CALCULUS II MATH 72

LECTURE NOTES BY SYLVIA SILBERGER AND STEFAN WANER

Contents

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1. Sets and Functions

A **set** is an undefined "primitive" notion. Intuitively, it refers to a collection of things called elements. If a is an element of the set S, we write $a \in S$. If a is not an element of the set S, we write $a \notin S$. Some important sets are:

$\mathbb{Z}$, the set of all integers
$\mathbb{N}$, the set of all natural numbers (including 0)
$\mathbb{Q}$, the set of all rational numbers
$\mathbb{R}$, the set of all real numbers
$\mathbb{C}$, the set of all complex numbers

We can describe a set in several ways:

(1) by listing its elements; e.g.. $S = \{6, 66, 666\}$
(2) in the form $\{\, x \mid P(x)\}$, where $P(x)$ is a predicate in x, for instance

$$S = \{\, x \mid x \text{ is a real number other than } 6\}$$

or

$$T = \{\, x \in \mathbb{R} \mid x \leq 4\} = (-\infty, 4]$$

Note: Two sets are equal if they have the same elements. That is,

$$\boxed{A = B \text{ means } x \in A \Leftrightarrow x \in B}$$

Definitions 1.1. Let A and B be sets.

We say that A is a **subset** of B and write $A \subseteq B$ if $x \in A \Rightarrow x \in B$.

$A \cap B$ is the intersection of A and B, and is given by
$$A \cap B = \{\, x \mid x \in A \text{ and } x \in B\}$$

$A \cup B$ is the union of A and B, and is given by
$$A \cup B = \{\, x \mid x \in A \text{ or } x \in B\}$$

$\emptyset$ is the empty set; $\emptyset = \{\, x \mid F(x)\}$, where $F(x)$ is any false predicate in x, such as "$x = 3$ and $x \neq 3$" or "Your math instructor drives a red mustang with x doors."

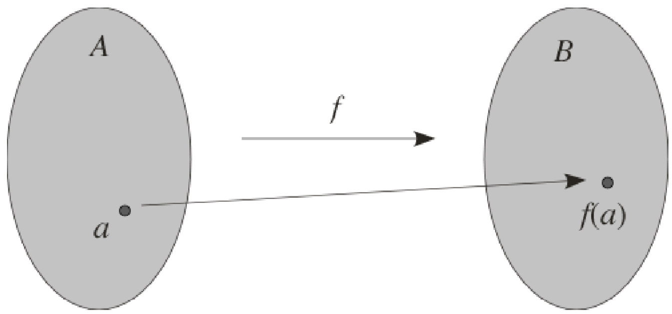
$A - B$ is the complement of B in A, and is given by
$$A - B = \{\, x \mid x \in A \text{ and } x \notin B\}$$

$A \times B$ is the set of all ordered pairs,
$$A \times B = \{\, (a, b) \mid a \in A, b \in B\}$$

Definition 1.2. Let A and B be sets. A **function** $f \colon A \to B$ is a triple (A, B, f) where f is a subset of $A \times B$ such that for every $a \in A$, there exists a unique $b \in B$ (that is, one and only one $b \in B$) with $(a, b) \in f$. We refer to this element b as $f(a)$. A is called the **domain** or **source** of f and B is called the **codomain** or **target** of f.

Notes:

(1) We think of f a rule which assigns to every element of A a unique element $f(a)$ of B, and we can picture a function $f\colon A \to B$ as shown in the figure.



(2) Sometimes, the domain of a function may not be specified, so we use the largest possible domain (sometimes called the *natural domain*). For instance, $f(x) = \sqrt{x-1}$ has natural domain $[1, +\infty)$.

(3) The codomain of f is not the "range" of f; that is, not every element of B need be of the form $f(a)$.

(4) The sets A and B are *part of the information* of f. For instance, specifying f by saying only "$f(x) = 2x - 1$" is not sufficient because we have not specified the domain and codomain. We should instead say something like this:

"Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x - 1$."

Definition 1.3. Let $f : A \to B$ be a function. The **graph of** f is the subset of $A \times B$ consisting of all pairs $(x, f(x))$ where x is in the domain A of f :

$$\text{Graph of } f = \{\, (x, f(x)) \mid x \in A\}$$

Examples 1.4.

A. Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.
This is not the same function as:

B. $g\colon [0, +\infty) \to \mathbb{R};\ g(x) = x^2$
This in turn is not the same function as:

C. $h\colon [0, +\infty) \to [0, +\infty);\ g(x) = x^2$

D. $\sin\colon \mathbb{R} \to \mathbb{R}$, the usual sine function

E. Define $\sin_r\colon [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ by $\sin_r(x) = \sin(x)$. We will call this the **restricted sine function.**

F. Define $\cos_r\colon [0, \pi] \to [-1, 1]$ by $\cos_r(x) = \cos(x)$. We will call this the **restricted cosine function.**

G. Define $\tan_r\colon (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ by $\tan_r(x) = \tan(x)$. We will call this the **restricted tan function.**

H. **Exponential maps,** such as $f\colon \mathbb{R} \to (0, +\infty)$; $f(x) = 2^x$. We denote this specific map by $\exp_2\colon \mathbb{R} \to (0, +\infty)$. In general, if $a > 0$, define $\exp_a\colon \mathbb{R} \to (0, +\infty)$ by $\exp_a(x) = a^x$.

Important Example 1.5. First we have to believe, for now, in the existence of a certain limit: Consider the sequence of numbers

$$\left(1+\frac{1}{1}\right)^1, \left(1+\frac{1}{2}\right)^2, \left(1+\frac{1}{3}\right)^3, \ldots, \left(1+\frac{1}{n}\right)^n, \ldots$$

You will see later in the course that it is an increasing function of n, and also **bounded above,** and we will also conclude that it must have a limit as $n \to \infty$. Accepting this on faith this for now, we define this limit to be the number e. That is,

$$e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n.$$

Definition 1.6. The (standard) **exponential function** $\exp\colon \mathbb{R} \to (0, +\infty)$ is defined by $\exp(x) = e^x$. (Notice that exp is another way of writing $\exp_e$.) In class, we illustrate its graph and compare it with those of the other exponential functions $\exp_a$.

Definition 1.7. Let $f\colon A \to B$ be a map. Then f is **injective** (or **one-to-one**) if

$$\boxed{f(x) = f(y) \Rightarrow x = y}$$

In other words, if $x \neq y$, then $f(x)$ cannot equal $f(y)$. This amounts to the **horizontal line test** condition on the graph of f (illustrated in class).

Showing That a Function f is (or is not) Injective

Showing f is injective:
Assume $f(x) = f(y)$, and then show that $x = y$.

Showing f is not injective:
Produce two distinct numbers x and y in the domain of f with $f(x) = f(y)$.

Examples 1.8.

A. $f\colon \mathbb{R} \to \mathbb{R}$; $f(x) = 2x - 1$ is injective.

B. $f\colon \mathbb{R} \to \mathbb{R}$; $f(x) = x^2 + 1$ is not.

C. $\sin\colon \mathbb{R} \to \mathbb{R}$ is not injective, whereas $\sin_r\colon [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ is. Similarly for the restricted cosine and tan functions.

D. $\exp\colon \mathbb{R} \to (0, +\infty)$ is injective, as are the other exponential maps $\exp_a$ if $a > 0,\ a \neq 1$.

E. Identity maps are always injective. (What are they? Answered in class.)

Definition 1.9. Let $f\colon A \to B$ be a map. Then the **range,** or **image,** of f is defined as

$$\operatorname{Range} f = \{\, f(x) \mid x \in A \,\}.$$

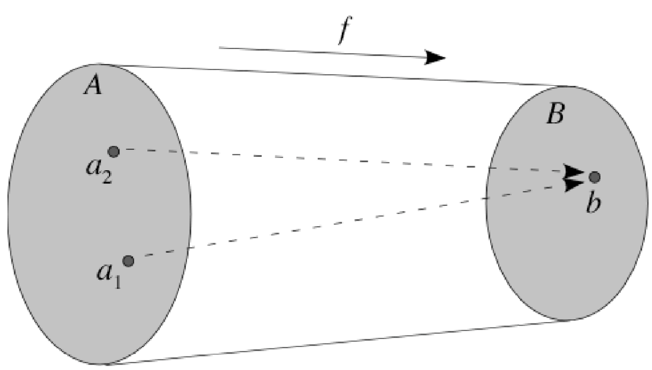
Examples 1.10.

A. $f\colon \mathbb{R} \to \mathbb{R}$; $f(x) = x^2 + 1$. Find Range f.
B. Range of $\sin\colon \mathbb{R} \to \mathbb{R}$
C. Range of $\exp_a\colon \mathbb{R} \to \mathbb{R}^+$ if $a = 1$ and $a \neq 1$

Definition 1.11. Let $f\colon A \to B$ be a map. Then f is **surjective** (or **onto**) if Range $f = B$. In other words,

$$\boxed{b \in B \Rightarrow \exists a \in A \text{ such that } f(a) = b}$$

Thus, f "hits" every element in the target B (see figure). Also, we will see what this says about the graph in class.



f is surjective iff Range $f = B$

Showing That a Function *f* is (or is not) Surjective

Showing f is surjective:
Write down the equation $y = f(x)$, and show that it can be solved for x for every choice of y is in the target.

Showing f is not surjective:
Produce a number y in the target for which the equation $y = f(x)$ cannot be solved for x.

Examples 1.12.

A. Identity maps are always surjective.

B. $\sin\colon \mathbb{R} \to \mathbb{R}$ is not surjective, whereas $\sin_r\colon [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ is.
C. $\exp\colon \mathbb{R} \to (0, +\infty)$ is surjective, as are the other exponential maps $\exp_a$ if $a > 0,\ a \neq 1$.

Definition 1.13. $f\colon A \to B$ is **bijective** if it is both injective and surjective.

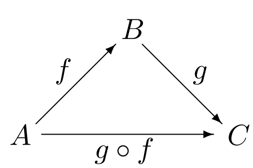
Examples 1.14.

A. The identity map 1_A on any given set set A
B. Square root function
C. Restricted Trig functions
D. Multiplication by a non-zero real number
E. The exponential maps $R \to [0, +\infty)$

Notes 1.15.

