2.5 Let f: $D \to \mathbb{C}$ and let z_0 be a limit point of D. Then we say that $f(z) \to L$ as $z \to z_0$ if for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(z) - L) < \varepsilon \text{ whenever } 0 < |z - z_0| < \varepsilon.$$

When this happens, we also write $\lim_{z \to z_0} f(z) = L$.

If $z_0 \in D$ as well, we say that f is **continuous at z_0** if $\lim_{z \to z_0} f(z) = f(z_0)$.

Fact: Every closed-form (single-valued) function of a complex variable is continuous on its domain.

Definition 2.6 Let f: $D \to \mathbb{C}$ and let z_0 be in the interior of D. We define the **derivative** of f at z_0 to be

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

f is called **analytic at** z_0 if it is differentiable at z_0 and also in some neighborhood of z_0. If f is differentiable at every complex number, it is called **entire.**

Consequences Since the usual rules for differentiation (power, product, quotient, chain rule) all follow formally from the same definition as that above, we can deduce that the same rules hold for complex differentiation.

Geometric Interpretation of $f'(z)$[†]
Question What does $f'(z)$ look like geometrically?
Answer We describe the magnitude and argument separately. First look at the magnitude of $f'(z_o)$. For z near z_0,

$$|f'(z_0)| \approx \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = \frac{|f(z) - f(z_0)|}{|z - z_0|}$$

In other words, the magnitude of $f'(z_0)$ gives us an *expansion factor*; The distance between points is expanded by a factor of $|f'(z_0)|$ near z_0.

Now look at the direction (argument) of $f'(z_o)$: [Note that this only makes sense if $f'(z_0) \neq 0$ -- otherwise the argument is not well defined.]

$$f'(z_0) \approx \frac{f(z) - f(z_0)}{z - z_0}$$

Therefore, the argument of $f'(z_0)$ is $\mathrm{Arg}[f(z) - f(z_0)] - \mathrm{Arg}[z - z_0]$. That is,

$$\mathrm{Arg}[f'(z)] \approx \mathrm{Arg}[\Delta f] - \mathrm{Arg}[\Delta z]$$

Therefore, the argument of $f'(z_0)$ gives the direction in which f is rotating near z_0. In fact, we shall see later that f preserves angles at a point if the derivative is non-zero there.

Question What if $f'(z_0) = 0$?
Answer Then the magnitude is zero, so, locally, f "squishes' everything to a point.

Examples 2.7
(A) Polynomials functions in z are entire.

(B) $f(z) = 1/z$ is analytic at every no-zero point.

(C) Find $f'(z)$ if $f(z) = \dfrac{z^2}{(z+1)^2}$

(D) Show that $f(z) = Re(z)$ *is nowhere differentiable!* Indeed: think of it geometrically as projection onto the x-axis. Choosing Δz as a real number gives the difference quotient equal to 1, whereas choosing it to be imaginary gives a zero difference quotient. Therefore, the limit cannot exist!

Cauchy-Riemann Equations

[†] Evidently not worth mentioning by the textbook

If f: $D \to \mathbb{C}$, write $f(z) = f(x, y)$ as $u(x, y) + iv(x, y)$

Theorem 2.8 (Cauchy-Riemann Equations)

If f: $D \to \mathbb{C}$ is analytic, then the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ all exist, and satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Conversely, *if* $u(x, y)$ and $v(x, y)$ are have continuous first-order partial derivatives in D and satisfy the Cauchy-Riemann conditions on D, then f is analytic in D with

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

Note that the second equation just above says that

$f'(z)$ is the complex conjugate of the gradient of $u(x, y)$

Proof Suppose f: $D \to \mathbb{C}$ is analytic. Then look at the real and imaginary parts of $f'(z)$ using $\Delta z = \Delta x$, and $\Delta z = i\Delta y$. We find:

$$\Delta z = \Delta x: \qquad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\Delta z = i\Delta y \qquad f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Equating coefficients gives us the result.

Proving the converse is beyond the scope of this course. (Basically, one proves that the above formula for $f'(z)$ works as a derivative.)

Examples

Show that $f(z) = x^2 - y^2 i$ is nowhere analytic.

Now let us fiddle with the CR equations. Start with

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and take $\partial/\partial x$ of both sides of the first, and $\partial/\partial y$ of the second:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$$

Combining these gives

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 1}$$

u is harmonic

Similarly, we see that v is harmonic. A pair u, v of harmonic functions that also satisfy C-R are called **conjugate harmonic functions**.

Example

Let $u(x, y) = x^3 - 3xy^2 - 5y$. Show that u is harmonic, and find a conjugate for it.

Example 2.9 Write $f(z) = 1/z$ in this form.

Exercise Set 2
p. 806, #1, 5, 9, 15, 19, 21, 23, 25, 31, 35
p. 810 #1, 5, 9, 15, 25, 32

Hand In
1. Using the fact (shown in class) that $f(z) = Re(z)$ is differentiable nowhere, and the formal rules for differentiation but not the C-R condition, deduce each of the following:
(a) f: $\mathbb{C} \to \mathbb{C}$ given by $f(z) = Im(z)$ is differentiable nowhere
(b) f: $\mathbb{C} \to \mathbb{C}$ given by $f(z) = \bar{z}$ is differentiable nowhere.
(c) f: $\mathbb{C} \to \mathbb{C}$ given by $f(z) = |z|^2$ is differentiable nowhere except possibly at zero.

2. Now show that $f(z) = |z|^2$ is, in fact, differentiable at $z = 0$.

3. Transcendental Functions

Definition 3.1. The e**xponential complex function** exp: $\mathbb{C} \to \mathbb{C}$ is given by

$$\exp(z) = e^x(\cos y + i \sin y),$$

for $z = x+iy$. This is also written as e^z, for reasons we saw in the last section.

Properties of the Exponential Function
1. For x and y real, $e^{iy} = \cos y + i \sin y$ and e^x is the usual thing.
2. $e^z e^w = e^{z+w}$
3. $e^z / e^w = e^{z-w}$
4. $(e^z)^w = e^{zw}$
5. $|e^{iy}| = 1$
6. Periodicity: $e^z = e^{z + 2\pi i}$
7. Derivative: $\frac{d}{dz}(e^z) = e^z$.
This follows by either using the Taylor series, or by using the formula

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

Examples 3.2
(a) We compute e^{3+2i}, and e^{3+ai} for varying a.
(b) The geometric action of the exponential function: it transforms the complex plane. Vertical lines go into circles. The vertical line with x-coordinate a is mapped onto the circle with radius e^a. Thus the whole plane is mapped onto the punctured plane.
(c) The action of the function $g(z) = e^{-z}$.

Definition 3.3 Define the **trigonometric sine and cosine functions** by

$$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz})$$
$$\sin z = \tfrac{1}{2i}(e^{iz} - e^{-iz})$$

(Reason for this: check it with z real.) Similarly, we define

$$\tan z = \frac{\sin z}{\cos z},$$

etc.

Examples 3.4
(A) We compute the sine and cosine of $\pi/3 + 4i$

(B) Determine all values of z for which $\sin z = 0$ and $\cos z = 0$.

Properties of Trig Functions

1. Adding $\cos z$ to $i \sin z$ gives **Euler's Formula** $\boxed{e^{iz} = \cos z + i \sin z}$

2. The traditional identities work as usual

$$\sin(z+w) = \sin z \cos w + \cos z \sin w$$
$$\cos(z+w) = \cos z \cos w - \sin z \sin w$$
$$\cos^2 z + \sin^2 z = 1$$

3. **Real and Imaginary Parts of Sine & Cosine**
Some more interesting ones, using (2):

$$\sin(z) = \sin(x + iy) = \sin x \cos(iy) + \cos x \sin(iy)$$

$$\boxed{\sin z = \sin x \cosh y + i \cos x \sinh y}$$

and similarly

$$\boxed{\cos z = \cos x \cosh y - i \sin x \sinh y}$$

4. $\dfrac{d}{dz}(\sin z) = \cos z$ etc.

Definition 3.5 We also have the **hyperbolic sine and cosine,**

$$\cosh z = \tfrac{1}{2}(e^z + e^{-z})$$
$$\sinh z = \tfrac{1}{2}(e^z - e^{-z})$$

Note that $\cosh(iz) = \cos z$, $\sinh(iz) = i \sin z$.

Logarithms
Definition 3.6 A **natural logarithm,** $\ln z$, of z is defined to be a complex number w such that $e^w = z$.

Notes
1. There are *many* such numbers w; For example, we know that $e^{i\pi} = -1$. Therefore,

$$\ln(-1) = i\pi.$$

But, $e^{i\pi + i2\pi} = -1$ as well, therefore,

$$\ln(-1) = i\pi + i.(2\pi)$$

Similarly,

$$\ln(-1) = i\pi +$$

In general, if

$$\ln z = w,$$

then

$$\ln z = w + i(2n\pi)$$

2. We calculate $w = \ln z$ as follows: First write z in the form $z = re^{i\theta}$. Now let $w = u+iv$. Then

$$e^w = z$$

gives $e^{u+iv} = z = re^{i\theta}$.

Thus, $e^u e^{iv} = re^{i\theta}$.

Equating magnitudes and arguments,

$$e^u = r, \ v = \theta,$$

or $u = \ln r, \ v = \theta.$

Thus, **Formula for ln z**

$$\boxed{\ln z = \ln r + i\theta, \quad r = |z|, \ \theta = \arg(z)}$$

3. If θ is chosen as the *principal value* of $\arg(z)$, that is, $-\pi < \theta \le \pi$, then we get the principal value of $\ln z$, called $\text{Ln}\, z$. Thus,

Formula for Ln z

$$\boxed{\text{Ln}\, z = \ln r + i\theta, \ r = |z|, \ \theta = \text{Arg}(z)}$$

$\boxed{\text{Also } \ln z = \text{Ln}\, z + i(2n\pi); \ n = 0, \ \pm1, \ \pm2, \ \dots}$

What about the domain of the function Ln?

Answer: Ln: $\mathbb{C} - \{0\} \to \mathbb{C}$. However, Ln is discontinuous everywhere along the negative x-axis (where $\text{Arg}(z)$ switches from π to numbers close to $-\pi$. If we want to make the Ln continuous, we remove that nasty piece from the domain and take

$$\text{Ln}: \{z \mid z \ne 0 \text{ and } \arg(z) \ne \pi\} \to \mathbb{C}$$

4. $\ln 0$ is still undefined, as there is no complex number w such that $e^w = 0$.

Examples 3.7

(a) $\ln 1 = 0 + 2n\pi i = 2n\pi i$; $\quad \text{Ln } 1 = 0$;

(b) $\ln 4 = 1.386\dots + 2n\pi i$; $\quad \text{Ln } 4 = 1.386\dots$

(c) If r is real, then
$\ln r =$ the usual value of $\ln r + 2n\pi i$; $\quad \text{Ln } r = \ln r$

(d) $\ln i = \pi i/2 + 2n\pi i$; $\quad \text{Ln } i = \pi i/2$;

(e) $\ln(-1) = \pi i + 2n\pi i$; $\quad \text{Ln } (-1) = \pi i$;

(f) $\ln(3-4i) = ln 5 + i \arg(3-4i) + 2n\pi i$
$= \ln 5 + i \arctan(4/3) + 2n\pi i$; $\quad \text{Ln}(3-4i) = \ln 5 + i \arctan(4/3)$.

More Properties

1. $\ln(z\, w) = \ln z + \ln w$; $\ln(z/w) = \ln(z) - \ln(w)$.

This doesn't work for Ln; eg., $z = w = -1$ gives

$$\text{Ln}\, z + \text{Ln}\, w = \pi i + \pi i = 2\pi i,$$

but $\text{Ln}(zw) = \text{Ln}(1) = 0$.

2. $\text{Ln}\, z$ jumps every time you cross the negative $x-$axis, but is continuous everywhere else (except zero of course). If you want it to remain continuous, you must switch to another **branch** of the logarithm. ($\text{Ln} z$ is called the **principal branch** of the logarithm.)

3. $e^{\ln z} = z$, and $\ln(e^z) = z + 2n\pi i$;

$e^{\text{Ln}\, z} = z$, and $\text{Ln}(e^z) = z + 2n\pi i$;

(For example, $z = 3\pi i$ gives $e^z = -1$, and $\text{Ln}(e^z) = \pi i \ne z$.)

Exercise Set 3
p. 817 #1, 3, 5, 11, 13, 17, 21, 23–31 odd, 35, 37, 45
p. 821 #1, 5, 7, 11, 13, 15, 23

Hand In
1. Find functions f that do the following:
(a) Map the region $\{z \mid 0 \leq \arg(z) \leq \pi/2\}$ onto the whole plane
(b) Map the upper half plane to the lower half plane
(c) Maps the second quadrant onto the right-half plane
(d) What happens to the strip $\{x+iy \mid 0 \leq y \leq 1,\ x \geq 0\}$ under the map $f(z) = ie^{-z}$?
2. A **Möbius transformation** is a complex function of the form

$$f(z) = \frac{az + b}{cz + d}.$$

(a) Find a Möbius transformation f with the property that $f(1) = 1$, $f(0) = i$, and $f(-1) = -1$.
(b) Prove that your function is the only possible Möbius transformation with this property. (It is suggested you do some research in the Section 12.9 of the textbook.)

4. Contour Integrals & the Cauchy-Goursat Theorem

(§18.1–18.4 in the text)
A **curve** C in the complex plane $\mathbb{C}$ is a pair of piecewise continuous functions $x = x(t)$, $y = y(t)$ for $a \leq t \leq b$. (This is just a piecewise continuous curve in 2-dimensional space).
Given a curve C in a domain $D \subset \mathbb{C}$ and a function $f: D \to \mathbb{C}$, we can define the corresponding **contour integral,**

$$\int_C f(z)\, dz$$

as the limit of a Riemann sum of the form

$$\Sigma f(z_i{}^*)\Delta z_i$$

associated with a partition $a = t_0 < t_1 \leq \ldots \leq t_n = b$, where the limit is taken as max $\{|\Delta z_i|\} \to 0$. If we write $f(z)$ as $u(x, y) + iv(x, y)$ and dz as $dx + idy$ we obtain

$$\int_C f(z)\, dz = \int_C (u + iv)(dx + idy)$$

$$= \int_C u\, dx - v\, dy + i\int_C u\, dy + v\, dx$$

where the real and imaginary parts are just ordinary path-integrals, as in Calc 3. In fact, if we think of $f(z)$ as a vector field $\langle u, v\rangle$, then

$$\int_C f(z)\, dz = \int_C \overline{\mathbf{f(z)}} \cdot \underline{\mathbf{dr}} - i\int_C \overline{\mathbf{i\,f(z)}} \cdot \underline{\mathbf{dr}}$$

(Note that the is do not cancel since we are thinking of things as vector fields here.)