(a) A function is injective if and only if its graph passes the "horizontal" line test: Each horizontal line corresponding to a point in the target passes through *at most* one point on the graph.
(b) A function is surjective if and only if its graph passes another "horizontal" line test: Each horizontal line corresponding to a point in the target passes through *at least* one point on the graph.
(c) A function is bijective if and only if its graph passes the "super horizontal" line test: Each horizontal line corresponding to a point in the target passes through *exactly* one point on the graph.
(d) If I is some interval on the real line, and $f\colon I \to \mathbb{R}$ is continuous on I and differentiable inside I, then f is injective on I if and only if f' is nowhere zero inside I.This gives us another way of showing that a nice function f is injective on an interval I. Just show that that its derivative never vanishes. (You might find this useful in the exercises).

Definition 1.16. If $f\colon A \to B$ and $g\colon B \to C$, then their **composite,** $g \circ f\colon A \to C$, is the function specified by $g \circ f(a) = g(f(a))$.



(Example in class.)

Definition 1.17. $f\colon A \to B$ and $g\colon B \to A$ are called **inverse functions** if $g \circ f = 1_A$ and $f \circ g = 1_B$. In this event, we write $g = f^{-1}$ (and say that g **is the inverse of** f) and $f = g^{-1}$. If f has an inverse, we say that f is **invertible.**

Theorem 1.18 (Inverse of a Function)**.**

(a) $f\colon A \to B$ is invertible iff f is bijective
(b) The inverse of an invertible map is unique. □

Notes 1.19.

(a) Since the inverse of a function f is unique (if it exists), we refer to the inverse of $f\colon A \to B$ as $f^{-1}\colon B \to A$.
(b) The graph of the f^{-1} is related to the graph of$f$ in a particularly nice way — sketch in class.
(c) By the theorem, it is enough to know that a function f is bijective to conclude that it must have an inverse; there is no need (and it is not always possible) to come up with an algebraic formula for the inverse.
(d) To find a formula for $f^{-1}(x)$ from that of $f(x)$, all you do is set $y = f(x)$, solve the equation for x, and then switch x and y. (We will see this in some of the examples.)

Examples 1.20.

A. Inverse of $f\colon \mathbb{R} \to \mathbb{R};\ f(x) = x^3 - 4$
B. Inverse of $f\colon [0, +\infty) \to [0, +\infty);\ f(x) = x^2$
C. Inverse of $f\colon \mathbb{R} \to (-1, 1);\ f(x) = \frac{x}{1+|x|}$
D. The inverse of $\sin_r\colon [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ is called **the inverse sine function:**

$$\arcsin\colon [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$$

Put another way (and see the exercises),

$$y = \sin x \Longleftrightarrow x = \arcsin y$$

whenever $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. We sketch its graph in class, and also go through some subexamples.
E. The inverse of $\cos_r\colon [0, \pi] \to [-1, 1]$ is called **the inverse cosine function,** $\arccos\colon [-1, 1] \to [0, \pi]$. We sketch its graph in class.
F. The inverse of $\tan_r\colon (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is called **the inverse tan function,** $\arctan\colon \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$. We sketch its graph in class.
G. If $a > 0$, the inverse of $\exp_a\colon \mathbb{R} \to \mathbb{R}^+$ is called the **logarithm to base *a*,** and written as

$$\log_a\colon \mathbb{R}^+ \to \mathbb{R}$$

Put another way (again see the exercises),

$$y = a^x \Longleftrightarrow x = \log_a y$$

Subexamples in class.

Exercise Set 1.

1. Which of the following functions is injective? Justify your claim in each case.

(a) $f\colon \mathbb{R} \to \mathbb{R};\ f(x) = e^{x-4}$
(b) $f\colon \mathbb{R}^+ \to \mathbb{R};\ f(x) = e^{x^2}$
(c) $f\colon \mathbb{R}^+ \to \mathbb{R};\ f(x) = \sin x$
(d) $f\colon \mathbb{R} \to \mathbb{R};\ f(x) = (3x+4)^3 - 8$

2. Which of the following functions is surjective? Justify your claim in each case. Also, if a specific function is not surjective, obtain its range.
(a) $f\colon \mathbb{R} \to \mathbb{R}^+;\ f(x) = e^{x-4}$
(b) $f\colon \mathbb{R} \to \mathbb{R};\ f(x) = e^{x^2}$
(c) $f\colon [-1,1] \to \mathbb{R};\ f(x) = x + \arcsin x$
(d) $f\colon \mathbb{R} \to \mathbb{R};\ f(x) = (3x+4)^3 - 8$

3. Which of the following functions is invertible? In each case, if a function is invertible, produce the inverse. If it is not, show why.
(a) $f\colon \mathbb{R} \to \mathbb{R}^+;\ f(x) = e^{x-4}$
(b) $f\colon \mathbb{R}^+ \to (0,1);\ f(x) = \frac{x}{x+1}$
(c) $f\colon \mathbb{R} \to \mathbb{R}^+;\ f(x) = \frac{e^x+e^{-x}}{2}$
(d) $f\colon \mathbb{R}^+ \to \mathbb{R}^+;\ f(x) = e^{x^2}$

4. If A and B are subsets of $\mathbb{R}$ and $f\colon A \to B$ has inversef^{-1}, prove in two lines or less that

$$y = f(x) \Longleftrightarrow x = f^{-1}(y)$$

whenever $x \in A$.

5. Peek ahead to the properties of the logarithm and prove the following identities (where x and a are positive real numbers):
(a) $a^x = e^{x \ln a}$ (b) $\log_a x = \dfrac{\ln x}{\ln a}$

6. The **hyperbolic sine and cosine functions** are defined by

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$
$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

(a) Show that $\cosh^2(x) - \sinh^2(x) = 1$.
(b) Supply a domain and target for each of these functions so that it becomes invertible. The inverses are called **the inverse hyperbolic trig functions.**
(c) Sketch the graphs of these functions and their inverses.

7. A little Calculus Show that, of $x \neq 0$,

$$\frac{d}{dx}|x| = \frac{|x|}{x}$$

[This is easier than you think..] Having established this formula, now use the chain rule to find a formula for $\dfrac{d}{dx}|u|$, where u is a differentiable function of x.

8. Obtain algebraic formulas for $\sinh^{-1} x$, $\cosh^{-1} x$, $\tanh^{-1} x$. (In one of the cases, you should restrict the domain appropriately before taking the inverse.)

Some Answers for Section 1

1.(a) injective (b) injective (c) not injective ($f(\pi) = f(2\pi)$) (d) injective
2. (a) surjective (solving for x yields $x = \ln y + 4$) (b) not surjective (for instance, -1 is not in the range of f (c) not surjective ($-1 \leq x \leq 1 \Rightarrow -1 - \pi/2 \leq x + \arcsin x \leq 1 + \pi/2$, so that, for instance, 3 is not in the range of the given function.) (d) surjective
3. (a) invertible; inverse: $g\colon \mathbb{R}^+ \to \mathbb{R}; g(x) = \ln x + 4$ (b) invertible; inverse $g\colon (0,1) \to \mathbb{R}^+; g(x) = x/(1-x)$ (c) not surjective, and hence not invertible ($1/2$ is not in the range of f) (d) not surjective, and hence not invertible ($1/2$ is not in the range of f)
4. $y = f(x) \Rightarrow f^{-1}(y) = f^{-1}(f(x))$ (Applying f^{-1} to both sides) $\Rightarrow f^{-1}(y) = x$. Conversely, $x = f^{-1}(y) \Rightarrow f(x) = f(f^{-1}(y))$ (Applying f to both sides) $\Rightarrow f(x) = y$.
5. (a) To show they are equal take the natural logarithm of both sides. (b) Multiply both sides by $\ln a$ and then raise e to both sides.
6. (a) This is just algebra manipulation. (b) For cosh an appropriate domain and target are $[0, +\infty) \to [1, +\infty)$. For sinh they are $\mathbb{R} \to \mathbb{R}$.
7. To verify the formula, check that the left- and right-hand sides agree for $x < 0$ and $x > 0$. The desired formula is $\frac{d}{dx}|u| = \frac{|u|}{u}\frac{du}{dx}$.
8. $\cosh^{-1}(x) = \ln[x + \sqrt{x^2 - 1}]$, $\sinh^{-1}(x) = \ln[x + \sqrt{x^2 + 1}]$, $\tanh^{-1}(x) = \ln\sqrt{\frac{x+1}{x-1}}$. The function tanh should be restricted to a mapping $\mathbb{R} \to (-1, 1)$.

2. Properties of Log, Exponential, and Inverse Trig Functions

We first need a result about inverse functions and their derivatives.

Theorem 2.1 (Continuity and Differentiability of the Inverse)**.**
Let $f : A \to B$, be invertible, and let $a \in A$ with $f(a) = b$. If f is continuous at a, then its inverse $f^{-1} : B \to A$, is continuous at b. If f is differentiable at a, f^{-1} is differentiable at b, and

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

Note This is a fancy way of saying that

$$\frac{dx}{dy} = \frac{1}{dy/dx}.$$

Notation
$\log_e x = \ln x$
$\log_{10} x = \log x$

Properties of Log and Exponential Functions

Let x, y, r and a be real numbers with x, y and a, b positive and $a, b \neq 1$. Then:
Algebraic Properties (We will do quick examples of each in class)

(a) $\log_a xy = \log_a x + log_a y$
(b) $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
(c) $\log_a a = 1; \quad \log_a 1 = 0$
(d) $\log_a x^r = r \log_a x$
(e) $a^x = e^{x \ln a}$
(f) $\log_a x = \frac{\log_b x}{\log_b a}$
(g) $\lim_{x \to +\infty} \log_a x = +\infty$
(h) $\lim_{x \to 0^+} \log_a x = -\infty$

Derivatives
The functions exp and $\log_a$ are differentiable on their domains with:

(a) $\frac{d}{dx} e^x = e^x$
(b) $\frac{d}{dx} a^x = a^x \ln a$
(c) $\frac{d}{dx} \ln x = \frac{1}{x}$ and $\frac{d}{dx} \ln |x| = \frac{1}{x}$
(d) $\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$

In class, we write down the corresponding integral formulas, and also obtain the derivative formulas.

Examples 2.2.

A. Let $r = \ln 2$, $s = \ln 3$, and $t = \ln 37$. Express $\ln 9$, $\ln 1.5$, and $\ln 666$ in terms of r and s.

B. Compute $\dfrac{d}{dx}\ln(3x^2 - x)$

C. Prove that $\boxed{\dfrac{d}{dx}\ln|x| = \dfrac{1}{x}}$ if $x \neq 0$.

D. Compute $\dfrac{d}{dx}\left[\,\ln|3x^2 - x|\,\right]$

E. Compute $\dfrac{dy}{dx}$ by implicit differentiation if $y = \ln|x\tan y|$

F. Compute $\displaystyle\int \frac{3\ dx}{-5x+9}$

G. Compute $\displaystyle\int \frac{x\sin(x^2+1)}{1+\cos(x^2+1)}dx$

H. Use logarithmic differentiation to compute $\dfrac{d}{dx}[\,x^x\,]$.

I. Obtain the antiderivatives of tan, cot, sec, and cosec.

Properties of Inverse Trig Functions

Algebraic Properties (We will do quick examples of each in class)

(a) $\sin(\arcsin x) = x$ for every $x \in [-1, 1]$.
(b) $\arcsin(\sin x)) = x$ if $-\pi/2 \le x \le \pi/2$;
otherwise, it could equal $\pi - x + 2n\pi$ or $x + 2n\pi$
(c) $\cos(\arccos x) = x$ for every $x \in [-1, 1]$.
(d) $\arccos(\cos x)) = x$ if $0 \le x \le \pi$;
otherwise, it could equal $\pm(x + 2n\pi)$
(e) $\tan(\arctan x) = x$ for every real number x.
(f) $\arctan(\tan x)) = x$ if $-\pi/2 < x < \pi/2$;
otherwise, it could equal $x + n\pi$.

Derivatives

The inverse trig functions are differentiable on their domains with:

(a) $\dfrac{d}{dx}[\,\arcsin x\,] = \dfrac{1}{\sqrt{1-x^2}}$

(b) $\dfrac{d}{dx}[\,\arccos x\,] = \dfrac{-1}{\sqrt{1-x^2}}$

(c) $\dfrac{d}{dx}[\,\arctan x\,] = \dfrac{1}{1+x^2}$

Other Inverse Trig Functions One has the following inverse trig functions:

(a) $\cot^{-1} = \text{arccot}\colon \mathbb{R} \to (0, \pi)$
(b) $\sec^{-1} = \text{arcsec}\colon (-\infty, -1) \cup (1, +\infty) \to (0, \pi/2) \cup (0, \pi)$
(c) $\csc^{-1} = \text{arccosec}\colon (-\infty, -1) \cup (1, +\infty) \to (-\pi/2, 0) \cup (0, \pi/2)$
You will obtain their derivatives in the homework!

In class, we write down the corresponding integral formulas, and also obtain the derivative formulas.

Exercise Set 2.

1. **Online Practice with Logarithm Identities**
   Go to AppliedCalc.com and follow the links
   Student Web Site → Online Text → Using and Deriving Algebraic Properties of Logarithms.
   There you will find a link to "Exercises for this topic" which will take you to some practice exercises.
2. Use the identities from Exercise Set 1 # 5 to deduce derivative properties (b) and (d) of log and exponential functions.
3. Verify the following identities. [Hint: One way to show two differentiable functions with a connected domain are the same is to show (1) that they have the same domain and target; (2) that they have the same derivative and (3) agree at at least one point.]
   (a) $\cos^{-1} x = \dfrac{\pi}{2} - \sin^{-1} x$
   (b) $\sin^{-1} x = \dfrac{\pi}{2} - \cos^{-1} x$
4. Obtain the derivatives of all six inverse trig functions.
5. Evaluate the following:
   (a) $\dfrac{d}{dx}(\ln x)^2$
   (b) $\dfrac{d}{dx}\,[\ln|\tan x|)\,]$
   (c) $\dfrac{d}{dx}\sqrt{\ln x}$
   (d) $\dfrac{dy}{dx}$ if $y = x^3 \ln|3 - 2x|$

(e) $\dfrac{dy}{dx}$ if $y + \ln xy = 1$

(f) $\dfrac{d}{dx}\ln(\ln x)$

(g) $\dfrac{dy}{dx}$ if $y = \dfrac{x^2+3}{(x-1)(4x+6)(x^3-6x^2+1)}$

(h) $\dfrac{dy}{dx}$ if $y = x^{x^2-4x+1}$

(i) $\dfrac{d}{dx}\left[e^{x^2-1}\right]$

(j) $\dfrac{d}{dx}\left[2^{\sin x}\right]$

(k) $\dfrac{d}{dx}\left[2^{2^x}\right]$

(l) $\dfrac{d}{dx}\left[\sin(e^{x+\ln|x-3|})\right]$

(m) $\dfrac{d}{dx}\left[3^x\sin(2x^2-1)\right]$

(n) $\dfrac{d}{dx}\left[x^{x^x}\right]$

6. Simplify the following if possible, and state for which values of x the expressions are valid (that is, determine their natural domains):

(a) $e^{-\ln x}$

(b) $e^{\ln(x^2)}$

(c) $\ln(e^{-x^2})$

(d) $\ln\left(\dfrac{1}{e^x}\right)$

(e) $e^{3\ln x}$

(f) $\ln(xe^x)$

(g) $\ln(e^x - 3\sqrt{x})$

(h) $e^x - \ln x$

7. Solve for x:

(a) $\ln(\sqrt{x}) + \ln(x^{3/2}) = 1$

(b) $e^{2\pi x} = \sqrt{2}$

(c) $\ln(\cos x) = 0;\ 0 \le x < \pi/2$

(d) $e^{2x} + 2e^x + 1 = 9$

(e) $\arcsin(x^2 - x - 11) = \pi/2$

(f) $\arcsin x - \arccos x = \pi/2$

8. Evaluate the following integrals:

(a) $\displaystyle\int \frac{\sin(\ln x)}{x}dx$ $[-\cos(\ln x) + C]$

(b) $\displaystyle\int \frac{1}{x \ln x}dx$ $[\ln|\ln x| + C]$

(c) $\displaystyle\int \frac{1}{x \ln x \ln(\ln x)}dx$ $[\ln|\ln(\ln x)| + C]$

(d) $\displaystyle\int \frac{x^2 + x}{2x^3 + 3x^2 + 5}dx$ $[\frac{1}{6}\ln|2x^3 + 3x^2 + 5| + C]$

(e) $\displaystyle\int \frac{y}{y^2 - 1}dy$ $[\frac{1}{2}\ln|y^2 - 1| + C]$

(f) $\displaystyle\int_0^2 \frac{x + 1}{x + 2}dx$ $[2 - \ln 2]$

(g) $\displaystyle\int_0^1 \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$ $[\ln(\cosh 1)]$

9. Obtain the integrals of all six hyperbolic trig functions.

10. Evaluate, showing all work:

(a) $\displaystyle\int \frac{dx}{1 + (ax + b)^2}$ $(a \neq 0)$

(b) $\displaystyle\int \frac{p\,dx}{q + (ax + b)^2}$ $(a \neq 0,\ q > 0)$

(c) $\displaystyle\int \frac{dx}{ax^2 + bx + c}$ $(a > 0,\ b^2 - 4ac < 0)$

(d) $\displaystyle\int \frac{dx}{\sqrt{1 - (ax + b)^2}}$ $(a \neq 0,\ x \neq -b/a)$

(e) $\displaystyle\int \frac{dx}{\sqrt{ax^2 + bx + c}}$ $(a < 0,\ b^2 - 4ac > 0)$

(f) $\displaystyle\int \frac{dx}{x\sqrt{x^2 - 1}}$

Some Answers for Section 2

4. $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$, $\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$, $\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$, $\frac{d}{dx}\csc^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}}$

5.(a) $\frac{2}{x}\ln x$ (b) $\frac{\sec^2 x}{\tan x}$ (c) $\frac{1}{2x\sqrt{\ln x}}$ (d) $3x^2\ln|3-2x| - \frac{2x^3}{3-2x}$ (e) $-\frac{y}{x(y+1)}$ (f) $\frac{1}{x\ln x}$ (g) $\frac{x^2+3}{(x-1)(4x+6)(x^3-6x^2+1)}[\frac{2x}{x^2+3} - \frac{1}{x-1} - \frac{4}{4x+6} - \frac{3x^2-12x}{x^3-6x^2+1}]$ (h) $x^{x^2-4x+1}[(2x-4)\ln x + \frac{x^2-4x+1}{x}]$ (i) $2xe^{x^2-1}$ (j) $2^{\sin x}\cos x\,\ln 2$ (k) $2^{2^x}2^x(\ln 2)^2$ (l) $\cos(e^{x+\ln|x-3|})e^{x+\ln|x-3|}[1+\frac{1}{x-3}]$ (m)$3^x\ln 3\,\sin(2x^2-1) + 3^x 4x\cos(2x^2-1)$ (n) $x^{x^x}[x^x[\ln x+1]\ln x + \frac{x^x}{x}]$

6. (a) $1/x$; $(0,+\infty)$ (b) x^2; $(-\infty,0)\cup(0,+\infty)$ (c) $-x^2$; $\mathbb{R}$ (d) $-x$; $\mathbb{R}$ (e) x^3; $(0,+\infty)$ (f) $x+\ln x$; $(0,+\infty)$ (g) Cannot be simplified (h) Cannot be simplified

7. (a) $\sqrt{(e)}$ (b) $\frac{\ln 2}{4\pi}$ (c) 0 (d) $\ln 2$ (e) 4, -3 (f) If you look at the graphs of $y = \arcsin x$ and $y = \arccos x$ you find that negating the second and adding $\pi/2$ gives the first. In other words, for *every* x, we have $\arcsin x + \arccos x = \pi/2$. Subtracting this equation from the given one yields $\arccos x = 0$, so that $x = 1$ is the only solution.