However, to evaluate it, we need not go so far, but instead stay with complex numbers:

$$\int_C f(z)\, dz = \int_a^b f(z(t))\, z'(t)\, dt$$

where $z(t) = x(t) + iy(t)$. A consequence of this is that, if $f(z)$ has an antiderivative in D, then

$$\oint_C f(z)\, dz = 0$$

over any closed contour C.

Question Why?
Answer Write $f(z) = F'(z)$, and so

$$\oint_C f(z)\, dz = \int_a^b f(z(t))\, z'(t)\, dt = \int_a^b F'(z(t))\, dt$$
$$= F(z(b)) - F(z(a)) = 0$$

since $z(b) = z(a)$ for a closed contour.

Examples 4.1

(A) Evaluate $\int_C \bar{z}\, dz$, where C: $x = 3t,\ y = t^2;\ -1 \le t \le 4$

(B) Evaluate $\oint_C \frac{1}{z}\, dz$, where C is the unit circle centered at the origin, traversed counter-clockwise. To make it easier, use polar coordinates: Write the curve as $z = e^{it}$ with $0 \le t \le 2\pi$. Then $z'(t) = ie^{it}$ and so the integral reduces to

$$\oint_C \frac{1}{z}\, dz = \int_0^{2\pi} e^{-it}\, i\, e^{it}\, dt = 2\pi i$$

Properties of Contour Integrals:

Linearity:	$\int_C [\alpha f(z) + \beta g(z)]\, dz = \alpha \int_C f(z)\, dz + \beta \int_C g(z)\, dz \quad (\alpha, \beta \in \mathbb{C})$
Linearity in C:	$\int_{C\#D} f(z)\, dz = \int_C f(z)\, dz + \int_D f(z)\, dz$

$$\int_{C_{\text{reversed}}} f(z)\,dz = -\int_C f(z)\,dz$$

Bound for Absolute Value:

If $|f(z)| \le M$ everywhere on C, then

$$\left| \int_C f(z)\,dz \right| \le ML$$

where L is the length of C.

A **simple** closed curve is a closed curve with no self-intersections. The domain D is **simply connected** if every loop can be continuously contracted to a point within D. (Illustrations in class)

Theorem 4.2 (Cauchy-Gorsat)
If f is any analytic function defined on the simply connected region D and if C is any simple closed contour in D, then

$$\oint_C f(z)\,dz = 0$$

Sketch of Proof:[1] We first need a little fact:
Fact: Let R be the region interior to a positively oriented simple contour C, together with the points of C itself. Then for any $\varepsilon>0$, R can be covered by a finite number of (partial) squares so that each (partial) square S_i contains a fixed point z_i such that for each $z \in S_i$, one has

$$\left| \frac{f(z) - f(z_i)}{z - z_i} - f'(z_i) \right| < \varepsilon$$

Remarks on why that is true: Certainly, we can cover the region R by infinitely many such squares, and the result now follows by the fact that the region R is **compact.**
Now do a little algebra to write

$$f(z) = f(z_i) + f'(z_i)(z-z_i) + \delta(z)(z-z_i)$$

where k is the expression inside the absolute values above. One therefore has

$$\oint_{S_i} f(z)\,dz = \oint_{S_i} f(z_i)\,dz + \oint_{S_i} f'(z_i)(z-z_i)\,dz + \oint_{S_i} \delta(z)(z-z_i)\,dz$$

However, $f(z_i)$ is a constant, and $\oint_{S_i} 1\,dz$ and $\oint_{S_i} z\,dz = 0$ for any closed contour, since the functions $f(z) = 1$ and $f(z) = z$ posses antiderivatives. . Therefore, we are left with

[1] Don't bother with the textbook's proof -- they only prove a special case by citing Green's theorem, which few instructors have time to prove in calc 2 anyway..

$$\oint_{S_i} f(z)\, dz = \oint_{S_i} \delta(z)(z-z_i)\, dz$$

Now $|\delta(z)| < \varepsilon$, and $|z-z_i| \le \operatorname{diam} S_i$. This gives

$$\begin{aligned}
\left|\oint_{S_i} f(z)\, dz\right| &= \left|\oint_{S_i} \delta(z)(z-z_i)\, dz\right| \\
&\le \varepsilon \times \operatorname{diam} S_i \times \text{length } S_i \\
&\le \varepsilon\sqrt{2}\, s_i \times 4s_i \text{ in the case of squares totally inside } R \\
&\qquad = 4\sqrt{2}\,\varepsilon \times \text{Area of } S_i \\
&\text{or} \le \varepsilon \times \sqrt{2}\, s_i \times [s_i + \text{length } (C_i)] \\
&\qquad = \sqrt{2}\,\varepsilon(\text{Area of } S_i + s_i \text{Length}(C_i)]
\end{aligned}$$

where s_i = length of an edge in S_i and C_i is the portion of C inside S_i.

Adding these up gives a total not exceeding

$$4\sqrt{2}\,\varepsilon \times \text{Total area of } R + \sqrt{2}\,\varepsilon \times \text{Total area of } R + \sqrt{2}\,\varepsilon(S \times \text{Length(C)}]$$

where S is the length of some square that totally encloses R. Now, since is arbitrarily small, we are done.

Consequences:

1. If f is analytic throughout a simply connected region R containing two non-intersecting contours C and D with the same endpoints, then

$$\int_C f(z)\, dz = \int_D f(z)\, dz$$

2. If R is any old region (not necessarily simply connected) and C and D are closed simple contours with C enclosing D, such that the region in between C and D is simply connected, then

$$\oint_C f(z)\, dz = \oint_D f(z)\, dz$$

3. If C is a closed contour (not necessarily simple)) lying inside a simply connected domain D, and f is analytic on D, then

$$\oint_C f(z)\, dz = 0$$

(We show this for the case of finitely many self-intersection points).

4. If f is analytic throughout a simply connected domain D, then f has an antiderivative in D. (We construct the antiderivative by brute force.)

Examples

(A) $\oint_C e^z \, dz = 0$ for any old closed curve C.

(B) $\oint_C \frac{dz}{z^2} = 0$ for any closed curve C not including 0.

(C) $\oint_C \frac{dz}{z} = 2\pi i$ for every simple contour enclosing 0. (Consequence 2)

(D) $\oint_C \frac{dz}{z-\zeta} = 2\pi i$ for any simple closed contour about ζ. We can evaluate this using Consequence 2 and taking C to be the circle $\zeta + e^{it}$.

(E) $\oint_C \frac{dz}{(z-\zeta)^n} = 0$ if n is any integer other than 1. (Evaluate it directly for a circle).

(F) Evaluate $\oint_C \frac{5z + 7}{z^2 + 2z - 3} \, dz$ where C is the circle $|z-2| = 2$ (Use partial fractions)

In general, we have

Consequence 5. if f is not defined at $z_1, \ldots, z_k$, and C is a simple contour surrounding them all, then

$$\oint_C f(z) \, dz = \oint_{C_1} f(z) \, dz + \ldots + \oint_{C_k} f(z) \, dz$$

where the C_i are simple contours around the z_i.

Example Apply this to $\oint_C \frac{1}{1+z^2} \, dz$ where C is the circle $|z| = 3$.

Exercise Set 4
p. 832 #1–7 odd, 17, 23, 29
p. 837 # 1, 5, 9, 11, 13, 15
p. 842 # 1, 3, 5, 7, 11, 21

5. Cauchy's Integral Formula

This theorem gives the value of an analytic function at a point in terms of its values in a contour surrounding that point.

Theorem 5.1 (Cauchy's Integral Formula)
Let f be analytic on simply connected D, let $z_0 \in D$ and let C be any simple closed path in D around z_0. Then.

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$$

Proof The trick is replace $f(z)$ by the constant $f(z_0)$. So:

$$f(z) = f(z_0) + f(z) - f(z_0).$$

The integrand becomes

$$\frac{f(z)}{z - z_0} = \frac{f(z_0) + f(z) - f(z_0)}{z - z_0}$$

$$= \frac{f(z_0)}{z - z_0} + \frac{f(z) - f(z_0)}{z - z_0}$$

The integral of the first term is $2\pi i f(z_0)$ by Example (D) of the previous section. This will give us the result if we can show that the integral of the second term is zero. By Consequence 3, we can use a circle about z_0 as small as we like. Choose $\varepsilon > 0$ as small as you like. Since f is analytic, we have

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) + f'(z_0)$$

Since the integral of the constant term $f'(z_0)$ is zero, we are left with the integral of

$$\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)$$

whose magnitude is less than ε for z sufficiently close to z_0 (which we can assume by choosing a small enough circle). Therefore

$$\left| \oint_C \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| \leq \varepsilon \times \text{Length of } C < 2\pi\varepsilon$$

(the circle can be assumed to have a radius smaller than 1..) Since ε is arbitrarily small, the given integral must be zero, and we are done.

Examples 5.2

(A) Evaluate $\oint_C \frac{e^z}{z-2}\, dz$, where z is any circle enclosing 2.

(C) Evaluate $\oint_C \frac{\tan z}{z^2-1}$ dz where C is any simple contour enclosing 1 but non of the points ±π/2, ±3π/2, ...

(D) Evaluate $\oint_C \frac{z}{z^2 + 9}\, dz$ where C is the circle $|z - 2i| = 4$.

[To evaluate this, rewrite the integrand as $\frac{z/(z+3i)}{z-3i}$.]

Corollary 5.3 (Analytic Functions have Derivatives of All Orders)
Let f be analytic on simply connected D, let $z_0 \in D$ and let C be any simple closed path in D around z_0. Then $f^{(n)}(z_0)$ exists, and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

Proof: Let us start with $n = 1$: Write

$$f'(z_0) = \lim_{w \to \square} \frac{f(w) - f(z_0)}{w - z_0}$$

Applying the Integral Formula theorem to each term gives:

$$f(w) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - w}\, dz \qquad \text{and} \qquad f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$$

Combining them gives

$$f(w) - f(z_0) = \frac{1}{2\pi i} \oint_C f(z) \frac{w - z_0}{(z - w)(z - z_0)}\, dz$$

Noting that the term $w - z_0$ is constant, and dividing by it gives

$$\frac{f(w) - f(z_0)}{w - z_0} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - w)(z - z_0)}\, dz$$

Now the integrand is a continuous function of w, so letting $w \to z_0$ gives

$$f'(z_0) = \lim_{w \to \square} \frac{f(w) - f(z_0)}{w - z_0} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2}\, dz,$$

showing the case for $n = 1$. To show the proof for $n = 2$, use the same technique as for $n = 1$, except that we use the formula for $n = 1$ instead of the Cauchy integral formula. Then continue the proof inductively.

Corollary 5.4 (An important Inequality)

$$|f^{(n)}(z_0)| \le \frac{n!M}{r^n} \quad \text{for all } n \ge 0$$

where M is an upper bound of $|f(z)|$ on a circle centered at z_0 with radius r.

Proof:

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \right| = \frac{n!}{2\pi} \left| \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \right|$$

But, for z on C,

$$\left| \frac{f(z)}{(z - z_0)^{n+1}} \right| \le \frac{M}{|z - z_0|^{n+1}} = \frac{M}{r^{n+1}}$$

where r is the radius of the circle C. Therefore

$$\frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right| \le \frac{n!}{2\pi}\frac{M}{r^{n+1}}\,2\pi r = \frac{n!M}{r^n}$$

as required.

Corollary 5.5 (Louville's Theorem)
Entire bounded functions are constant.

Proof: S'pose that f is bounded on the entire complex plane, so that $|f(z)| \le K$ for some constant K. We now use the case $n = 1$ of the above theorem, giving

$$|f'(z_0)| \le \frac{K}{r}$$

where r is the radius of an arbitrary circle with center z_0. Since r is arbitrarily large, it must be the case that $f'(z_0) = 0$. Since this is true for every $z_0 \in \mathbb{C}$, it must be the case that $f(z) = $ constant. (If $f'(z) = 0$, then the partial derivatives of u and v must all vanish, and so u and v are constant.)

$$\le \frac{n!}{2\pi}\oint_C \frac{M}{|z-z_0|^{n+1}}\,dz$$

The integrand is now constant, since $|z - z_0| = r$, the radius of the circles. Therefore, the integral on the right boils down to

$$\frac{n!M}{2\pi r^n}\oint_C dz$$

Corollary 3 (Fundamental Theorem of Algebra)
Every polynomial function of a complex variable has at least one zero.

Proof S'pose $p(z)$ is a polynomial with no zeros. Then $f(z) = \dfrac{1}{p(z)}$ is entire. But it is also bounded, since $|f(z)| \to 0$ as $|z| \to \infty$. Thus, $f(z)$—and hence $p(z)$—are constant; a contradiction.

Exercise Set 5
p. 848 #1, 3, 7, 11, 15, 23

We now skip to Chapter 20

6. Conformal Mappings

Definition 6.1 A mapping f: $D \to \mathbb{C}$ is called **conformal** if it preserves angles between curves.

Theorem 6.2 If f is analytic, then f is conformal at all points where $f'(z) \ne 0$.

Proof. If C is any curve in D through z_0, we show that f rotates its tangent vector at z_0 through a fixed angle. First think of C as being represented by $z = z(t)$. The derivative, $z'(t)$, in vector form, evaluated at $z_0 = z(t_0)$ is tangent there, and the angle it makes with the x-axis is given by its argument. The image curve f_*C, is given by $z = f(z(t))$. The tangent vector to any path $z = z(t)$ is its derivative with respect to t, thought of as a vector, rather than a complex number. Therefore, the tangent to f_*C at z_0 is given by $f'(z(t_0))z'(t_0)$, and its angle is its argument, given by

$\arg f'(z_0) + \arg z'(t_0)$

= Angle independent of the path through z_0 + Angle of original tangent.

Done.

Question What happens when $f'(z) = 0$?
Answer Looking at the above argument, we find that the tangent vector at the image of such a point is the zero vector, and so we can say nothing about the direction of the path at that point—anything can happen.

Examples 6.3
(A) $f(z) = z + b$, or $w = z + b$ Translation by b.