8. Answers are shown next to the problems.

9. $\int \cosh x\, dx = \sinh x + C$, $\int \sinh x\, dx = \cosh x + C$, $\int \tanh x\, dx = \ln(\cosh x) + C$, $\int \operatorname{cotanh} x\, dx = \ln(\sinh x) + C$, $\int \operatorname{sech} x\, dx = \ln(\operatorname{sech} x + \tanh x) + C$, $\int \operatorname{cosech} x\, dx = \ln(\operatorname{cosech} x + \operatorname{cotanh} x) + C$

10. (a) $\frac{1}{a}\tan^{-1}(ax+b) + C$ (b) $\frac{p}{a\sqrt{q}}\tan^{-1}(\frac{ax+b}{\sqrt{q}}) + C$ (c) Complete the square and then rewrite the integrand as $\frac{4a}{4ac-b^2+(2ax+b)^2}$. At this point, you can apply (b) directly. (Note that the requirement $b^2 - 4ac < 0$ gives us the condition $q > 0$ in part (b). (d) $\frac{1}{a}\sin^{-1}(ax+b) + C$ (e) Complete the square again and rewrite the integrand as $\sqrt{\frac{-4a}{(b^2-4ac)-(2ax+b)^2}}$ Notice that the $-4a$ is in fact positive by assumption, and so we have reduced the integral to one that can be expressed in terms of $\sin^{-1}(2ax+b)$, similar to (c). (f) $\sec^{-1}x + C$

3. Integration by Parts

We derive the following formula in class using the product rule:

Integration by Parts Formula

$$\int f \cdot g \; dx = I(f) \cdot g - \int I(f) \cdot D(g) \; dx$$

We can use the following tabular representation:

$$\begin{array}{cccc} & D & & I \\ \oplus & f & & g \\ & & \searrow & \\ \ominus \int & Df & \rightarrow & Ig \end{array}$$

If the product on the bottom row cannot be readily integrated, continue the process for as many steps as convenient:

$$\begin{array}{cccc} & D & & I \\ \oplus & f & & g \\ & & \searrow & \\ \ominus & Df & & Ig \\ & & \searrow & \\ \oplus \int & D^2 f & \rightarrow & I^2 g \end{array}$$

Examples 3.1.

A. $\displaystyle\int xe^x \; dx$

B. $\displaystyle\int x^2 e^{-x} \; dx$

C. $\displaystyle\int (3x^3 - x^2) 2e^{2x} \; dx$

D. $\displaystyle\int x \ln x \; dx$

E. $\displaystyle\int \ln x \; dx$

F. $\displaystyle\int_0^1 (x^2 + 1) e^{-2x+1} \; dx$

G. $\displaystyle\int_0^1 (2x - 1) e^{x^2 - x} \; dx$

H. $\displaystyle\int e^{ax} \sin bx \; dx$

Example 3.2. What Happens When We Integrate "Backwards" Suppose that f is has derivatives of all orders[1] in an interval containing $[a,b]$, and that $x \in [a,b]$. Start with the formula

$$\int_a^x f'(t)\ dt = f(x) - f(a)$$

so that

$$f(x) = f(a) + \int_a^x f'(t)\ dt$$

We now use integration by parts in a funny way to evaluate the integral:

$$\begin{array}{cccc}
 & \underline{D} & & \underline{I} \\
\oplus & f'(t) & & 1 \\
 & & \searrow & \\
\ominus & f''(t) & & (t-x) \\
 & & \searrow & \\
\oplus & f'''(t) & & \dfrac{(t-x)^2}{2} \\
 & & \searrow & \\
\ominus & \dots & & \dots \\
 & & \searrow & \\
(-1)^{n-1} & f^{(n)}(t) & & \dfrac{(t-x)^{n-1}}{(n-1)!} \\
 & & \searrow & \\
(-1)^n \int & f^{(n+1)}(t) & \to & \dfrac{(t-x)^n}{n!}
\end{array}$$

In class, we see what this gives us...

Exercise Set 3.

Evaluate the following integrals:

1. $\displaystyle\int x \sin x\ dx$

2. $\displaystyle\int 3x^2 \cos(4x-1)\ dx$

3. $\displaystyle\int 3x \cos(4x^2-1)\ dx$

4. $\displaystyle\int (x^2+1)e^{3x+1}\ dx$

5. $\displaystyle\int \frac{2x+1}{e^{3x}}\ dx$

6. $\displaystyle\int \frac{x\ dx}{(x-2)^3}$

[1] We call such a function a **smooth** or $\boldsymbol{C^\infty}$ function.

7. $\displaystyle\int_1^e x^2 \ln x \; dx$

8. $\displaystyle\int_0^1 x \ln(x+1) \; dx$

9. $\displaystyle\int \arctan x \; dx$

10. $\displaystyle\int \arcsin x \; dx$

11. $\displaystyle\int_0^\pi e^{3x} \sin 2x \; dx$

12. $\displaystyle\int_0^\pi \sin x \cosh x \; dx$

13. $\displaystyle\int \sec^3 x \; dx$ [Hint: Write the integrand as $\sec x \; \sec^2 x$.]

Some Answers for Section 3

1. $-x\cos x + \sin x + C$ **2.** $\sin(4x-1)[\frac{3x^2}{4} - \frac{3}{32}] + \frac{3}{8}\cos(4x-1) + C$ **3.** $\frac{3}{8}\sin(4x^2-1) + C$ **4.** $\frac{e^{3x+1}}{27}(9x^2 - 6x + 11) + C$ **5.** $-\frac{1}{9e^{3x}}(6x+5) + C$ **6.** $-\frac{x}{2(x-2)^2} - \frac{1}{2(x-2)} + C$ **7.** $\frac{2e^3-1}{9}$ **8.** $\frac{1}{4}$ **9.** $x\arctan x - \frac{1}{2}\ln(1+x^2) + C$ **10.** $x\arcsin x + \sqrt{1-x^2} + C$ **11.** $\frac{2}{13}[1 - e^{3\pi}]$ **12.** $\frac{1}{2}[\cosh\pi + 1]$ **13.** $\frac{1}{2}[\sec x \tan x + \ln|\sec x + \tan x|] + C$

4. Integrating Powers of Trig Functions

(a) Integrating Powers of Sine and Cosine

Integrals of the form $\int \sin^m x \cos^n x \; dx$ where at least one of m or n is odd:

If m is odd, convert all but one one the sines into cosines and use $u = \cos x$.
If n is odd, do it the other way 'round. (If both are odd, take your pick.)
If they're both even, see below.

Examples 4.1.

A. $\int \sin^2 x \cos^3 x \; dx$

B. $\int \sin^3 x \; dx$

C. $\int \sin x \cos x \; dx$

Integrals of the form $\int \sin^m x \cos^n x \; dx$ where both m and n are even:

Use the trig identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

to reduce all the exponents by 2.

Examples 4.2.

A. $\int \sin^2 x \; dx$

B. $\int \sin^4 x \; dx$

C. $\int \sin^6 x \; dx$

D. $\int \sin^2 x \cos^2 x \; dx$

(b) Integrating Powers of Tan and Secant

Integrals of the form $\int \tan^n x \; dx$**:**

Here, replace two of the tan's with sec's by the formula

$$\tan^2 x = \sec^2 x - 1.$$

This gives you two integrals—one can be done by putting $u = \tan x$ and the other by repeating the process if necessary.

Examples 4.3.

A. $\int \tan x \; dx$

B. $\int \tan^2 x \; dx$

C. $\int \tan^3 x \; dx$

D. $\int \tan^4 x \; dx$

Integrals of the form $\int \sec^n x \; dx$**:**

For these, we develop a reduction method using integration by parts.

Examples 4.4.

A. $\int \sec x \; dx$

B. $\int \sec^2 x \; dx$

C. $\int \sec^3 x \; dx$

D. $\int \sec^4 x \; dx$

Integrals of the form $\int \tan^n x \; \sec^m x \; dx$**:**

If n is even, replace all the tans by secants and use the preceding rule. If n is odd, replace all but one of the tans by secants, and obtain two integrals; one a power of secant, and the other doable by putting $u = \sec x$.

Examples 4.5.

A. $\int \tan x \ \sec^2 x \ dx$

B. $\int \tan^2 x \ \sec^2 x \ dx$

C. $\int \tan^3 x \ \sec x \ dx$

Exercise Set 4.

1. Evaluate the following integrals.

a. $\int \sin^4 x \ \cos^5 x \ dx$

b. $\int \frac{\sin^5 x}{\cos^2 x}$

c. $\int \cos^2 x \ dx$

d. $\int \cos^4 x \ dx$

e. $\int \cos^6 x \ dx$

f. $\int \tan^6 x \ dx$

g. $\int \cot^2 x \ dx$

h. $\int \cot^3 x \ dx$

i. $\int \sec^6 x \ dx$

j. $\int \tan^3 x \ \sec^2 x \ dx$

k. $\int \tan^3 x \ \sec^3 x \ dx$

2. Use integration by parts to obtain the following reduction formulas:

$$\int \sin^n x \ dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \ dx$$

$$\int \cos^n x \ dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \ dx$$

3. Come up with formulas (depending on n) for the integrals

$$\int_0^{n\pi/2} \sin^2 x\ dx \qquad \text{and} \qquad \int_0^{n\pi/2} \cos^2 x\ dx$$

[Hint: You can evaluate these integrals by drawing pictures...]

4. Integrals of the form $\int \cos mx \, \sin nx \, dx$

For these, we use the following identities:

$$\sin A \cos B = \tfrac{1}{2}[\sin(A-B) + \sin(A+B)]$$
$$\cos A \cos B = \tfrac{1}{2}[\cos(A-B) + \cos(A+B)]$$
$$\sin A \sin B = \tfrac{1}{2}[\cos(A-B) - \cos(A+B)]$$

a. Evaluate $\int \sin 5x\ \cos 4x\ dx$

b. Prove that $\int_0^{2\pi} \sin mx\ \cos nx\ dx = 0$ for each pair of positive integers m, n.

c. Prove that $\int_0^{2\pi} \cos mx\ \cos nx\ dx = \begin{cases} \pi & \text{if } m = n \text{ (positive integers)}, \\ 0 & \text{if } m \neq n. \end{cases}$

d. Prove that $\int_0^{2\pi} \sin mx\ \sin nx\ dx = \begin{cases} \pi & \text{if } m = n \text{ (positive integers)}, \\ 0 & \text{if } m \neq n. \end{cases}$

5. Write $\sin x + \cos x$ in the form $\sqrt{2}[\sin x(\frac{1}{\sqrt{2}}) + \cos x(\frac{1}{\sqrt{2}})]$ and hence in the form $\sin(x+\phi)$. Use this to evaluate $\int \frac{dx}{\sin x + \cos x}$.