(B) $f(z) = az$, or $w = az$ Expansion/Contraction + Rotation

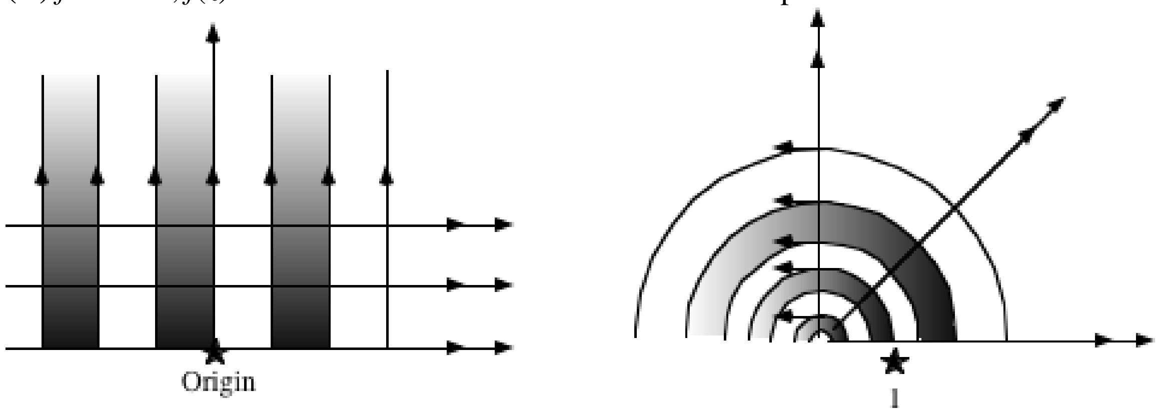
If $a = r$ is real, we get expansion or contraction. If $a = e^{i\theta}$ we get rotation by θ. Therefore, in general, we get a composite of the two.

(C) $f(z) = az + b$ or $w = az + b$ Affine: A combination of all 3

This is the stuff of geometry.

Note that, in geometry, two objects in the plane are congruent iff one can be obtained from the other using an affine transformation.

(D) $f: \mathbb{C} \to \mathbb{C}$; $f(z) = e^z$. Here is a better illustration than that pathetic one in the book:

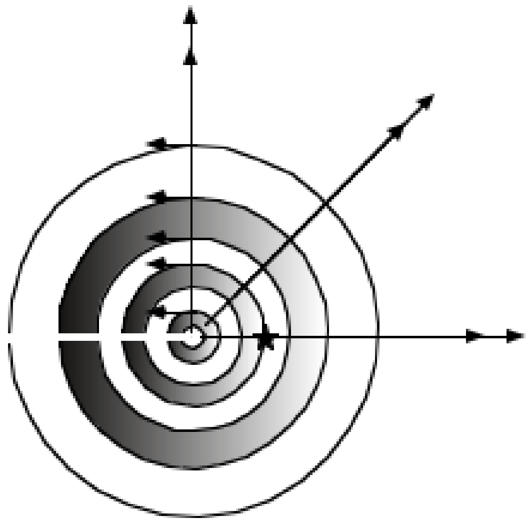


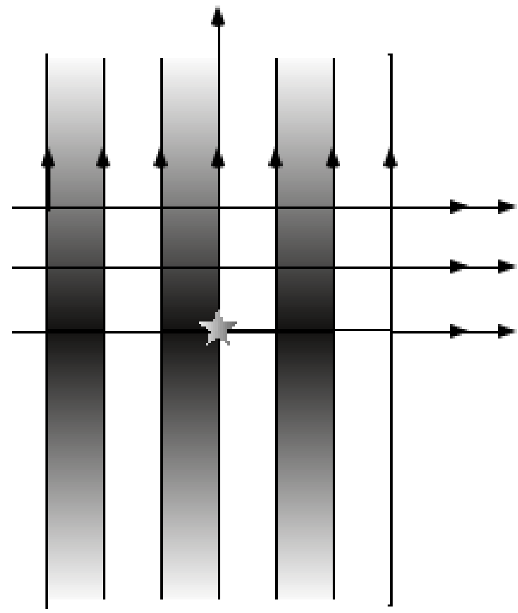
Vertical lines $\to$ circles Horizontal lines $\to$ rays

(E) What about the inverse mapping, $\mathrm{Ln}(z)$? Recall that

$\mathrm{Ln}: \{z \mid z \neq 0 \text{ and } \arg(z) \neq \pi\} \to \mathbb{C}$

Think of it as the above map in reverse: The above picture on the right shows the top half the domain, and we get:



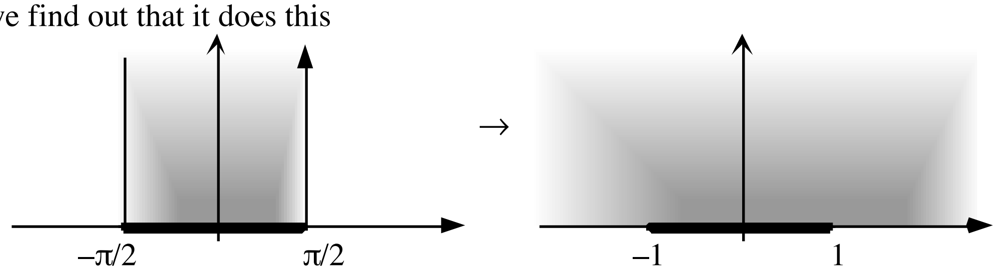


(F) $f: \mathbb{C} \to \mathbb{C}$; $f(z) = \sin z$

For this it is useful to remember that

$$f(x + iy) = \sin x \cosh y + i \cos x \sinh y$$

and we find out that it does this



the next block over $(\pi/2 \leq x \leq \pi)$ goes underneath the axis, and then it repeats as we go across the left-hand

(G) $f: \mathbb{C} - \{0\} \to \mathbb{C}$; $f(z) = \dfrac{1}{z}$ or $w = \dfrac{1}{z}$.

Look at what happens to the general point $z = x + iy$

$$w = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = u + iv$$

A vertical line in the w-plane corresponds to $u = k$

$$\frac{x}{x^2 + y^2} = k, \text{ a constant}$$

But this is the equation to a circle For instance, taking $k = \frac{1}{2}$ gives the circle center $(1, 0)$ radius 1. In general, all these circles pass through the origin (where f is not defined)., since the above equation, when cross-multiplied, is satisfied by $(0, 0)$.

Similarly, horizontal lines also correspond to circles, but this time centered on the y-axis.

In general, we have the following:

Proposition 6.4 The transformation $w = 1/z$ takes circles or straight lines to circles or straight lines.

Proof One can represent circles and straight lines by

$$A(x^2+y^2) + Bx + Cy + D = 0$$

Now $x^2 + y^2 = z\bar{z}$, and $x = (z+\bar{z})/2$, $y = (z-\bar{z})/2i$. So the above equation can be rewritten as

$$Az\bar{z} + \frac{B(z+\bar{z})}{2} + \frac{C(z-\bar{z})}{2i} + D = 0$$

Now write this in terms of $w = 1/z$. Substituting $z = 1/w$, $\bar{z} = 1/\bar{w}$ and multiplying by $w\bar{w}$ gives us

$$A + \frac{B(w+\bar{w})}{2} - \frac{C(w-\bar{w})}{2i} + Dw\bar{w} = 0$$

or

$$A + Bu - Cv + D(u^2 + v^2) = 0,$$

again the equation of a circle or straight line. □

More generally:

Theorem 6.5 Every map of the form $f(z) = \dfrac{az + b}{cz + d}$ takes circles or straight lines to circles or straight lines

Proof We can manipulate $f(z)$ to write it in the form

$$f(z) = A\left[1 + \frac{B}{c + d/z}\right]$$

which is a composite affine maps and inversions.

Continuing with the examples..

(G) $f(z) = z^2$ is conformal everywhere except at the origin. In fact, it doubles angles at the origin.

Some reverse ones:

Examples
(A) Find a complex function that maps the upper half plane into the wedge $0 \le \text{Arg } z \le \pi/4$.
(B) Ditto for the Strip $0 \le y \le \pi \to$ Wedge $0 \le \text{Arg } w \le \pi/4$. (Look at the exponential map.)

Exercise set 6
p. 893 # 1–13 odd, , 21–27 odd
p. 900 #1, 3, 11, 13, 15, 17

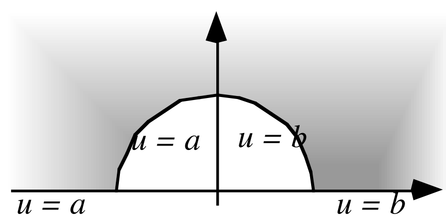
Hand-In:
1. Find an analytic complex function that maps the interior of the unit disc centered at (0, 0) onto the interior of the first quadrant. [Use composites of the conformal mappings in the Appendix of the book.] (1) Translate the center to z = 1 (2) apply 1/z (mapping in

onto the right of the vertical line $x = 1/2$ (3) Translate by adding $-1/2$ and rotate through $\pi/2$, giving the top half of the plane. (4) Take the square root.
2. p. 894 # 31. Joukowski airfoil [Hint for (b): Start with the given equation in the w-plane, and substitute for u and v to reduce it to the equation of a circle.]

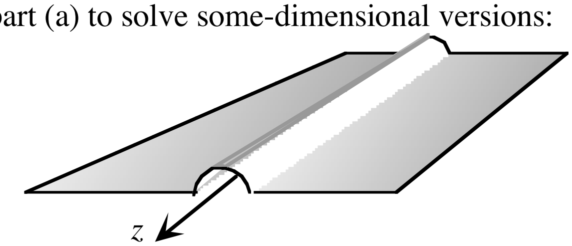
7. More on Conformal Mappings and Harmonic Functions

Question What use are these quaint conformal mappings?
Answer We can use then to solve the 2-dimensional Dirichlet problem with complicated boundary conditions. This, in turn, can be used to solve the 3-dimensional one. Recall that the steady sate heat equation with given boundary conditions is just Dirichlet's problem. We sill look at some examples to illustrate this

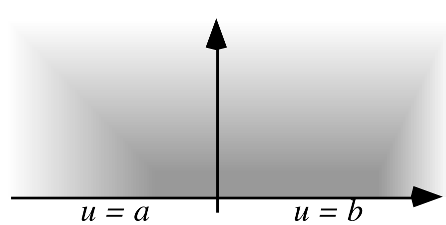
Example (A) Solve the two-dimensional heat equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for u (the temperature) specified as in the following figure. (u is actually the temperature.)



(b) Use the result of part (a) to solve some-dimensional versions:



Solution
(a) Solving it directly would be a nightmare—in fact none the usual methods would be at all tractable. Therefore, we use the appendix to transform this region into a simpler one, and we find that the map $w = z + 1/z$ maps this into the upper-half place taking the boundary of the above region onto the x-axis. and gives us the following region in the w-plane:



Now, we can solve the Dirichlet problem for the w-region: It is radially symmetric, and Dirichlet's problem in radial coordinates is:

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

So, any linear function in θ will work, like $u = b + (a-b)\theta/\pi$. In complex notation, this is

$$U(w) = b + (a-b)\mathrm{Arg}(w)/\pi$$

Now notice that $\mathrm{Arg}(w)$ is the imaginary part of the analytic function $f(z) = \mathrm{Ln}\, z$ (which is another reason that it is satisfies Laplace's equation). So, let us take

$$F(w) = b + \frac{(a-b)}{\pi}\mathrm{Ln}(w)$$

Since $w = z + 1/z$, we have

$$F(z) = b + \frac{(a-b)}{\pi}\mathrm{Ln}(z + 1/z)$$

its imaginary part is a function of x and y that satisfies the original equation.

(b) If u on the boundary is independent of z, then the same solution (independent of z) will suffice for the 3-dimensional solution. If, on the other hand, a and b above are linear functions of z, then if we simply substitute them in the above formula, and notice that the imaginary part is linear, we get $u_{zz} = 0$ as well.

It would be nice not to have to rely so much on tables for our work, and for this, we specialize to **Linear Fractional Transformations**. These have the form

$$w = \frac{az + b}{cz + d} \qquad \text{LFT}$$

where a, b, c, and d are complex constants. For it to be conformal, we need to ensure that its derivative is non-zero and exists. This amounts to two conditions:

Condition 1: $ad - bc \neq 0$
Condition 2: $z \neq -d/c$

We now look at what happens to regions of the z-plane under these transformations. First note that we can divide top & bottom by a or b (depending on which one is nonzero) and thereby eliminate one of these constants. Thus there are only three constants in the formula. This suggests that if we know where we want to map three points, we can plug them in and solve for the constants uniquely. In other words, we can always find an LFT that takes any three points to any other three points.

Note: S'pose we want the LFT to take $z_1 \to w_1$, $z_2 \to w_2$ and $z_3 \to w_3$. Consider the LFT:

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} \qquad (*)$$

Since plugging in the values (z_i, w_i) make it hold, it must be the one we're after.

Examples
(A) Mapping the upper half-plane onto the unit disc.
Since we need only say what happens to three points, we shall choose them to be $z = -1$, 0 and 1 on the real axis. Since these points are on the boundary of the half-plane, they must be mapped to points on the boundary of unit disc, and we let $-1 \to -1$, $0 \to -i$, $1 \to 1$ under the mapping. Substituting in (*) and solving for w gives:

$$w = \frac{z - i}{-iz + 1}$$

Question Why does it do what we claimed?
Answer We use the fact that lines or circles go to lines or circles.
1. Because of what happens to the three points, we know that the real axis $\to$ unit circle.

2. By continuity, nearby parallel lines must also go to (nearby) circles.
3. We know $0 \to -i$. Also, we can check that $i \to 0$. Thus the positive imaginary axis goes to the line starting at $-i$ and going up.
4. By 2 & 3, lines above the real axis must go to circles inside the given circle.
5. Since ∞ goes to i,(look at the highest powers of z) all these circles must touch the point i.

(B) ***Mapping the unit disk into the right-half plane***
Here, we choose $-1 \to 0$, $i \to i$ and $1 \to \infty$ Looking at (*), we get

$$\frac{(w\ )(i-\infty)}{(w-\infty)(i)} = \frac{(z+1)(i-1)}{(z-1)(i+1)}$$

To evaluate the left-hand side, we treat it as a limit:

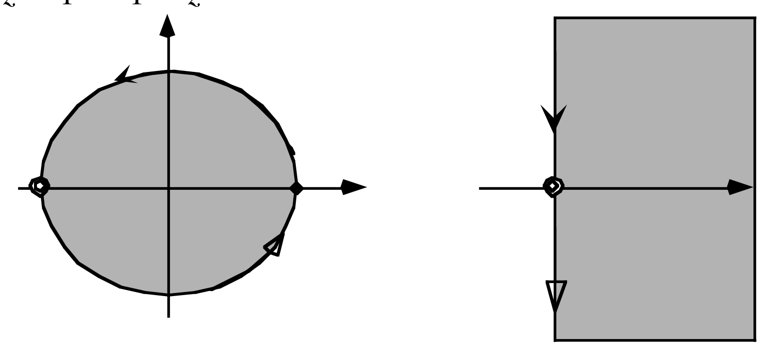
$$\lim_{z \to \infty} \frac{i-z}{w-z} = 1,$$

so we get

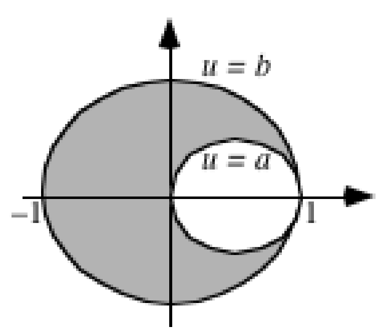
$$\frac{w}{i} = \frac{(z+1)(i-1)}{(z-1)(i+1)} = \frac{i(z+1)}{z-1}$$

giving

$$w = -\frac{z+1}{z-1} = \frac{1+z}{1-z}$$



(C) Mapping a moon-shaped region into the top-half plane
Using a map into the top-half plane, solve the following Dirichlet problem:



Solution The easiest is to map the given region into a horizontal strip $0 \le y \le 1$ by sending the inner circle to the x-axis and the outer circle to the line $y = 1$. this means sending the point 1 to ∞. Let us therefore take $1 \to \infty$, $0 \to 0$, and $-1 \to i$.‡ Using (*), we get

‡ For some inexplicable reason, the textbook does something more complicated, requiring a lot more algebra to deal with

$$\frac{(w - \infty)(0 - i)}{(w - i)(0 - \infty)} = \frac{(z - 1)(0 + 1)}{(z + 1)(0 - 1)}$$

Taking limits and simplifying gives

$$\frac{-i}{w - i} = \frac{z - 1}{-(z + 1)}$$

Solving for w gives

$$w = \frac{2iz}{z - 1}.$$

We can also check that $\pm i \to i \pm 1$.

We now solve the Dirichlet problem for this. In the horizontal strip in the w-plane, it is the real-valued function given by

$$U(w) = a + (b-a)Im(w)$$

This is the imaginary part of

$$F(w) = a + (b - a)Im(w)$$

So

$$F(z) = a + (b - a)Im\left[\frac{2iz}{z - 1}\right]$$

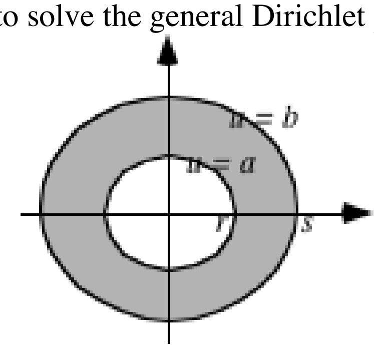
solves the Dirichlet problem.

Exercises Set 7
p. 907 #9, 11, 18

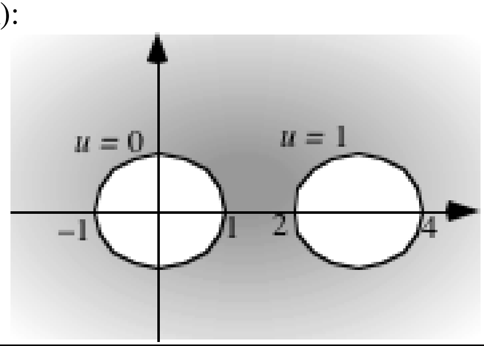
Hand-In

1. Express the solution in Example (C) in terms of x and y, verify directly that it satisfies Laplace's equation , and check that it has the given boundary values given on three different boundary points.

2. Use a conformal mapping to solve the general Dirichlet problem on the annulus:



3. Use a conformal mapping to solve the following Dirichlet problem (see Conformal Mapping #C1 in the book):



8. Poisson Integral Formula

We saw that, once we know a potential on the upper half plane, we can find it for any region. So now, the question is to solve Dirichlet's problem on the upper half plane.

> **Theorem (Poisson Integral Formula for *H*)**
> The (unique) potential $u(x, y)$ on the upper half plane with $u(x, 0) = f(x)$ is
>
> $$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2 + y^2}\, dt$$

Sketch of Proof

We prove it first for a simple step function of the form $f(x) = \begin{cases} 1 & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$.

For this simple kind of function, we take

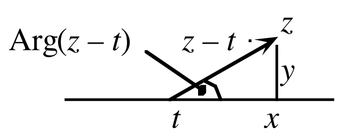
$$u(x, y) = \frac{1}{\pi}[\mathrm{Arg}(z - b) - \mathrm{Arg}(z - a)] = \frac{1}{\pi}\mathrm{Arg}\left[\frac{z - b}{z - a}\right] \qquad \text{(I)}$$

(Note that, since we are on the upper half plane, all arguments are between 0 and π, and the above equality holds because none of the angles in question cross zero.)
The reason this works is:
(1) u is the imaginary part of an analytic function, so that u does satisfy Laplace's equation.
(2) For z on the real axis outside of $[a, b]$, $(z-b)/(z-a)$ is a real positive number, and so its argument is zero.
(3) For z on the real axis between a and b, $(z-b)/(z-a)$ is a real negative number, and so its argument is π.

Now $\quad u(x, y) = \frac{1}{\pi}[\mathrm{Arg}(z - b) - \mathrm{Arg}(z - a)]$

$$= \frac{1}{\pi}\int_a^b \frac{d}{dt}[\mathrm{Arg}(z - t)]\, dt$$

Now we use a little trick:



$$\mathrm{Arg}(z - t) = \tan^{-1}\left[\frac{y}{x - t}\right]$$

Therefore,

$$u(x, y) = \frac{1}{\pi}\int_a^b \frac{d}{dt}\tan^{-1}\left[\frac{y}{x - t}\right] dt$$

$$= \frac{1}{\pi}\int_a^b \frac{y}{(x-t)^2 + y^2}\, dt$$

$$= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2 + y^2}\, dt \ ,$$

because of the definition of $f(t)$. Therefore, the formula works for this simple function. BUT:

(1) Potential functions are additive, as are integrals

(2) Any function can be approximated arbitrarily closely by a linear combination of steps functions.

Therefore, it works for all integrable functions. QED.

Example Solve Dirichlet's problem on H with $u(x, 0) = \begin{cases} x & \text{if } -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

Solution We get $u(x, y) = \displaystyle\int_{-1}^{1} \frac{t}{(x-t)^2 + y^2}\, dt$

Substituting $s = x - t$ transforms this to

$$u(x, y) = \int_{x-1}^{x+1} \frac{x - s}{s^2 + y^2}\, ds$$

$$= \frac{1}{\pi}\left[\frac{y}{2}\ln[(x-t)^2+y^2] - x\tan^{-1}\left(\frac{x-t}{y}\right)\right]_{-1}^{1}$$

$$= \frac{y}{2\pi}\left[\ln\left(\frac{(x-1)^2+y^2}{(x+1)^2+y^2}\right)\right] + \frac{x}{\pi}\left[\tan^{-1}\left(\frac{x+1}{y}\right) - \tan^{-1}\left(\frac{x-1}{y}\right)\right]$$

Note

We can also use this method for regions other than H. Take a conformal map onto H from another region, see what it does on the boundary, solve it on H as above, and then compose it with the conformal map to get the solution (see the homework)

Exercise Set 8

p. 916 #1, 3 (Express $u(x, 0)$ as a sum of step functions and use (I) on each one (rather then doing the integral computation)

#5, 6,

Hand In:

p. 916 #7 (use z^2) and #8 (use H-1)

Change-of Reference: At this point, we abandon Zill’s book (since it is inadequate) and go to Erwin Kreyzsig, *Advanced Engineering Mathematics,* 8th Edition, Wiley.

9. Complex Potentials (Based on Kreyszig)

The electrostatic force of attraction between charged objects is the gradient of an **electrostatic potential function** Φ that satisfied Laplace's equation.

Examples

(A) Find the potential Φ between two parallel plates extending to infinity, which are kept at potentials Φ_1 and Φ_2 respectively.
Solution: This is just Dirichlet's problem. If the parallel plates are vertical, we can take the vertical axis to be the y-axis with the lower plate $x = 0$, and take

$$\Phi = \Phi_1 + \frac{x}{h}(\Phi_2 - \Phi_1)$$

where h is the distance between the plates. A **complex potential** function corresponding to the real potential function $\Phi(z)$ is an analytic function $F(z) = \Phi(z) + i\Psi(z)$. Notice that Φ and Ψ are **conjugate harmonic functions**.

In this example, the associated complex potential function is

$$F(z) = \Phi_1 + \frac{z}{h}(\Phi_2 - \Phi_1)$$

The complex conjugate of Φ is therefore

$$\Psi = Im(F(z)) = \frac{y}{h}(\Phi_2 - \Phi_1)$$

In short:

For parallel plates, the complex potential $F(z) = Az + B$ is just a linear function of z (A, B real)

(B) Potential between two Coaxial Cylinders
Here we need to solve Laplace's equation in the xy-plane using polar coordinates:

$$\nabla^2\Phi = \Phi_{rr} + \frac{1}{r}\Phi_r + \frac{1}{r^2}\Phi_{\theta\theta} = 0$$

Since Φ depends only on r by symmetry, we are reduced to

$$\nabla^2\Phi = \Phi_{rr} + \frac{1}{r}\Phi_r = 0$$

We can write this as

$$\frac{\Phi_{rr}}{\Phi_r} = -\frac{1}{r}$$

or $\quad \dfrac{d}{dr}\ln(\Phi_r) = -\dfrac{1}{r}$

Thus $\quad \ln(\Phi_r) = -\ln(r) + K = \ln(A) - \ln(r)$, say

so $\quad \Phi_r = \dfrac{A}{r}$

giving $\Phi = A\ln(r) + B$
We can now solve for A and B by knowing the potentials on each of the two cylinders and their radii.

For the associated complex potential, we use that fact that $\ln(r)$ is the real part of $\mathrm{Ln}(z)$, and so

$$F(z) = A\mathrm{Ln}(z) + B$$

with associated conjugate potential

$$\Psi(z) = A\,\mathrm{Arg}(z)$$

For a circular symmetry situation (potential independent of θ), the complex potential $F(z) = A\mathrm{Ln}(z) + B$ is just a linear function of $\mathrm{Ln}(z)$.

(C) Potential in an Angular Region

Here we have two plates at an angle α with a different potential at each plate. This time, Laplace's equation depends only on θ and is therefore $\Phi_{\theta\theta} = 0$, meaning that Φ is a linear function of θ; that is,

$$\Phi = A\theta + B$$

We can now solve for A and B by knowing the potentials on each of the two plates, and the angle between them.

The associated complex potential can be found by rewriting the above as

$$\Phi = A\,\mathrm{Arg}(z) + B$$

$A\,\mathrm{Arg}(z)$ is the imaginary part of $A\mathrm{Ln}(z)$ or the real part of $-iA\mathrm{Ln}(z)$. Thus Φ is the real part of

$$F(z) = -iA\mathrm{Ln}(z) + B$$

and so the associated conjugate potential is the imaginary part:

$$\Psi(z) = -A\ln(|z|)$$

For a symmetry that depends only on θ, the complex potential $F(z) = -i\,ALn(z) + B$ is just a linear function of $-iLn(z)$ (A, B real)

(D) Using Superposition: Potential due to two oppositely charged parallel wires normal to the complex plane

The complex potential for a *single* wire at the origin is given by Example (B):

$$F(z) = A\mathrm{Ln}(z) + B$$

where we have $B = 0$ (since the potential is 0 at infinity. Also, the potential is undefined at the origin, since the potential is infinite on the actual wire. We can obtain A by measuring at the potential close to the wire, or by doing a brute force integration using the electrostatic force law and the charge on the wire.

By superposition, if we now have two wires located at $z = c$ and $z = -c$, and oppositely charged (positive at c and negative at $-c$), we obtain

$$F(z) = A[\mathrm{Ln}(z-c) - \mathrm{Ln}(z+c)]$$

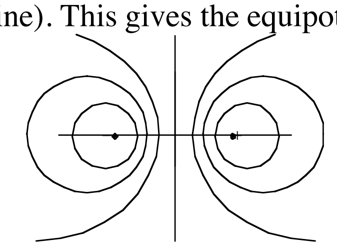
This gives the real part (actual potential) as

$$\Phi(z) = A\ln\left|\frac{z-c}{z+c}\right|$$

The equipotentials are therefore given by

$$\left|\frac{z-c}{z+c}\right| = \text{constant}$$

If we use a little coordinate geometry, and find all (x, y) whose distance from one point = constant time its distance from another, we get the equation of a circle (or in one special case when the constant is 1, a line). This gives the equipotential lines as shown:



Question What is the significance of the *conjugate* potential?

Answer. Since Φ and Ψ are conjugate,

$$\frac{\partial\Phi}{\partial x} = \frac{\partial\Psi}{\partial y} \qquad \text{and} \qquad \frac{\partial\Phi}{\partial y} = -\frac{\partial\Psi}{\partial x}$$

In other words, $\underline{\nabla}\Phi$ and $\underline{\nabla}\Psi$ are orthogonal. However, since these gradients are themselves orthogonal to the lines Φ = const and Ψ = const, we see that the lines Ψ = constant are at right angle to the equipotentials. Put another way:

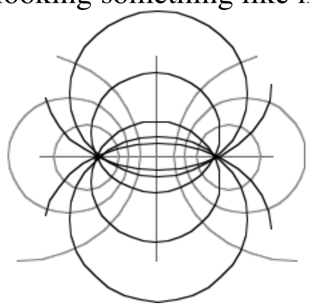
The lines Ψ= const are *parallel* to $\underline{\nabla}\Phi$

and are therefore *lines of force* (showing the direction of the force)

Looking at the example we just did, the lines of force are given by

$$\Psi(z) = A[\text{Arg}(z-c) - \text{Arg}(z+c)] = \text{Const}$$

In the homework, you will see that these too are circles (except for the one degenerate case when the constant is zero), looking something like magnetic lines of force:



(In fact the are the same thing...)