6. Use the method in the previous exercise to evaluate

$$\int \frac{dx}{A \sin x + B \cos x}$$

for constants A and B, not both zero.

Some Answers for Section 4

1. (a) $\frac{\sin^5 x}{5} - 2\frac{\sin^7 x}{7} + \frac{\sin^9 x}{9} + C$ (b) $\sec x + 2\cos x - \frac{\cos^3 x}{3} + C$ (c) $\frac{x}{2} + \frac{1}{4}\sin 2x + C$ (d) $\frac{1}{4}[\frac{3x}{2} + \sin 2x + \frac{1}{8}\sin 4x] + C$ (e) $\frac{1}{8}[\frac{5x}{2} + \frac{5\sin 2x}{2} + \frac{3\sin 4x}{8} - \frac{\sin^3 2x}{6}] + C$ (f) $\frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + C$ (g) $-\cot x - x + C$ (h) $-\frac{\cot^2 x}{2} - \ln|\sin x| + C$ (i) $\frac{1}{5}\sec^4 x \tan x + \frac{4}{15}\sec^2 x \tan x + C$ (j) $\frac{\tan^4 x}{4} + C$ (k) $\frac{\sec^5}{5} - \frac{\sec^3}{3} + C$ **3.** Both integrals are equal to $n\pi/4$. **4.** (a) $-\frac{1}{2}[\cos x + \frac{\cos 9x}{9}] + C$ **5.**

$-\frac{1}{\sqrt{2}} \ln|\csc(x+\pi/4)+\cot(x+\pi/4)|+C$ **6.** $-\frac{1}{\sqrt{A^2+B^2}} \ln|\csc(x+\theta)+\cot(x+\theta)|+C$, where $\theta = \begin{cases} \tan^{-1}(B/A) & \text{if } A>0; \\ \tan^{-1}(B/A)+\pi & \text{if } A<0; \\ \pi/2 & \text{if } A=0. \end{cases}$

5. Trig Substitution

We use trig substitution to deal with powers of quadratic expressions as in, for example, $\sqrt{x^2+k}$, $(k-x^2)^3$, and so on. Here are the substitution rules that always work:

Trig Substitutions

Expression	**Substitution**
$a^2 - b^2x^2$	$x = \frac{a}{b}\sin\theta$
$a^2 + b^2x^2$	$x = \frac{a}{b}\tan\theta$
$a^2x^2 - b^2$	$x = \frac{b}{a}\sec\theta$

Examples 5.1.

A. $\displaystyle\int \sqrt{x^2+4}\,dx$

B. $\displaystyle\int \sqrt{4x^2-9}\,dx$

C. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+4}}$

D. $\displaystyle\int \frac{x^2\,dx}{\sqrt{5-x^2}}$

E. $\displaystyle\int_0^1 \frac{dx}{1+x^2}$

In order to deal with those annoying middle terms in quadratics, we first get rid of them by completing the square, as in the following examples.

Examples 5.2.

A. $\displaystyle\int \sqrt{x^2+x+4}\,dx$

B. $\displaystyle\int \frac{dx}{x^2-2x+5}$

Exercise Set 5.

1. Evaluate the following integrals (taken from Anton's book)

a. $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

b. $\displaystyle\int \frac{dx}{(4+x^2)^2}$

c. $\int e^x\sqrt{1-e^{2x}}\;dx$

d. $\int_{\sqrt{2}}^{2} \frac{dx}{x^2\sqrt{x^2-1}}$

e. $\int_{1}^{3} \frac{dx}{x^4\sqrt{x^2+3}}$

f. $\int \sqrt{x^2-1}$. Do this one two ways: (i) (easier!) using the regular substitution and (ii) (harder) using an appropriate hyperbolic trig substitution. Check that the two answers you obtain agree. [You might need to consult Exercise 1 # 8 to check the equivalence of the two results.]

g. $\int \frac{dx}{x^2-4x+13}$

h. $\int \frac{dx}{\sqrt{x^2-6x+10}}$

i. $\int \sqrt{3-2x-x^2}\;dx$

Some Answers for Section 5

1. (a) $\frac{9}{2}\sin^{-1}(x/3)-\frac{1}{2}x\sqrt{9-x^2}+C$ (b) $\frac{1}{16}\tan^{-1}(x/2)+\frac{x}{8(4+x^2)}+C$ (c) $\frac{1}{2}\sin^{-1}(e^x)+\frac{1}{2}e^x\sqrt{1-e^{2x}}+C$ (d) $(\sqrt{3}-\sqrt{2})/2$ (e) $\frac{10\sqrt{3}+18}{243}$ (f) (ii) $\frac{1}{4}\sqrt{x^2-1}-\frac{1}{2}\cosh^{-1}x+C$ (g) $\frac{1}{3}\tan^{-1}(\frac{x-2}{3})+C$ (h) $\ln|x-3+\sqrt{(x-3)^2+1}|+C$ (i) $2\sin^{-1}(\frac{x+1}{2})+\frac{1}{2}(x+1)\sqrt{3-2x-x^2}+C$

6. Partial Fractions

A rational function is a ratio of two polynomials, eg.

$$\frac{3}{x^2-1}, \qquad \frac{5x^2+4x-2}{x^6-x^5}, \qquad \frac{x}{(x+1)(x^2+4)}.$$

We see how to break these up into sums of simpler rational functions. This takes two steps:

Step 1: A **proper rational function** has the degree of the numerator < degree of the denominator. If that's not the case, Step 1 is to divide, and rewrite the rational function in the form:

$$\text{quotient } + \frac{\text{remainder}}{\text{denominator}}$$

The term remainder/denominator is then proper.

Example 6.1. Find $\int \frac{3x^2}{x^2+1}\,dx$

Answer: We do Step 1 (since that's all we can do at this stage) & hope for the best.

Step 2: We now have a proper rational function to work with. By the Fundamental Theorem of Algebra, no matter how messy the denominator is, it can be completely factored into linear terms-of the form $(ax+b)$ and irreducible quadratic terms-of the form (ax^2+bx+c) with $^{-}4ac < 0$. So Step 2 is simply to factor the denominator completely.

Step 3: *The meat of the matter.* We now consider various cases, depending on what the factorization looks like.

Case 1 All the factors of the form $(x-a)$ are distinct.

Example: Find $\int \frac{dx}{x^2+x-2}$.

Case 2 All factors of the form $(x-a)$ but with repeating terms.

Example: Find $\int \frac{2x+4}{x^3-2x^2}\,dx$.

Case 3 Irreducible quadratics

Example: Find $\int \frac{x^2+x-2}{(3x-1)(x^2+1)}\,dx$.

Finally, we look at a HUGE fake example which has everything.

Exercise Set 6.

1. Evaluate the following integrals (taken from Anton's book)

a. $\int \frac{dx}{x^2+3x-4}$

b. $\int \frac{2x^2-9x-9}{x^3-9x}\,dx$

c. $\int \frac{3x^2-10}{x^2-4x+4}\,dx$

d. $\displaystyle\int \frac{2x^2+3}{x(x-1)^2}\,dx$

e. $\displaystyle\int \frac{x^2}{(x+2)^3}\,dx$

f. $\displaystyle\int \frac{x^3+3x^2+x+9}{(x^2+1)(x^2+3)}\,dx$

g. $\displaystyle\int \frac{\cos\theta}{\sin^2\theta+4\sin\theta-5}\,d\theta$

2. List all the possible kinds of integrals that can arise from the use of partial fractions, and how you would integrate them (providing enough details to convince a skeptic. Is it true that the method of partial fractions permits one, in principle, to integrate every rational function?

3. **a.** Show that $\displaystyle\int_0^1 \frac{x}{x^4+1}\,dx = \frac{\pi}{8}$.

b. Now use "long division" or a geometric series to represent the integrand $x/(1+x^4)$ as a geometric series with common ratio $-x^4$. Next, integrate this series term-by-term, and hence obtain a formula for π as an infinite sum.

4. **a.** Use the technique of the preceding exercise to calculate a formula for computing $\ln 2$. [Hint: Consider $1/(x+1)$.]

b. Now obtain an expression for $\ln a$ where $0 < a \le 1$

c. Use the formula in part (b) to obtain an expression for the natural logarithm of any positive real number. [Hint: What is the easiest algebraic way to convert a number bigger than 1 to one smaller than 1?]

Some Answers for Section 6

1. **a.** $\frac{1}{5}\ln|\frac{x-1}{x+4}|+C$ **b.** $\ln|\frac{x(x+3)^2}{x-3}|+C$ **c.** $3x+12\ln|x-2|-\frac{2}{x-2}+C$ **d.** $3\ln|x|-\ln|x-1|-5/(x-1)+C$ **e.** $\ln|x+2|+4/(x+2)-2/(x+2)^2+C$ **f.** $3\tan^{-1}x+\frac{1}{2}\ln(x^2+3)+C$ **g.** $\frac{1}{6}\ln|\frac{1-\sin\theta}{5+\sin\theta}|+C$

2. Since the denominators all factor into irreducible quadratics or linear factors, we can arrange, after making the partial fractions "proper," that the numerators are all linear and the denominators are all linear or quadratic, we are reduced to things of the following forms:

$$\text{Any polynomial,}\quad \frac{a}{(bx+c)^n},\ \text{or}\ \frac{ax+b}{(cx^2+dx+e)^n}$$

The only integrals that might present issues are those of the last type, but those break up further, after completing squares, into things like

$$\frac{a}{(b(x-c)^2+d)^n},\ \text{and}\ \frac{x-c}{(b(x-c)^2+d)^n}$$

For the first kind, we can just write down the antiderivative. For the second, trig substitution yields, after some manipulation, one of the following types:

$$\frac{\cos^2 x}{\sin^n x}, \quad \frac{\sec^2 x}{\tan^m x}, \quad \text{or} \quad \frac{\tan^2 x}{\sec^m x}$$

which can all be done by substitution (convert to sines and cosines for the last one...)