Using the Complex Potential to get the Electrostatic Field

We know that we can recover the electrostatic field by just taking the gradient of Φ:

$$\underline{E} = \underline{\nabla}\Phi$$

However,

$$\underline{\nabla}\Phi = \left\langle\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}\right\rangle = \left\langle\frac{\partial\Phi}{\partial x}, -\frac{\partial\Psi}{\partial x}\right\rangle$$

$$= \frac{\partial\Phi}{\partial x} - i\frac{\partial\Psi}{\partial x} \qquad \text{In complex notation}$$

$$= \overline{F'(z)}$$

where F is the complex potential. Conclusion

Conservative Vector Fields and Complex Potentials (Not in Kreyzsig)
If $\underline{E}$ is a conservative field independent of the ζ-coordinate (or in the complex plane) then

$$\underline{E} = \overline{F'(z)},$$

where $F(z)$ is the associated complex potential.

Example

Find the electric field corresponding to Example (D).

Notes

1. ***The Third Dimension*** In all of the above, we take the third coordinate to be the $z-$coordinate, which we cannot call z for obvious reasons! So, I suppose we can call it ζ, and write

$$\Phi(x, y, \zeta) = \text{The same formula for } \Phi \text{ we used in the above examples}$$

since it is independent of ζ.

2. ***Haven't we done this before?*** Earlier, we solved Dirichlet's problem using conformal mappings, but had to first solve it on a simpler region —usually H. Here we are doing it again, from first principles, and interpreting it as electrostatic force. Also, it is good to do things several ways.

Exercise Set 9

1. Find the potential, complex potential, equipotential lines, and lines of force of between two parallel plates at $x = -5$ and $x = 10$ having potentials 200 and 500 volts respectively.

2. Find the potential, complex potential, equipotential lines, and lines of force between two coaxial cylinders with radii 1 and 5 cm with inner cylinder charged to 10 volts and the outer cylinder charged to 100 volts.

3. Find the associated electric fields in each of these cases.

Hand In

1. Repeat 1 of the non-hand in work for plates along $y = 2 - x$ and $y = 4 - x$ with voltages as above.

2. Show that $F(z) = \sin^{-1} z$ may be regarded as the complex potential associated with the two horizontal lines $(-\infty, -1]$ charged with one potential and $[1, \infty)$ charged with another. Sketch some equipotential lines and lines of force. Hence find the associated electric field.

3. Verify the claim that $A[\mathrm{Arg}(z-c) - \mathrm{Arg}(z+c)] = \mathrm{Const}$ are circles. [Hint: Express $\mathrm{Arg}(z-c) + \mathrm{Arg}(z+c)]$ in terms of the angle between $z-c$ and $z+c$]

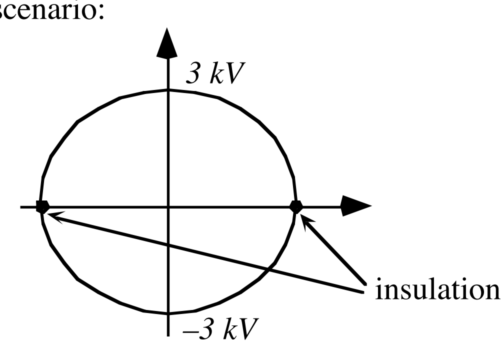
10. Using Conformal Maps to Find Electric Potentials and Fields: Based on Kreyszig's Excellent Book

We know that harmonic functions are the real (or imaginary) part of an analytic function. Therefore, if Φ is harmonic on the upper half-plane; Φ: $H \to \mathbb{R}$, then Φ is the real part of a complex (analytic) potential function F: $H \to \mathbb{C}$. Precomposing this with another analytic map $D \to H$ therefore gives us a complex potential $D \to \mathbb{C}$, and hence a harmonic function $\underline{\Phi}$:$D \to \mathbb{R}$. This is exactly what we were doing two sections ago, and now we do it some more, this time in the context of complex potentials.

Examples

(A) ***Potential between two semicircular plates***

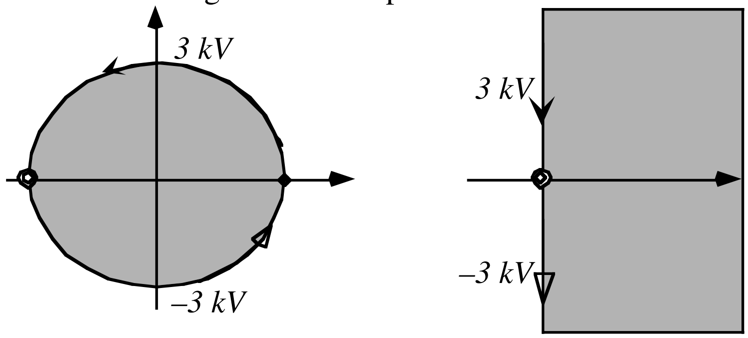
Consider the following scenario:



We would like a conformal mapping sending the disk to H. Without being too demanding, let us go back to Example (B) on p. 23 of these notes, where we saw that

$$w = \frac{1 + z}{1 - z}$$

takes the above disc onto the right-hand half-plane as shown:



So what we need now is a nice potential for the right-hand region. But this is an angular one, so our potential is given by (see the last section)

$$\Phi = A\theta + B$$

$$= \frac{6}{\pi}\text{Arg}(z) \quad \text{(Recall that Arg is fine for the right-hand plane; } -\pi < \text{Arg}(z) \le \pi\text{)}$$

This is the imaginary part of $\frac{6}{\pi}\text{Ln}(z)$ or the real part of $-i\frac{6}{\pi}\text{Ln}(z)$. Thus Φ is the real part of

$$F(z) = -i\frac{6}{\pi}\text{Ln}(z)$$

Transferring this over the left-hand region gives the desired complex potential:

$$G(z) = -i\frac{6}{\pi}\text{Ln}\left[\frac{1+z}{1-z}\right] kV$$

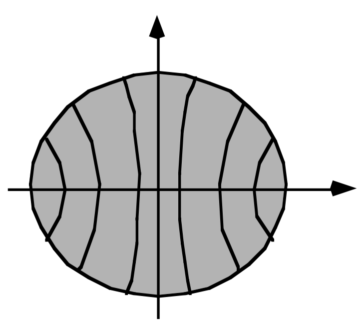
Its real part, $\frac{6}{\pi}\text{Arg}\left[\frac{1+z}{1-z}\right] kV$, is our desired potential function.

Question What are the equipotentials:

Answer Φ = constant iff $\text{Arg}\left[\frac{1+z}{1-z}\right] = \text{Arg}(w) =$ constant. But these are just rays from the origin in the right--hand region. Since these rays extend from 0 to infinity, they must give circular arcs in the left-hand region extending from -1 to 1.

Question What are the lines of force?

Answer Setting the imaginary part of the complex potential equal to constants gives Ψ = constand iff $|w| = const$, giving semicircles centered at the origin on the right-hand side, corresponding to circular arcs, roughly as shown:



Note that the y-axis itself is one of those arc, corresponding to the unit semicircle on the right.

Question What is the vector form of the electric field?

Answer We use $\underline{E} = \overline{F'(z)}$; actually $\overline{G'(z)}$ in this case.

$$G(z) = -i\frac{6}{\pi}\mathrm{Ln}\left[\frac{1+z}{1-z}\right] = -i\frac{6}{\pi}[\mathrm{Ln}(1+z) - \mathrm{Ln}(1-z)]$$

$$G'(z) = \frac{12z}{\pi(1-z^2)}$$

$$\overline{G'(z)} = \frac{12\bar{z}}{\pi(1-\bar{z}^2)}$$

The vector components are then the real and imaginary parts of this.

(B) *Potential between two non-coaxial cylinders*.

Find the potential between two cylinders C_1: $|z| = 1$ being grounded (potential 0) and C_2: $|z - 0.4| = 0.4$ having a potential of 110 volts.

This is hard to solve without some trick: First, consider the general FLT

$$r(z) = \frac{z - z_0}{cz - 1} \quad \text{where } c = \overline{z_0} \text{ and } |z_0| < 1.$$

Then I claim that r maps the unit disc onto itself, but takes z_0 to 0. The latter claim is obvious. Let us check that first claim: Mapping the unit disc onto itself:

$$\begin{aligned} |z| = 1 \Rightarrow |z - z_0| &= |\bar{z} - \overline{z_0}| && \text{Since } |w| = |\bar{w}| \text{ for every } w \\ &= |z|\,|\bar{z} - \overline{z_0}| && \text{Since } |z| = 1 \\ &= |z\bar{z} - z\overline{z_0}| \\ &= |1 - z\overline{z_0}| && \text{Again, since } |z| = 1 \end{aligned}$$

Therefore, $|r(z)| = 1$, as claimed.

Notice one further thing about this strange map: If we choose z_0 to be real; $z_0 = b$, say, then

$$r(z) = \frac{z - b}{bz - 1}$$

and $\quad r(1) = \dfrac{1 - b}{b - 1} = -1$, and also $r(-1) = \dfrac{-2}{-2} = 1$ so that *r flips the unit circle over*.

Notice that, since this is an FLT, circles inside the unit disc must map to circles inside the unit disc (they can hardly map to infinite lines!) and it certainly looks like circles centered at z_0 inside the disc map to circles centered at 0 (look at very small circles, for instance).

Now back to the example at hand: We try to adjust this so that the non-concentric circles are moved onto the concentric circles $|z| = 1$ and $|z| = r$ for some $r < 1$. For this, we take $z_0 = b$, a point somewhere on the x-axis in order to map the off-centered inner circle onto the circle centered at 0 radius r. Since $b = \bar{b}$, we have

$$r(z) = \frac{z - b}{bz - 1}$$

We would also like 0 to map to r and 0.8 to map to $-r$ (remember the flipping effect—draw a picture).

$$r = \frac{-b}{-1} \text{ giving } b = r$$

$$-r = \frac{0.8 - b}{0.8b - 1}$$

Substituting the first in the second gives, after some fiddling, the quadratic

$$2b^2 - 5b + 2 = 0$$

$$(b - 2)(2b - 1) = 0$$

$b = 2$ (no good; this will give $r = 2$ -- too big) and $b = 0.5$*, which we use*.

Therefore, our FLT is

$$w = r(z) = \frac{z - 0.5}{0.5z - 1} = \frac{2z - 1}{z - 2}$$

This happens to take the inner circle into a circle of radius 0.5 centered at the origin. We now find the potential for the nice coaxial cylinders (Example (B) in the previous section):

$$\Phi = \text{A}\ln(r) + B$$

$$0 = A\ln(1) + B$$

$$110 = A\ln(0.5) + B$$

This gives $B = 0$ and $A = 110/\ln(0.5) \approx -158.7$.

So, $\quad \Phi = 110\ \ln(r)/\ln(0.5)$

The associated complex potential (See Example (B) above) is

$$F(z) = \frac{110}{\ln(0.5)} \text{Ln}(z)$$

Therefore, the complex potential on the non-coaxial region is

$$F(r(z)) = \frac{110}{\ln(0.5)} \text{Ln}\left[\frac{2z - 1}{z - 2}\right] \approx -158.7\ \text{Ln}\left[\frac{2z - 1}{z - 2}\right]$$

The real potential is

$$\Phi = -158.7 \ln\left|\frac{2z - 1}{z - 2}\right|$$

We can now express the RHS in terms of (x, y) if we really want,

Question What are the equipotentials:

Answer $\Phi = $ constant iff $\left|\frac{2z - 1}{z - 2}\right| = |w| = $ constant. But these are just circles in the nice coaxial region, which correspond to off-centered circles in the non-coaxial region.

Question What are the lines of force?

Answer Set the imaginary part of the complex potential equal to constants:

$$-158.7\ \text{Arg}\left[\frac{2z - 1}{z - 2}\right] = const$$

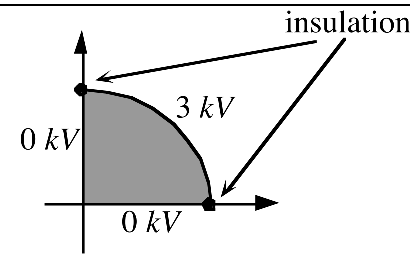
giving $\text{Arg}(w) = const$

so that, in the coaxial plane, they are straight lines. Since they cannot be straight lines in the original plane (since the circles they cross are not concentric) they have to be arcs of circles!

Exercise Set 10

Hand In

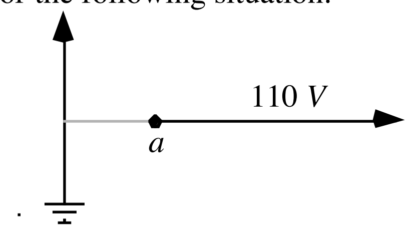
1. Find the potential, equipotential, and lines of force for the following situation. Also find a three-dimensional field corresponding to an appropriate three-dimension version of the following:



The disc has radius 5

[Suggestion: First shrink the radius. Then use z^2 to go to a simpler region. Follow by something we have already done...]

2. Repeat the first question for the following situation:



[Suggestion: The given region is, with a slight scale adjustment, the image of nice region under the sine function.]

11. Heat Problems (Still from Kreyzsig)
Heat conduction in a homogeneous material is governed by

$$\frac{\partial T}{\partial t} = c^2\nabla^2 T$$

where T = temperature, and c^2 is a positive constant that varies from material to material. When the temperature stops changing, we have **steady state**, and

$$\nabla^2 T = 0 = T_{xx} + T_{yy}$$

in the two-dimensional case. Since T satisfies Laplace's equation, it is also called the **heat potential**, and is, as usual, the real part of a complex potential

$$F(z) = T(x, y) + i\cong(x, y)$$

The equipotentials $T = const$ are called **isotherms** and the curves $\Psi = const$ are **heat flow lines.** The conjugate derivative $\overline{F'(z)}$ gives the heat flow vector field, measured in units of energy per unit time.

Examples
(A) ***Temperature between parallel plates***
Going back to Topic 9 on around p. 28, we find that the complex potential is just linear:

$$F(z) = Az + B$$

where A and B can be found from the temperatures of the two plates and their distance apart.

(B) ***Insulated Hot Wire***
A Hot Wire (500º) of radius 1 mm. in the center inside a cool cylinder (60º) or radius 100 mm. on the outside: Again going back to Topic 9, we use

$$F(z) = A\mathrm{Ln}(z) + B$$

Looking at the real part:

$$500 = A\ln(1) + B = B$$
$$60 = A\ln(100) + B = A\ln(100) + 500$$

So we get $A \approx -95.54$ and so

$$F(z) = -95.54\text{Ln}(z) + 500$$

This gives the temperature as the real part:

$$T(x, y) = -95.54 \ln r + 500,$$

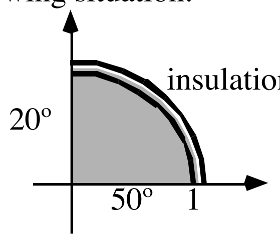
where r is the distance from the center.