3 a. Put $u = x^2$ to do the integration. **b.** The geometric series is

$$\frac{x}{1+x^4} = x - x^5 + x^9 - \cdots + (-1)^n x^{4n+1} + \ldots$$

The desired formula for π is

$$\pi = 8\Big(\frac{1}{2} - \frac{1}{6} + \frac{1}{10} - \cdots + \frac{(-1)^n}{4n+2} + \ldots\Big).$$

4 $\ln a = a - \frac{a^2}{2} + \frac{a^3}{3} - \cdots + (-1)^{n+1}\frac{a^n}{n} + \ldots$

7. Numerical Integration

1. Trapezoid Rule: This is the average of the left- and right-hand sums:

> If $x_k = a + k\Delta x = a + k\frac{b-a}{n}$, then
> $$\int_a^b f(x)\ dx \approx \frac{b-a}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Example 7.1. Evaluate $\int_0^1 x^2\ dx$, (a) using 4 intervals and (b) using 10 intervals.

2. Simpson's Rule: Based on using parabolic curve passing through successive triples:

> If $x_k = a + k\Delta x = a + k\frac{b-a}{n}$, then
> $$\int_a^b f(x)\ dx \approx \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2} + 4f(x_{n-1}) + f(x_n)]$$

Question Why?
Answer As with the trapezoid rule, we want to approximate the areas in each strip by something more complicated than a rectangle. This time we take the strips in pairs (which is why we need an even number of them) and draw a parabola through the three points $(x_{k-1}, f(x_{k-1})), (x_k, f(x_k))$, and $(x_{k+1}, f(x_{k+1}))$, as shown in a figure we will see in class.

Example 7.2. Use four intervals in Simpson's Rule to approximate $\int_0^1 x^2\ dx$.

> **The Errors in the Trapezoid Rule and Simpson's Rule**
> **Error in Trapezoid Rule:** If $f''(x)$ is continuous in $[a, b]$, then the error in the trapezoid rule is no larger than
> $$\frac{(b-a)^3}{12n^2}|f''(M)|,$$
> where $|f''(M)|$ is the largest value of $|f''(x)|$ in $[a, b]$.

Error in Simpson's Rule: If $f^{(4)}(x)$ is continuous in $[a, b]$, then the error in Simpson's rule is no larger than

$$\frac{(b-a)^5}{180n^4}|f^{(4)}(M)|,$$

where $|f^{(4)}(M)|$ is the largest value of $|f^{(4)}(x)|$ in $[a, b]$.

Examples 7.3.

A. How accurate is the answer to the Example 7.1 above?
B. How accurate is the answer to the Example 7.2 above?
C. Going back to Example 7.1, how large must n be to approximate the answer to 5 decimal places?

Exercise Set 7.

Go online to AppliedCalc.com and follow the path
Student Web Site → Online text → Numerical Integration → Exercises for This Topic
Do all the exercises listed, especially the "Communication and reasoning" ones.

8. Improper Integrals

All the definite integrals we have seen have the form $\int_a^b f(x)\,dx$, where a and b are finite and $f(x)$ is piecewise-continuous on the closed interval $[a, b]$. There are occasions when we would like to relax these requirements, and when we do so we obtain what are called improper integrals. There are various types of improper integrals.

Improper Integral with an Infinite Limit of Integration

We define

$$\int_a^{+\infty} f(x)\ dx = \lim_{M\to\infty} \int_a^M f(x)\ dx,$$

provided the limit exists. If the limit exists, we say that $\int_a^{+\infty} f(x)\ dx$ **converges**. Otherwise, we say that $\int_a^{+\infty} f(x)\ dx$ **diverges.** Similarly, we define

$$\int_{-\infty}^{b} f(x)\ dx = \lim_{M\to-\infty} \int_M^b f(x)\ dx,$$

provided the limit exists. Finally, we define

$$\int_{-\infty}^{+\infty} f(x)\ dx = \int_{-\infty}^{a} f(x)\ dx + \int_a^{+\infty} f(x)\ dx,$$

for some convenient a, provided both integrals on the right converge.

Examples 8.1.

A. $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\ dx$

B. $\displaystyle\int_{-\infty}^{0} \frac{dx}{(3x-1)^{4/3}}$

C. For which p does $\displaystyle\int_1^{+\infty} \frac{1}{x^p}\ dx$ converge?

D. (An application) It was estimated that from 2000, sales of freon would decrease continuously with a fractional rate of decrease of 15% per year. Further, sales of freon at the start of 2000 were estimated to be 35 million pounds per year.[2] What will be the total future sales of freon starting in 2000? [Use $s(t) = 35e^{-0.15t}$ (t is the number of years since 2000)]

Improper Integrals where the Integrand Becomes Infinite

If $f(x)$ is defined for all x with $a < x \le b$ but becomes infinite at $x = a$,

[2]These figures are approximations based on published data. (Source: The Automobile Consulting Group/The New York Times, December 26, 1993, p. F23

we define

$$\int_a^b f(x)\, dx = \lim_{r\to a^+} \int_r^b f(x)\, dx,$$

provided the limit exists. Similarly, if $f(x)$ is defined for all x with $a \le x < b$ but becomes infinite at $x = b$, we define

$$\int_a^b f(x)\, dx = \lim_{r\to b^-} \int_a^r f(x)\, dx,$$

provided the limit exists. In either case, if the limit exists, we say that $\int_a^b f(x)\, dx$ **converges.** Otherwise, we say that $\int_a^b f(x)\, dx$ **diverges.**

If the integrand becomes infinite *between* the limits of integration, we break it into two or more improper integrals, each having at most one bad point at one of the end-points.

Examples 8.2.

A. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\, dx$

B. $\displaystyle\int_{-1}^3 \frac{x}{x^2-9}\, dx$

C. $\displaystyle\int_{-3}^3 \frac{1}{x^2}\, dx$

D. $\displaystyle\int_{-8}^8 \frac{1}{x^{1/3}}\, dx$

The first of the following tests is mentioned in textbooks; the second usually is not:

Theorem 8.3. *Comparison Tests for Improper Integrals*

(a) S'pose f is piecewise continuous on $[a,+\infty)$ and $|f(x)| \le g(x)$ for $x \ge$ some M with $\int_M^{+\infty} g(x)\, dx$ convergent. then $\int_a^{+\infty} f(x)\, dx$ converges too.

(b) S'pose f is piecewise continuous on $[a,b)$ and $|f(x)| \le g(x)$ for $x \ge$ some $m \in [a,b)$ with $\int_m^b g(x)\, dx$ convergent. then $\int_a^b f(x)\, dx$ converges too.

(c) S'pose f is piecewise continuous on $(a,b]$ and $|f(x)| \le g(x)$ for $x \le$ some $m \in (a,b]$ with $\int_a^m g(x)\, dx$ convergent. then $\int_a^b f(x)\, dx$ converges too.

Examples 8.4.

A. $\displaystyle\int_0^{+\infty} e^{-x^2}\, dx$

B. $\displaystyle\int_0^1 \frac{sinx}{x}\, dx$

C. For what s does the integral $\int_0^{+\infty} e^{-st} f(t)\ dt$ converge for every function f with $|f(t)| \leq e^{kt}$?

An Interesting Type of Indefinite Integral: The Laplace Transform

If f is a function of t , we then define a certain associated function $F(s)$ of s by the formula:

$$F(s) = \int_0^{+\infty} e^{-st} f(t)\ dt$$

This new function of s is called the **Laplace transform of** f, provided it exists. Notice that Example 8.4 C gives a sufficient condition for the Laplace Transform to exist for $k \leq s < +\infty$.

Examples 8.5. Find F(s) in each of the following cases, giving the domain of each:

A. $f(t) = 1$
B. $f(t) = t$
C. $f(t) = t^n$
D. $f(t) = e^{\alpha t}$; $\alpha \neq 0$
E. $f(t) = \sin \beta t$; $\beta \neq 0$

Exercise Set 8.

1. Determine whether each of the following integrals converge, justifying your claims. (You can justify the convergence of an integral by either evaluating it, or by using a comparison.)

a. $\int_0^{+\infty} e^{-x}\ dx$

b. $\int_{-\infty}^{-1} \frac{1}{x^{1/3}}\ dx$

c. $\int_0^{+\infty} (2x-4)e^{-x}\ dx$

d. $\int_0^{+\infty} \sin 4x\, e^{-x}\ dx$

e. $\int_0^{+\infty} (3x^4 - x^3 + 1)e^{-x^2}\ dx$

f. $\int_{-1}^{2} \frac{3}{(x+1)^2}\ dx$

g. $\int_{-1}^{2} \frac{3}{(x+1)^{1/2}}\ dx$

h. $\int_{-1}^{2} \frac{3x}{x^2-1}\ dx$

i. $\displaystyle\int_0^{+\infty} \frac{1}{x\ln x}\,dx$

j. $\displaystyle\int_0^{+\infty} \sin x\ dx$

k. $\displaystyle\int_0^{+\infty} \frac{\sin x}{xe^x}\,dx$

2. Compute the Laplace Transform F of each of the given functions $f\colon [0,+\infty) \to \mathbb{R}$, indicating the domain of F in each case.

a. $f(t) = k$ (k constant)

b. $f(t) = \cos\alpha t$ ($\alpha \neq 0$ constant)

c. $f(t) = \sinh\alpha t$ ($\alpha \neq 0$ constant)

d. $f(t) = \cosh\alpha t$ ($\alpha \neq 0$ constant)

e. $f(t) = u_c(t)t$ ($c > 0$ constant), where u_c is the *unit step function:* $u_c(t) = \begin{cases} 0 & \text{if } t < c; \\ 1 & \text{if } t \geq c. \end{cases}$

3. *Martian Education* Let $M(t)$ be the number of high school students graduated in the Republic of Mars in year t. This number is projected to change at a rate of about

$$M'(t) = 0.321t^{-1.10} \text{ thousand graduates per year} \quad (0 \leq t \leq 50)$$

where t is time in years since 2020. In 2021, there were about 1300 high school students graduated. By extrapolating the model, what can you say about the number of high school students graduated in a year far in the future?