Note on Insulation:

By definition, heat cannot pass through (ideal) insulation, therefore the heat flow lines can have no component perpendicular to an insulation barrier, In other words, heat flow must be parallel to insulation. Or, put another way, the heat flow lines Ψ = constant are the same as the insulation lines.

(C) ***Mixed Boundary value Problem***

Solve for temperature in the following situation:



This is a classical situation with T independent of r. Referring again to Topic 9, we find

$$F(z) = -iA\text{Ln}(z) + B$$

Looking at the real part,

$$T(z) = A \text{ Arg}(z) + B$$

and we get

$$B = 50 \text{ and } A = -60/\pi,$$

giving

$$T(z) = 50 - \frac{60}{\pi}\theta$$

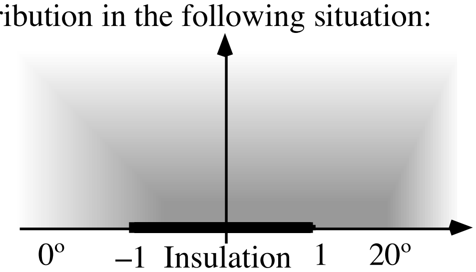
Notice that the heat flow lines are

$$\ln|r| = constant$$

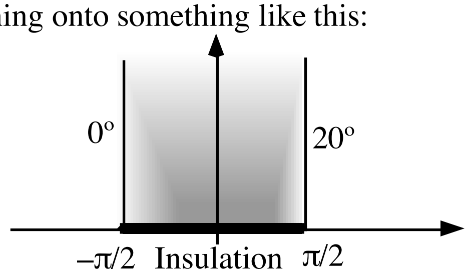
which is consistent with the above drawing (insulation = semicircles)

(D) ***Using Conformal Mappings***

Find the temperature distribution in the following situation:



It looks best to map this thing onto something like this:



which we can do with the inverse sine function (see the discussion of what the sine function does muuuch earlier). On the target strip, the temperature is given by

$$T(x,\ y) = \frac{10}{\pi}(2x + \pi) = \frac{20}{\pi}x + 10$$

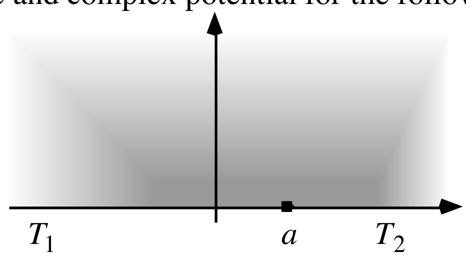
and so

$$F(z) = \frac{20}{\pi}z + 10$$

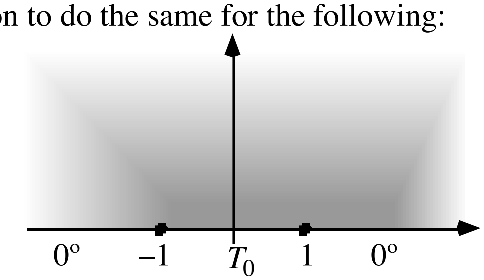
Therefore, on the original region, we have

$$G(z) = \frac{20}{\pi}\sin^{-1}(z) + 10$$

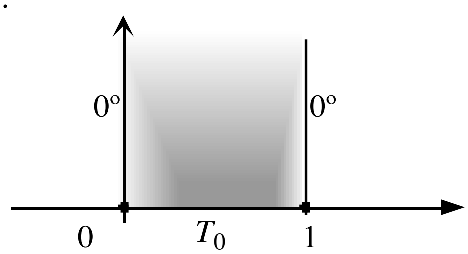
Exercise Set 11
Hand In: (Based on Kreyzsig, p. 811)
1. (a) Find the temperature and complex potential for the following situation:



(b) Now use superposition to do the same for the following:



(C) Finally, repeat for this:



(Insulation is at the solid dots.)

12. Fluid & Air Flow (16.4 Kreyszig)
We know that electrostatic potentials and temperature are harmonic under suitable conditions. What about fluid flow? Let $\underline{\boldsymbol{v}}$ denote the velocity field for a fluid flow. The fluid flow in a particular region is called **irrotational** if $\underline{\boldsymbol{\nabla}}\times\underline{\boldsymbol{v}} = \underline{\mathbf{0}}$.† [Recall the interpretation of the curl from calculus]. If v depends on x and y only and has only two coordinates,

$$\underline{\boldsymbol{v}} = \langle P(x,\ y),\ Q(x,\ y)\rangle,$$

then this translates to

† Viscous fluids are not irrotational. Think of the a viscous fluid moving down a pipe, and choose a closed path going down the center, to the edge, and up the edge. The path integral will not be zero, so that there is a net circulation.

$P_y = Q_x$.

On the other hand, a fluid is **incompressible** (like water and oil) if $\underline{\nabla}.\underline{v} = 0$. [Again recall the interpretation...] This condition implies that

$P_x = -Q_y$.

These two equations look like the C-R equations with the wrong signs. In fact, they show that the pair $P, -Q$ satisfy the C-R equations, whence they are the real and imaginary parts of a complex analytic function: Write

$v(z) =$ velocity field in complex form $= P + iQ$

Then $\overline{v(z)} = P - iQ$ *i*s analytic, and is therefore has an (analytic) antiderivative, $F(z)$ such that

$$F'(z) = \overline{v(z)}$$

In other words,

$$\overline{F'(z)} = v(z),$$

just as in the case of the electrostatic field. F is called, as usual, the **complex potential** of the flow. If we write F as $\Phi + i\Psi$ as usual, then we see that the velocity of the gradient of Φ:

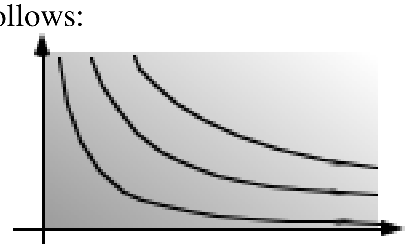
$$\underline{v} = \underline{\nabla}\Phi$$

so Φ is called the **velocity potential** and Ψ is called the **stream function**, since it gives the streamlines of $\underline{v}$. In other words, we just have the electric potential situation in disguise.

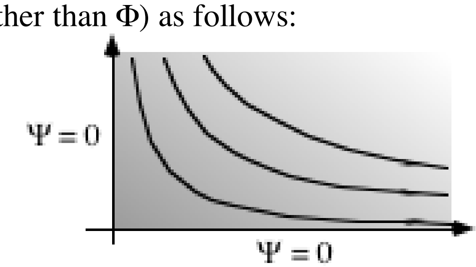
Examples

(A) ***Flow around a corner***

We want to model flow as follows:



Since the above picture is one of streamlines, Ψ = const, we can set up the Dirichlet problem as one for Ψ (rather than Φ) as follows:



We now use $w = z^2$ to map this onto H, and use

$\Psi = Ay$

for the associated potential in H. Remembering that this is the *imaginary* part of a complex potential in H, we simply use

$F = Aw = Az^2$

as our complex potential. Therefore,

$\Phi = A(x^2 - y^2)$

and $\Psi = 2Axy$

Equipotentials: These are the curves $\Phi = \text{constant}$, or

$$A(x^2 - y^2) = \text{const}$$

giving radial lines emanating from the origin.
Streamlines: These are the curves

$$2Axy = \text{const}$$

giving hyperbolas.
Velocity: $v(z) = F'(z) = 2Az$, so $\overline{F'(z)} = 2A\bar{z}$. In other words,

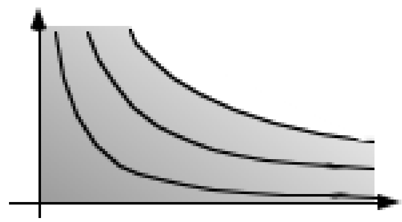
$$\underline{v} = 2A\langle x, -y\rangle.$$

This gives an interpretation of A:

$$\text{speed} = |\underline{v}| = 2A\sqrt{x^2 + y^2}$$

So by knowing the speed of the flow at any particular point away from the wall, we can compute A.

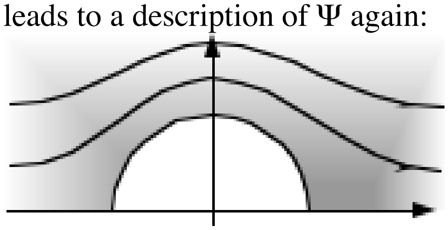
Note The speed is not constant along a streamline (hyperbola) but varies as the distance from the origin. The particle slow down the most nearest the origin, where the width of the flow channels is widest:

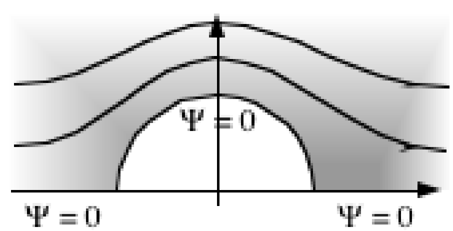


A typical flow channel

[The above potential also gives a model of the flow along any flow channel such as the one above.] also, the flow speeds up as the flow channel gets narrower and narrower. This is how water pistols work.

(B)*Flow around a cylinder*
This leads to a description of Ψ again:





This region D maps into H via $w = z + 1/z$, and, on H, Ψ can again be taken to be

$$\Psi = Ay$$

Giving us a complex potential

$$F(z) = A\left[z + \frac{1}{z}\right]$$

To see the real and imaginary parts, use polar form: $z = re^{i\theta}$. This gives

$$F(z) = A\left[re^{i\theta} + \frac{1}{r}e^{-i\theta}\right] = A\left[r + \frac{1}{r}\right]\cos\theta + i\,A\left[r - \frac{1}{r}\right]\sin\theta$$

So we can now get the potentials and streamlines in polar form:
Equipotentials:

$$\left[r + \frac{1}{r}\right]\cos\theta = \text{const}$$

Kind of complicated to draw these piggies - wait until we do things parametrically in the next section.

Streamlines:

$$\left[r - \frac{1}{r}\right] \sin\,\theta = \text{const}$$

Again, these are not standard curves. However, at large distances, $1/r \approx 0$, and so the streamlines are $1r \sin\,\theta = \text{const}$, or $y = \text{const}$—horizontal lines.

Velocity Field:

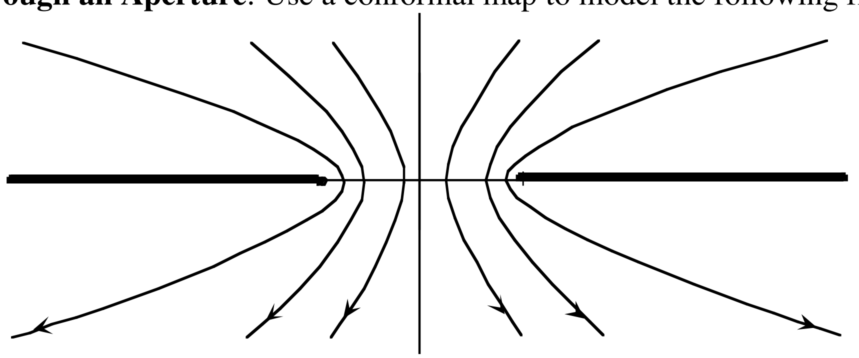
$$F'(z) = A\left[1 - \frac{1}{z^2}\right],$$

whence

$$\overline{F'(z)} = A\left[1 - \frac{1}{\bar{z}^2}\right]$$

We get **stagnation points** when the velocity equals zero, so we see that this gives $z = \pm 1$.

Exercise Set 12
Hand In:
1. Compute all the details for the flow around a corner of 60°.
2. Flow through an Aperture: Use a conformal map to model the following flow.



The width of the aperture is set to $2a$. To model this, [Suggestion: Consider what the inverse sine function does to this region.]

13. Parametrically Representing Streamlines, and Using Technology

Sometimes it is hard to draw the streamlines from the implicit equation that defines them. An analytic approach (short of seeing directly what the curves are, as we have done up to now) is to find an equation for dy/dx using implicit differentiation and then drawing the integral curves using technology. However, a more direct way is the following, which hinges on inverting the conformal mappings we have been using up to now.

Proposition ***Streamlines and Equipotentials go to Streamlines and Equipotential***
S'pose $F: D \to H$ is a conformal invertible map, with $P = \Phi + i\Psi$ is a complex potential on H; $P: H \to \mathbb{C}$. Then the image, under F^{-1} of the streamlines and equipotentials on H are the streamlines and equipotentials on D.

Proof The associated complex potential on D is, as we have seen, given by composition:

$$Q = P \circ F$$

Therefore its streamlines are specified by setting imaginary part equal to a constant:

$Im(P(F(z)) = \text{Const}$

That is,

$\text{Im}[\Phi(F(z)) + i\Psi(F(z))] = \text{Const}$ Since $P = \Phi + i\Psi$

or $\Psi[F(z)] = \text{const}$

So,

z is in a specific streamline on D

$\Leftrightarrow \Psi[F(z)] = K$ (K a specific constant)

$\Leftrightarrow w = F(z)$ is in the associated streamline $\Psi(w) = K$ on H.

In other words, streamlines go under F to streamlines. Put another way, streamlines in H map to streamlines of D under F^{-1} (since F^{-1} is the inverse of F, under which streamlines correspond to streamlines.) The argument for equipotentials is similar.

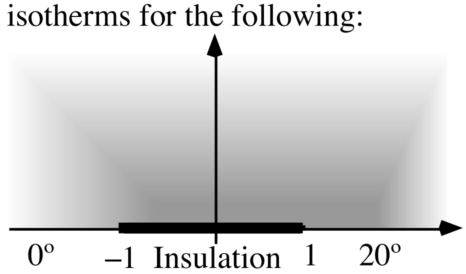
Consequence: ***How to graph streamlines & equipotentials***

Suppose that $z(t) = x(t) + iy(t)$ is a parametric representation of a streamline (or equipotential) on H. Then, by the proposition, $F^{-1}[z(t)]$ is a parametric representation of the associated streamline (or equipotential) on D.

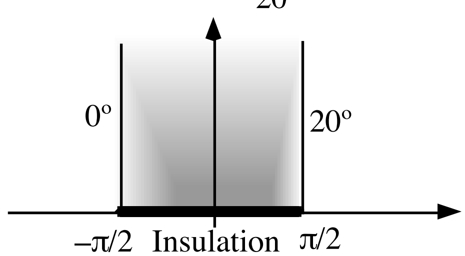
Examples

(A) ***Plotting Isotherms***

Use technology to plot the isotherms for the following:



Answer Recall that the isotherms are the equipotentials. Call the above region D. We saw two sections ago that D maps to a nicer representation of H using $F(z) = \frac{20}{\pi}\sin^{-1}(z) + 10$. The inverse of this is just $F^{-1}(z) = \sin(\frac{\pi}{20}[z-10])$. Here again is H:



On H, the isotherms are just vertical lines, which can be parameterized as $x = K$, $y = t$, where K is the temperature, and $t \geq 0$. In other words,

$$z(t) = K + it \qquad t \geq 0$$

Therefore, the corresponding isotherms in D are given by

$$F^{-1}[z(t)] = \sin(\frac{\pi}{20}[K-10 + it]) \qquad t \geq 0$$

To plot this, resolve into real and imaginary parts using the identity

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$$

Therefore

$$\sin(\frac{\pi}{20}[K-10 + it]) = \sin(\frac{\pi}{20}[K-10]) \cosh\frac{\pi}{20}t + i \cos(\frac{\pi}{20}[K-10]) \sinh\frac{\pi}{20}t$$

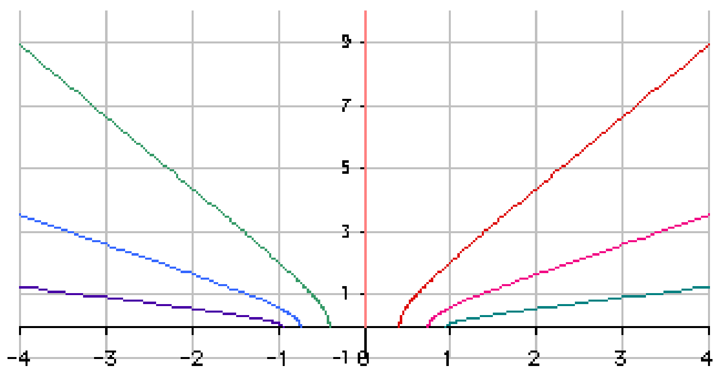
Therefore, we have the following parameterization of the isotherms:

$$x = \sin(\frac{\pi}{20}[K-10]) \cosh\frac{\pi}{20}t$$

$$y = \cos(\frac{\pi}{20}[K-10]) \sinh\frac{\pi}{20}t$$

K = temperature; $0 \le K \le 20$, $t \ge 0$.

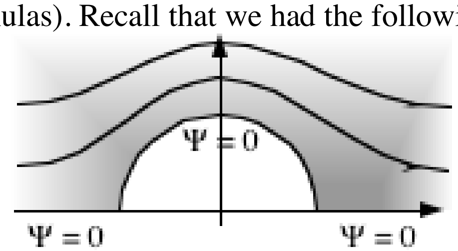
Here is what the lines look like in our little Excel plotter: (Compare with Exercise 2 in the preceding homework assignment.]



The 7 lines are $K = 2^{\circ}, 4.7^{\circ}, 7.3^{\circ}, 10^{\circ}, 12.7^{\circ}, 15.3^{\circ}, 18^{\circ}$

(B) ***Plotting Flow Lines***

Let us look again at the flow around a cylinder (which we failed to draw precisely due to the complexity of the formulas). Recall that we had the following intuitive picture:



and that the above region maps to H via

$$F(z) = A\left[z + \frac{1}{z}\right]$$

for some constant A. To invert this function, we set $w = F(z)$ and solve for z:

$$w = A\left[z + \frac{1}{z}\right]$$

$$\frac{w}{A} = z + \frac{1}{z}$$

Taking $1/A = B$, we get the quadratic

$$z^2 - Bwz + 1 = 0$$

Solving for z,

$$z = \frac{Bw \pm (B^2w^2 - 4)^{1/2}}{2}$$

so the inverse is

$$F^{-1}(z) = \frac{Bz \pm (B^2z^2 - 4)^{1/2}}{2}$$

Now you have to be careful, since there are two possible square roots to choose from (no such thing as a "positive" square root anymore). If z is in the first quadrant, then everything is in the upper half plane, and to get the inverse, we use the primitive square root (the one in the upper half plane) and also use the (+) sign.
The issue is now: How do we express this in terms of Cartesian coordinates?
Answer First look at $z^{1/2} = r^{1/2}[\cos(\theta/2) + i\ \sin(\theta/2)]$
where

$$\cos(\theta/2) = \sqrt{(1 + \cos\theta)/2} \quad \text{and} \quad \sin(\theta/2) = \sqrt{(1 - \cos\theta)/2}$$

so that, in Cartesian coordinates,

$$\begin{aligned} z^{1/2} &= r^{1/2}[\sqrt{(1 + \cos\theta)/2} + i\sqrt{(1 - \cos\theta)/2}\ ] \\ &= \sqrt{[r + r\cos\theta]/2} + i\sqrt{[r - r\cos\theta]/2} \\ &= \sqrt{[(x^2+y^2)^{1/2} + x]/2} + i\sqrt{[(x^2+y^2)^{1/2} - x]/2} \end{aligned}$$

(This is only valid in the upper half plane, since both of $z^{1/2}$ coords are positive..)
So now take $B = 1$ for simplicity (!) and look at

$$\begin{aligned} (z^2 - 4)^{1/2} &= (x^2-y^2 - 4 + i2xy)^{1/2} \\ &= \sqrt{(([x^2-y^2-4]^2 + 4x^2y^2)^{1/2} + x^2-y^2 - 4)/2} \\ &\qquad + i\sqrt{(([x^2-y^2-4]^2 + 4x^2y^2)^{1/2} - x^2+y^2 + 4)/2} \end{aligned}$$

adding this to $z = x+iy$ and dividing by 2 now gives $F^{-1}(z)$ in terms of Cartesian coordinates.
The flow lines we want are horizontal lines in H: $z(t) = t + iK$, where K is the height of the line. For simplicity (!) let us take $B = 1$, and then compute the real and imaginary parts of $F^{-1}(t + iK)$:

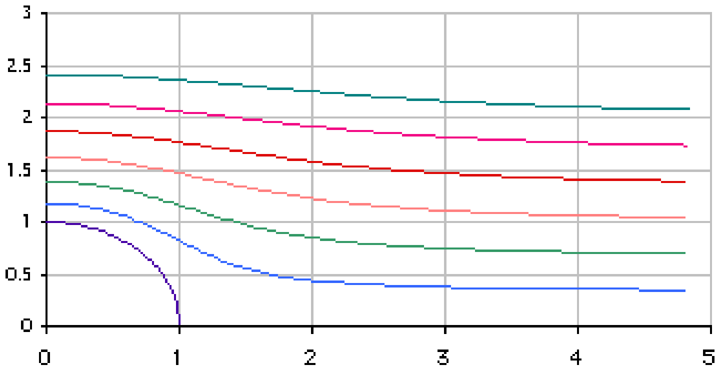
$$\text{Real part} = \frac{t + \sqrt{(\{[t^2-K^2-4]^2 + 4t^2K^2\}^{1/2} + t^2 - K^2 - 4)/2}}{2}$$

$$\text{Imaginary part} = \frac{K + \sqrt{(\{[t^2-K^2-4]^2 + 4t^2K^2\}^{1/2} - t^2 + K^2 + 4)/2}}{2}$$

To plot them technologically, use the following formulas for x and y:

```
x = (t + SQRT((((t^2-k^2-4)^2+4*t^2*k^2)^.5+t^2-k^2-
4)/2))/2
y = (k + SQRT((((t^2-k^2-4)^2+4*t^2*k^2)^.5-
t^2+k^2+4)/2))/2
```

Here is the plot for $t \geq 0$ (first quadrant):



7 values *of* k from 0 to 2

Question The curves are all wrong for negative t. Why?

Answer In the second quadrant, two things happen:

(1) We need to use the ($-$) sign in the formula for the inverse of F.

(2) The quantity under the square root in that formula is an imaginary number in the *lower half* plane, and so the square root we want to *add* is in the second quadrant, and so has a negative x-coordinate and a positive y-coordinate. The upshot of all of this is that, in the second quadrant, we use:

$$\text{Real part} = \frac{t - \sqrt{(\{[t^2-K^2-4]^2 + 4t^2K^2\}^{1/2} + t^2 - K^2 - 4)/2}}{2}$$

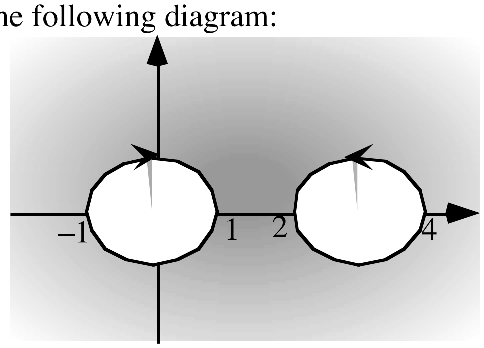
$$\text{Imaginary part} = \frac{K + \sqrt{(\{[t^2-K^2-4]^2 + 4t^2K^2\}^{1/2} - t^2 + K^2 + 4)/2}}{2}$$

Exercise Set 13

Hand In:

1. Obtain parametric equations for the streamlines of a flow around a 90° corner and use technology to plot a number of these lines.

2. (Refer to Exercise Set 7 #3.) Two rotating shafts with paddles, rotating in the opposite direction, are inserted into an incompressible and irrotational liquid and spinning at the same speed as shown in the following diagram:



(a) Find a mapping from the above diagram into an annulus, and then map the annulus into a vertical strip using the logarithm.

(b) Invert both maps to obtain a mapping from the vertical strip onto the given region.

(c) Now obtain the streamlines of the resulting flow.

(d) Using the non-inverted maps, obtain the complex potential and hence the velocity field, assuming that the outer surfaces of the shafts are rotating at unit speed.

14. Some Interesting Examples of Flows (Based partly on Kreyszig, p. 818)

When a fluid fails to be incompressible at an isolated point (usually when the divergence at that point is singular) we say it has a **source** or **sink** at that point depending on whether the divergence there—as measured by the total flux out of a small surface—is positive or negative. The **strength** of the source is equal to the total flux, if the flux integral exists.[2]

In physical terms, the flux of a vector field out of a surface measures the volume per unit time leaving that surface. Therefore, the strength of a source equals the *total amount of fluid per unit time* emerging out of that source.

Similarly, if the curl fails to vanish at an isolated point, then the fluid has a **vortex** at that point. The **moment** of the vortex is then the vector whose coordinates are the path integrals, of circles perpendicular to the axes around that point, if the integrals exists. The **vortex strength** is the magnitude of the moment. Physically, it measures the total angular momentum of a ring of fluid with unit mass per unit length rotating about the axis given by the moment vector.

In the real world, point sources are just sources of fluid placed somewhere, and sinks are drains or suctions. We gat vortices by inserting spinning cylinders in the liquid and waiting until it reaches a steady state.

How does this all effect the complex part of it? Since we are only interested here in fields that have isolated sinks and vortices, the field is represented everywhere else by a complex potential function $F(z)$, which will be singular at these points.

Examples

(A) ***Point Source*** Take $F(z) = \frac{c}{2\pi} \ln z$. The lines of flow are obtained by setting the imaginary part = 0, giving $\arg(z)$ = const, suggesting a source or sink at the origin. To determine which, we need to compute the outward flux in 2 dimensions. First, we get the resulting vector field:

$$\underline{v} = \overline{F'(z)} = \frac{c}{2\pi\bar{z}} = \frac{cz}{2\pi|z|^2} = \frac{c\langle x, y\rangle}{2\pi(x^2 + y^2)}$$

Which is a radially *outward* flow of magnitude $c/(2\pi r)$. We now compute its strength: Thinking of a 3-dimensional cylinder as out surrounding volume, we are reduced to computing the limit of

$$\int \frac{c}{2\pi r}\, ds,$$

taken around the circle, giving

$$\int_0^{2\pi} \frac{c}{2\pi r}(r\, d\theta) = c$$

So, the strength of the point source specified by $F(z) = \frac{c}{2\pi} \ln z$ is just c.

(B) ***Combining Sources and Sinks***

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[2] Note that we should get the same flux regardless of the shape or size or the surrounding surface, as long as that surface encloses only the given singular point. The reason for this is that the discrepancy between the integrals over two difference surfaces is itself a surface integral over a region where the divergence is zero, and so the difference is zero by the divergence theorem.

Since the flux integral is zero away from any singularity, it follows that we can just add fields like the above to get an arbitrary configurations of sources and sinks with specified strengths c_i by taking a sum of terms :

$$F(z) = \Sigma_i \frac{c_i}{2\pi} \ln(z - z_i)$$

(C) ***Point Vortex*** $F(z) = \frac{-Ki}{2\pi} \ln z$. Its imaginary part is given by the real part of the log, of the magnitude, which tells us that the flow is circular. To see it exactly (and in which direction it goes) compute the velocity field:

$$\underline{v} = \overline{F'(z)} = \frac{+Ki}{2\pi\bar{z}} = \frac{Kiz}{2\pi|z|^2} = \frac{K\langle -y, x\rangle}{2\pi(x^2 + y^2)}$$

which circulates counterclockwise if K is positive. To get the moment, we note that, since the circulation is in the xy-plane, it only has one coordinate: the z-coordinate. Therefore we need compute only one path integral (in the xy-plane). Now, actually we don't even need to evaluate the path integral, because of the following facts:

(1) The *complex* path integral of $1/z$ around such a circle is equal to $2\pi i$.

(2) The imaginary part of the integral of $1/z$ (namely, 2π) equals the negative of the ordinary path integral of the vector field represented by $\overline{1/z} = -i/\bar{z}$, (see the note just after the definition of the complex path integral). Therefore, the path integral of $i/\bar{z}$ equals 2π

It follows that the path integral of $\frac{Ki}{2\pi\bar{z}}$ is just K, so that the moment of the vortex is given by $K\langle 0, 0, 1\rangle$ or $K\underline{\mathbf{k}}$, and its strength is K.

(C) ***Combing Sources, Sinks, and Vortices***

Since all of the above functions only have singularities at isolated points, we can combine them to form velocity fields with vortices, sources and sinks as we desire. We can also combine these things with some of the other flows we have studied above.