4. *The Gamma Function* The **gamma function** is defined by the formula

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}\ dt$$

a. Show that the integral converges for every non-negative value of x.

b. Compute $\Gamma(1)$ and $\Gamma(2)$.

c. Use integration by parts to show that for every positive integer n, $\Gamma(n+1) = n\Gamma(n)$.

d. Deduce that $\Gamma(n) = (n-1)!$ for every positive integer n.

5. *Error Function* The **error function,** erf, is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\ dt$$

Estimate (numerically or otherwise) $\lim^{x\to+\infty} \operatorname{erf}(x)$.

6. Why can't the Fundamental Theorem of Calculus be used to evaluate $\int_{-1}^{1} \frac{1}{x}\,dx$?

7. It sometimes happens that the Fundamental Theorem of Calculus gives the correct answer for an improper integral. Does the FTC give the correct answer for improper integrals of the form

$$\int_{-a}^{a} \frac{1}{x^{1/r}}\, dr$$

if $r = 3, 5, 7, \ldots$?

Some Answers for Section 8

1. a. Converges to 1. **b.** Diverges **c.** Converges to -2 **d.** Converges **e.** Converges **f.** Diverges **g.** Converges to $6\sqrt{3}$ **h.** Diverges **i.** Diverges **j.** Diverges **k.** Converges **2 a.** $\frac{k}{s}$; $s > 0$ **b.** $\frac{s}{s^2+\alpha^2}$; $s > 0$ **c.** $\frac{\alpha}{s^2-\alpha^2}$; $s > |\alpha|$ **d.** $\frac{s}{s^2-\alpha^2}$; $s > |\alpha|$ **e.** $\frac{e^{-}cs}{s}$ **3.** The integral converges to 3.21 thousand, so the total number of graduates is projected to be $3210+1300 = 4510$ high school graduates per year. **4. a** $\Gamma(1) = 1$; $\Gamma(2) = 1$ **5.** 1 **6.** The integral does not converge, so the number given by the FTC is meaningless. **7.** Yes; the integrals converge to 0, and the FTC also gives 0.

9. Infinite Sequences

Definition 9.1. A **sequence** is an ordered set of numbers (with possible repetitions)

$$(a_0, a_1, \ldots, a_n, \ldots) = (a_n)_{n=0}^{\infty}$$

a_n is called the **general term** of the sequence.

Examples 9.2.

A. $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ or $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right)$

B. $(1)_{n=1}^{\infty}$ or $(1, 1, 1, \ldots, 1, \ldots)$

C. $(n)_{n=3}^{\infty}$ or $(3, 4, 5, \ldots, n, \ldots)$

D. $\left(\frac{1}{n^2}\right)_{n=1}^{\infty}$ or $\left(1, \frac{1}{4}, \frac{1}{9}, \ldots, \frac{1}{n^2}, \ldots\right)$

E. $\left(\frac{(-1)^n}{n}\right)_{n=1}^{\infty}$ or $\left(-1, \frac{1}{2}, -\frac{1}{3}, \ldots, \frac{(-1)^n}{n}, \ldots\right)$

Express the following in "bracket notation:"

F. $\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right)$

G. $\left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots\right)$

H. $(1, 3, 5, 7, \ldots)$

I. $\left(1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, \ldots\right)$

J. $\left(-1, \frac{1}{3}, -\frac{1}{5}, \frac{1}{7}, \ldots\right)$

K. $\left(-\frac{1}{2}, \frac{1}{4}, -\frac{1}{6}, \ldots\right)$

L. $\left(6, 3, \frac{3}{2}, \frac{3}{4}, \ldots\right)$

Graphical Representation and Limits of Sequences

Given a sequence $(a_1, a_2, ..., a_n, ...)$, we can represent it graphically by plotting the points $(1, a_1), (2, a_2), ..., (n, a_n), ...$ For instance, we can represent the sequence $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right)$ by plotting the points

$$(1,1),\left(2,\frac{1}{2}\right),\left(3,\frac{1}{3}\right),\ldots,\left(n,\frac{1}{n}\right)$$

as shown:

Notice that the points suggest the curve $y = 1/x$; in fact they lie exactly on that curve because this sequence is just the function $f(x) = 1/x$ with domain restricted to $\{1, 2, ...\}$, and we already know from our study of limits of functions that

$$\lim_{x\to+\infty}\frac{1}{x} = 0$$

So, by following the curve $y = 1/x$, the points also approach $y = 0$ asymptotically: The values $1/n$ approach 0 as $n \to \infty$. We express this property in limit notation by writing

$$\lim_{n\to\infty}\frac{1}{n} = 0,$$

and we say that the sequence $(1/n)_{n\geq 0}$ **converges to** 0**.**

Preliminary Definition of a Limit of a Sequence
The sequence (a_n) is said to **converge to the limit** L **as** $n \to +\infty$ if the values a_n approach the real number L as n gets large.
If instead, the values approach $\pm\infty$, we say that the sequence **diverges to** $\pm\infty$**.**
Otherwise we just say that the sequence **diverges,** period.

Generalizing the above example, gives:

Theorem 9.3. Theorem F: Functions and Sequences

(a) If f is a function such that $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} f(n) = L$ as well. That is, the sequence $(f(n))$ converges to L as well.
(b) If f is a function such that $\lim_{x\to\infty} f(x) = \pm\infty$, then $\lim_{n\to\infty} f(n) = \pm\infty$ as well. That is, the sequence $(f(n))$ also diverges to $+\infty$ or $-\infty$; as the case may be.

Examples 9.4.

A. $\lim\limits_{n\to\infty} n^2$

B. $\lim\limits_{n\to\infty} \dfrac{n}{1+2n}$

C. $\lim\limits_{n\to\infty} \dfrac{3n^2-2n+1}{\sqrt{2n^4+n-7}}$

D. $\lim\limits_{n\to\infty} \dfrac{\ln n}{n}$

E. $\lim_{n\to\infty} \frac{n}{e^n}$

Fact All the basic rules that hold for limits of functions of x also hold for limits of sequences. In particular, as we saw, we can use l'Hospital with due caution. We also have the following very nice one:

Theorem 9.5. Sandwich Rule *Assume (a_n), (b_n), and (c_n) are three sequences with*

$$a_n \leq b_n \leq c_n \texttt{ for } n \geq k$$

for some k. Then, if a_n and c_n both converge to L, so does b_n.

Examples 9.6.

A. $\lim_{n\to\infty} \frac{(-1)^n}{n}$

B. $\lim_{n\to\infty} r^n$ for $|r| < 1$, $|r| = 1$, and $|r| > 1$

C. $\lim_{n\to\infty} n\sqrt{n}$

D. $\lim_{n\to\infty} \frac{\cos[(-1)^n(3n^2 - 2\pi)]}{n^2 - 1}$

E. $\lim_{n\to\infty} \left(1 + \frac{\sin(n^3 - n)}{\sqrt{n}}\right)$

We also have the less inspiring—but also useful—result:

Theorem 9.7. Properties of Limits
If (a_n) and (b_n) are convergent and c is constant, then

(1) $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

(2) $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$

(3) $\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$

(4) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{lim_{n\to\infty} a_n}{lim_{n\to\infty} b_n}$, *assuming* $lim_{n\to\infty} b_n \neq 0$

(5) $\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p$ *if* $p > 0$ *and* $a_n > 0$

(6) $\lim_{n\to\infty} c = c$

Exercise Set 9.

1. Find a formula for the general term of each of the following:

a. $\left(-1, \frac{1}{8}, -\frac{1}{27}, \ldots\right)$

b. $\left(\frac{3}{4}, \frac{4}{8}, \frac{5}{16}, \ldots\right)$

c. $\left(\sqrt{3},\ \frac{3}{2},\ \frac{3\sqrt{3}}{4},\ \ldots\right)$

d. $\left(0,\ 1,\ 0,\ -1,\ 0,\ 1,\ \ldots\right)$

2. Determine whether each of the following sequences converges. If so, find the limit. if not, indicate why:

a. $\left(\frac{n}{2n+1}\right)_{n=0}^{\infty}$

b. $\left(\frac{(-1)^n}{2^n}\right)_{n=1}^{\infty}$

c. $\left\{\ln\left(\frac{1}{n}\right)\right\}_{n=1}^{\infty}$

d. $\left(\frac{2^n}{n!}\right)_{n=0}^{\infty}$

e. $\left(\frac{2\cdot 3^n}{5^n}\right)_{n=0}^{\infty}$

f. $\left\{\left(\frac{n+3}{n+1}\right)^n\right\}_{n=0}^{\infty}$

g. $\left[\frac{\sin^2(n^3-n)}{1+\sqrt{n}}\right]_{n=0}^{\infty}$

h. $\left[\cos(\frac{1}{n})\right]_{n=2}^{\infty}$

i. $\left[\sum_{k=1}^{n}\frac{1}{k(k+1)}\right]_{n=1}^{\infty}$ [Hint: Write out each term and use partial fractions.]

Some Answers for Section 9

1. a. $\left(\frac{(-1)^n}{n^3}\right)_{n=1}^{\infty}$ **b.** $\left(\frac{n+1}{2^n}\right)_{n=2}^{\infty}$ **c.** $\left(\frac{3^{n/2}}{2^{n-1}}\right)_{n=1}^{\infty}$ **d.** $\left(\sin\frac{n\pi}{2}\right)_{n=0}^{\infty}$

2. a. Converges to $\frac{1}{2}$. **b.** Converges to 0. **c.** Diverges to $-\infty$. **d.** Converges to 0. **e.** Converges to 0. **f.** Converges to e^2. **g.** Converges to 0. **h.** Converges to 1. **i.** Converges to 1.

10. Limit of a Sequence: Mathematical Definition

Getting to a Precise Limit of a Sequence *(In which we work our way delicately through successive refinements, avoiding treacherous pitfalls along the way)*

Start with the preliminary definition above: **1:** The sequence (a_n) is said to **converge to the limit** L **as** $n \to +\infty$ if the values a_n approach L as n gets large.

Teacher: But how closely to they need to approach L?

Johnny: Hey I guess within a millionth should do it.