(D) ***Flow with Circulation Around a Cylinder***

Start with the complex potential for basic flow around a cylinder: $F(z) = z + \frac{1}{z}$. Then add a circulation at the origin with some strength K:

$$F(z) = z + \frac{1}{z} + \frac{Ki}{2\pi} \ln z$$

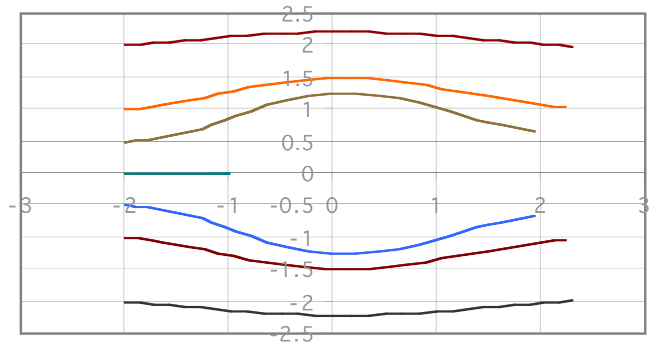
A **stagnation point** is a point where the velocity equals zero. Setting the speed equal to zero and solving for z gives (homework)

$$z = \frac{iK}{4\pi} \pm \sqrt{\frac{-K^2}{16\pi^2} + 1}$$

If $K = 0$, (no rotation) we have stagnation points at $z = \pm 1$. *As* k increases, the stagnation points creep up the unit circle. When K reaches 4π, the quantity under the square root turns imaginary, and so z creeps up the z-axis.

Note on finding streamlines: Inverting this function is not possible analytically. However, to draw the streamlines, all we really need to do is choose a small value of Δt, start at some point, find the velocity vector there, take a small step in that direction, and then continue. (We are actually solving a system of two differential equations in two unknowns: x and y as functions of t, numerically using Euler's method.)

Here is what you get using Excel with 30 points and $\Delta t = 0.12$ (a very large value):

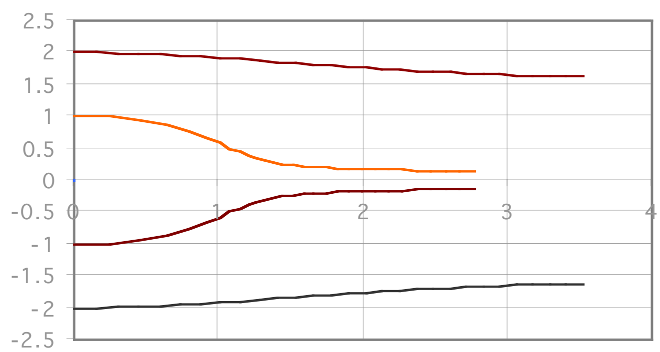


This uses $\underline{v} = \overline{F'(z)}$, where $F(z) = z + 1/z$. The Cartesian coordinates of $\overline{F'(z)}$ are

Real part: $v_x = \dfrac{(x^2+y^2)^2 - x^2 + y^2}{(x^2+y^2)^2}$ Imaginary part: $v_y = \dfrac{-2xy}{(x^2+y^2)^2}$

The increments are then given by $(v_x\Delta t,\ v_y\Delta t)$.

Warning: This method of plotting is not accurate unless one uses a very small Δt and plots lots of points. As you move along the curve, you are really changing from one streamline to another. As an example, if we started at (0, 1), the streamline should trace out a quadrant of the circle and then stop at the stagnation point, but instead we get: this with $\Delta t = .12$:



Notice that the line starting at $(-5, 0)$ stops when it hits the cylinder because of the stagnation point there

Exercise Set 14

1. Verify the claims in (D): **(a)** that the stagnation points are where claimed, **(b)** that they are on the unit circle id $K \le 4\pi$, and on the imaginary axis otherwise.

2. Set up an Excel spreadsheet to plot a as follows, so that you get around 200 points (allowing you to use a small value of Δt):

Δt	x	y	Δx	Δx
	x_0	y_0	$v_x\Delta t$	$v_x\Delta t$
	$=x_0 + \Delta x$	$=y_0 + \Delta y$		

(First copy the right-hand cells one level, and then copy everything.)For extra credit, you can set it up to plot several streamlines...

15. The Joukowski Airfoil

Start with the map $f(z) = z + \frac{1}{z}$, and look at the images of some circles. In general, we look at a circle whose center is offset a little to the right of a point on the y-axis passing through -1 That is,

Center = $i\mu + \varepsilon$

Radius = $\sqrt{\mu^2 + (1+\varepsilon)^2}$

Parametric equations for this circle are:

$x = \varepsilon + P\cos t$

$y = \mu + P\sin t$

$0 \le t \le 2\pi$

where $P = \sqrt{\mu^2 + (1+\varepsilon)^2}$

Its image under $f(z) = z + \bar{z}/|z|^2$ has parametric equations

$x = (\varepsilon + P\cos t)[1 + 1/[\varepsilon^2 + \mu^2 + P^2 + 2P(\varepsilon\cos t + \mu\sin t)]]$

$y = (\mu + P\sin t)[1 - 1/[\varepsilon^2 + \mu^2 + P^2 + 2P(\varepsilon\cos t + \mu\sin t)]]$

let us fix $\mu = 0.2$ and keep ε as a parameter. This gives

$P = \sqrt{0.04 + (1+\varepsilon)^2}$

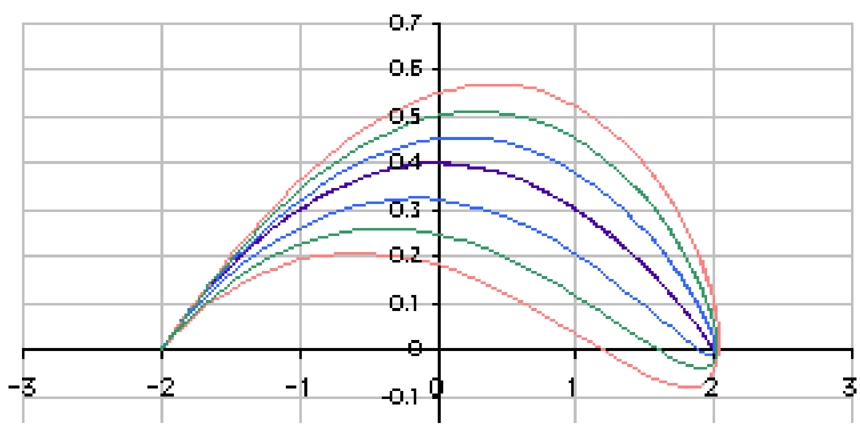
$x = \varepsilon + P\cos t\ [1 + 1/[\varepsilon^2 + 0.04 + P^2 + 2P(\varepsilon\cos t + \mu\sin t)]]$

$y = 0.02 + P\sin t\ [1 - 1/[\varepsilon^2 + 0.04 + P^2 + 2P(\varepsilon\cos t + \mu\sin t)]]$

Setting them up for Excel gives, with the parameter denoted by k:

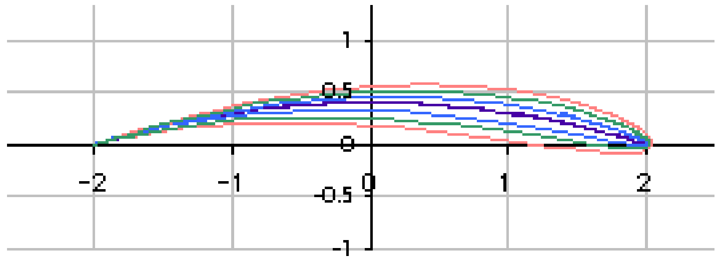
```
x =
(k+(0.04+(1+k)^2)^.5*cos(t))*(1+1/(k^2+0.08+(1+k)^2+
2*(0.04+(1+k)^2)^.5*(k*cos(t)+0.2*sin(t))))
y =
(0.2+(0.04+(1+k)^2)^.5*sin(t))*(1-1/(k^2+0.08+(1+k)^2+
2*(0.04+(1+k)^2)^.5*(k*cos(t)+0.2*sin(t))))
```

And here are the plots:

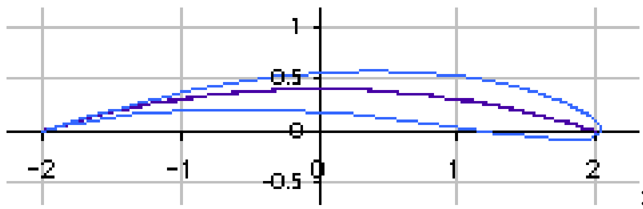


k = 0, 0.033, 0.067, 0.1

Each of the curves is a different airfoil, with the degenerate one corresponding to $k = 0$, the image of the circle passing through $(-1, 0)$ and $(1, 0)$ with center $(0, 0.2)$. Different values of k give different thicknesses. Notice that the cure is wildly exaggerated because of the y-scale. Squaring up the scale to give the correct proportions gives this:



Now let us choose one particular airfoil, $k = 0.1$ (the outmost one above):



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And now our next task will be to compute the air flow over this. Since this airfoil is the image of a cylinder with center (0.1, 0.2) and radius

$$P = \sqrt{0.04 + (1+0.2)^2} \approx 1.21655251$$

We will do them parametrically, in two stages:
(1) Start with horizontal lines:
$x = t,\ y = k$
or $z = t + ik$
(2) Map these via the inverse of $f(z) = z + 1/z$ as we did before
(3) Then scale up to give the unit disc a radius of P (above)
(4) Translate to take the origin to (0.1, 0.2)
(5) Finally, apply f to take that circle to the airfoil.

Steps 1–4 give, by previous calculations,

$$\text{Real part} = 0.1 + 1.2166\frac{t + \sqrt{(\{[t^2-K^2-4]^2 + 4t^2K^2\}^{1/2} + t^2 - K^2 - 4)/2}}{2}$$

$$= 0.1 + 0.6083\left[t + \sqrt{(\{[t^2-K^2-4]^2 + 4t^2K^2\}^{1/2} + t^2 - K^2 - 4)/2}\right]$$

$$\text{Imaginary part} = 0.2 + 1.2166\,\frac{K + \sqrt{(\{[t^2-K^2-4]^2 + 4t^2K^2\}^{1/2} - t^2 + K^2 + 4)/2}}{2}$$

$$= 0.2 + 0.6083\left[K + \sqrt{(\{[t^2-K^2-4]^2 + 4t^2K^2\}^{1/2} - t^2 + K^2 + 4)/2}\right]$$

For Step 5, we take its image under $f(z) = z + \bar{z}/|z|^2$. What is awful here is that we need to compute $|z|^2$. For the moment, call it A. Then we get

$$x = \left\{0.1 + 0.6083\left[t + \sqrt{(\{[t^2-K^2-4]^2 + 4t^2K^2\}^{1/2} + t^2 - K^2 - 4)/2}\right]\right\}(1 + 1/A)$$

$y \;=\; \left\{0.2 + 0.6083\left[K + \sqrt{(\{[t^2 - K^2 - 4]^2 + 4t^2K^2\}^{1/2} - t^2 + K^2 + 4)/2}\right]\right\}(1 \;-\; 1/A)$

Now with $R = \{[t^2 - K^2 - 4]^2 + 4t^2K^2\}^{1/2}/2$ and $S = (t^2 - K^2 - 4)/2$,
we wind up (after a lot of algebra) with

$|z|^2 = A$

$= 0.05 + 0.37(t^2 + k^2 + 2R) + 0.12166(t + 2k) + \sqrt{R + S}\,(0.74t + .12166) + \sqrt{R - S}\,(.37k + .24332)$

The two radicals are exactly those ugly terms that occur in the formula for x and y above.
To put this all together for an Excel formula, we must work in stages:

R = `((t^2-k^2-4)^2+4*t^2*k^2)^.5/2`

S = `(t^2-k^2-4)/2`

$\sqrt{R + S}$ = `(((t^2-k^2-4)^2+4*t^2*k^2)^.5/2+(t^2-k^2-4)/2)^.5`

$\sqrt{R - S}$ = `(((t^2-k^2-4)^2+4*t^2*k^2)^.5/2-(t^2-k^2-4)/2)^.5`

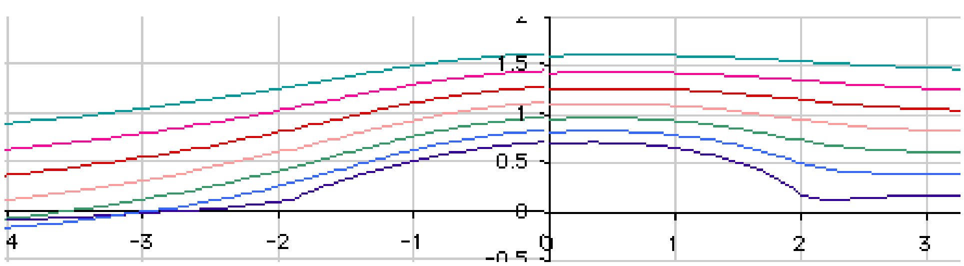
A =

```
0.05+0.37*(t^2+k^2+2*((t^2-k^2-
4)^2+4*t^2*k^2)^.5/2)+0.12166*(t+2*k)+(((t^2-k^2-
4)^2+4*t^2*k^2)^.5/2+(t^2-k^2-
4)/2)^.5*(0.74*t+0.12166)+(((t^2-k^2-4)^2+4*t^2*k^2)^.5/2-
(t^2-k^2-4)/2)^.5 *(.37*k+0.24332)
```

Finally, we can put everything together to get the streamlines!

```
x  =  (0.1+0.6083*(t+(((t^2-k^2-4)^2+4*t^2*k^2)^.5/2+(t^2-
k^2-4)/2)^.5))*(1+1/(0.05+0.37*(t^2+k^2+2*((t^2-k^2-
4)^2+4*t^2*k^2)^.5/2)+0.12166*(t+2*k)+(((t^2-k^2-
4)^2+4*t^2*k^2)^.5/2+(t^2-k^2-
4)/2)^.5*(0.74*t+0.12166)+(((t^2-k^2-4)^2+4*t^2*k^2)^.5/2-
(t^2-k^2-4)/2)^.5 *(.37*k+0.24332)))
```

```
y  =  (0.2+0.6083*(k+(((t^2-k^2-4)^2+4*t^2*k^2)^.5/2-(t^2-
k^2-4)/2)^.5))*(1-1/(0.05+0.37*(t^2+k^2+2*((t^2-k^2-
4)^2+4*t^2*k^2)^.5/2)+0.12166*(t+2*k)+(((t^2-k^2-
4)^2+4*t^2*k^2)^.5/2+(t^2-k^2-
4)/2)^.5*(0.74*t+0.12166)+(((t^2-k^2-4)^2+4*t^2*k^2)^.5/2-
(t^2-k^2-4)/2)^.5 *(.37*k+0.24332)))
```



(That was two pictures glued together!)

Notice that close to the front of the airfoil, the pressure (inverse distance between adjacent lines) is low, causing lift, but you would also have to plot the lower lines to see this...