2: The sequence (a_n) is said to **converge to the limit** L **as** $n \to +\infty$ if it gets to within $1/1,000,000$ of L and *stays there* for sufficiently large n. In other words:

3: The sequence (a_n) is said to **converge to the limit** L **as** $n \to +\infty$ if there exists an N such that $|a_n - L| < 1/1,000,000$ whenever $n > N$.

Teacher: But Johnny, $1/1,000,000$ could be awfully big-especially if measured in angstroms.[3] What are you going to do for more fussy people?

Johnny: Hey I guess we'll have to settle for a zillionth or something real small like that . . .

4: The sequence (a_n) is said to **converge to the limit** L **as** $n \to +\infty$ if there exists an N such that

$$|an - L| < 1/10^{100}$$

whenever $n > N$.

Teacher: But Johnny, no matter how small you choose your number—let's call it ϵ—there are going to be people who are so fussy that your simply won't be small enough.

Johnny: Yeah. Well, it's just too bad. I give up.

Teacher: No so fast; this is not too difficult a problem. All we have to say is that, for a sequence to converge to L, it must get so close, that, no matter how fussy someone is to begin with, they will be satisfied. In other words: no matter how small they insist you choose ϵ, the terms in the sequence will get that close to it.

Definition 10.1. The sequence (a_n) is said to **converge to the limit** L **as** $n \to +\infty$ if, given any number ϵ (no matter how small) there exists an integer N such that $|a_n - L| < \epsilon$ whenever $n > N$.

[Johnny: oic thats kewl hey I want to be a math major!]

Examples 10.2.

A. $\lim_{n\to\infty} \dfrac{1}{n} = 0$

[3] An angstrom is 10^{-10} meters, or 0.1 nanometers.

B. $\lim_{n\to\infty} 0 = 0$

C. $\lim_{n\to\infty} \dfrac{n}{1+n} = 1$

D. We show rigorously that $\frac{1}{n^2} \to 0$ as $n \to \infty$.

Exercise Set 10.

1. Prove that the following limits are as claimed:

a. $\lim_{n\to+\infty} \dfrac{1}{n^2} = 0$

b. $\lim_{n\to+\infty} \dfrac{n}{n^2+1} = 0$

c. $\lim_{n\to+\infty} \dfrac{n+1}{n} = 1$

2. Prove that, if the sequence (a_n) converges to L, then so does the sequence (a_{n+1}).

3. Prove Theorem 9.7 (1).

4. The **Fibonacci sequence** is the sequence $1, 1, 2, 3, 5, 8, 13, \ldots$ where each term is defined as the sum of the preceding two.

a. Denoting this sequence by $(a_1, a_2, \ldots)$ show that

$$\frac{a_{n+2}}{a_{n+1}} = 1 + \frac{a_n}{a_{n+1}} \text{ if } n \geq 1.$$

b. Assuming that the sequence (a_{n+1}/a_n) converges, deduce (using the result you proved in Exercise 2 above) that its limit is $(1+\sqrt{5})/2$.

11. Monotone Sequences and Bounded Sequences

Definition 11.1. A sequence (a_n) is **monotone increasing** if $a_{n+1} \geq a_n$ for every n; it is **eventually monotone increasing** if $a_{n+1} \geq a_n$ for every $n >$ some N; it is **strictly monotone increasing** if we can replace the inequality above by the strict one. Similar definitions hold for **monotone decreasing** sequences.

Tests for monotonicity

1. Take the ratio $\dfrac{a_{n+1}}{a_n}$. If it is > 1, the function is strictly increasing.
2. Take the difference $a_{n+1} - a_n$. If it is > 0, the function is strictly increasing.
3. If $a_n = f(n)$ for some differentiable function f, then take the derivative, $f'(x)$. If it is > 0, then the function is strictly increasing.

Examples 11.2. Are the following (eventually) monotone?

A. $\left(\dfrac{3^n}{8 \cdot 2^n}\right)_{n>0}$

B. $\left(\dfrac{n}{n+1}\right)$

C. $3 + \dfrac{(-1)^n}{n}$

D. $(n^2 - 10\sin n)_{n\geq 0}$

E. $\left(\dfrac{4(3^n)}{n!}\right)_{n\geq 0}$

F. $\left(1 + \dfrac{1}{n}\right)^n$

Procedure for doing Example F:

1. Let $f(x) = \left(1 + \dfrac{1}{x}\right)^x$.
2. Take $f'(x)$ and simplify it.
3. It boils down to a positive function times $\ln(x+1) - \ln x - \frac{1}{x+1}$. Now apply the MVT for $\ln x$ to the first two terms ...

Definition 11.3. The sequence (a_n) is **bounded above** if there is a number M with $a_n \leq M$ for every n. (We similarly define **bounded below,** and **bounded.**)

Examples 11.4.
All of the above except A.(See the exercises.)

Theorem 11.5. *On Monotone Sequences*

I if (a_n) is eventually monotone increasing, then

(a) *If it is bounded above, it is convergent.*

(b) *If it is not bounded above,* $\lim_{n\to\infty} a_n = +\infty$.

II if (a_n) is eventually monotone decreasing, then

(a) *If it is bounded below, it is convergent.*

(b) *If it is not bounded below,* $\lim_{n\to\infty} a_n = -\infty$.

Examples 11.6.
Check which of the above sequences converge.

Exercise Set 11.

1. Test each sequence for (eventual) monotonicity:

- **a.** $\left(1 - \dfrac{1}{n}\right)$
- **b.** $\left(1 - \dfrac{(-1)^n}{n}\right)$
- **c.** $\left(\dfrac{n}{2n+1}\right)$
- **d.** $\left(\dfrac{n}{e^n}\right)$
- **e.** $\left(\sin(\dfrac{n\pi}{2})\right)$
- **f.** $\left(\dfrac{\cos 2n\pi}{n}\right)$
- **g.** $\left(\dfrac{n!}{n^n}\right)$
- **h.** $\left(\dfrac{n!\, e^n}{n^n}\right)$

2. Which of the sequences in Examples 11.2 with the exception of (h) is bounded? *Justify your claims.* [The one in 1(h) actually unbounded, but proving it requires something called Stirling's Approximation of $n!$.]

3. S'pose you have a monotone sequence (a_n) with the property that $A < a_n < B$ for some fixed A and B.

- **a.** By quoting certain theorems, show that (a_n) converges to a limit L with $A \leq L \leq B$.
- **b.** Referring to part (a), give an example to show that is need not always be the case that $A < L < B$.

4. **a.** Use Riemann sums to obtain the following inequality:

$$\int_1^n \ln x \, dx < \ln(n!) < \int_1^{n+1} \ln x \, dx$$

[Hint: You may consult the figure in Anton's exercise set for this topic to see what we mean about using Riemann sums.]

b. Deduce that

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}$$

c. Now conclude that $\lim_{n\to+\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$.

Some Answers for Section 11

1. a. Increasing **b.** Not monotone **c.** Increasing **d.** Decreasing **e.** Neither **f.** Decreasing **g.** Decreasing **h.** Increasing

12. Infinite Series

Consider the expression

$$0.33333\cdots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots + \frac{3}{10^n} + \ldots$$

Note that it never terminates. An **infinite series** is a (possibly non-terminating) "expression" of the form

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k$$

In order to make sense of the infinite sum, we use partial sums as follows: Let

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\ldots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \end{aligned}$$

Since we think of these longer and longer **partial sums** as being more accurate approximations of the infinite sum, we take

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

if the limit exists. If it is finite ($= L$, say) we say that the infinite sum **converges to** L. If it is infinite, we say that the infinite sum **diverges to infinity.** If it does not exist at all, we say that it **diverges.**

Examples 12.1.

A. The one we started with:

$$\frac{3}{10} + \frac{3}{100} + \cdots + \frac{3}{10^n} + \ldots \ = \ \sum_{k=1}^{\infty} \frac{3}{10^k}$$

The partial sums are: $s_1 = .3$ $s_2 = .33$ $s_3 = .333$ $\ldots$. The limit of the s_n's *looks like* it is $1/3 = 0.\bar{3}$. We make this precise in the next example.

B. **Geometric Series** Consider a series of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \cdots + ar^n + \ldots$$

(Eg. Example [A] above with $a = 3/10$ and $r = 1/10$.) In class, we derive the following formula for the n^{th} partial sum:

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1} : \quad s_n = \frac{a(1-r^n)}{1-r}$$

As $n \to \infty$ we consider several cases:
Case 1. If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$, so that $\lim_{n\to\infty} s_n = \dfrac{a}{1-r}$.
Case 2. If $r = 1$, then $\lim_{n\to\infty} r^n = +\infty$, so that $\lim_{n\to\infty} s_n = \infty$.
Case 3. If $r = -1$, then $\lim_{n\to\infty} r^n$ does not exist, and hence nor does $\lim_{n\to\infty} s_n$.

Geometric Series

If $|r| < 1$, the geometric series $\displaystyle\sum_{k=0}^{\infty} ar^k$ converges to $\dfrac{a}{1-r}$. Otherwise it diverges.

C. Sum the series $5 + \dfrac{5}{4} + \dfrac{5}{16} + \cdots + \dfrac{5}{4^k} + \ldots$.

D. Ditto for $\displaystyle\sum_{k=1}^{\infty} \frac{3 \cdot 2^{2k}}{3^{k-1}}$.

E. Express the number $3.14784784784\ldots$ as a rational number.

F. Determine if $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ is convergent, and if it is, find the infinite sum.

G. Sum the harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$.

Theorem 12.2. The No-Hope Theorem

a. *If $a_n \not\to 0$ as $n \to \infty$, then there is* no hope *of the series $\sum_n a_n$ converging.*

b. *Furthermore, if the a_n are positive (resp. negative) from some point on, the series diverges to + (resp. −) ∞.*

Some people call this the *Divergence Theorem.*
Another way of saying part(a): If the series $\sum_n a_n$ converges, then we *must have* $a_n \to 0$ as $n \to \infty$ *(at the very least!)*

Sketch of proof:

a. S'pose the series above converges. Then write $a_n = s_n - s_{n-1}$ and take limits.

b. If the terms are positive, the sequence s_n of partial sums is an (eventually) increasing sequence. By Theorem 11.5, the partial sums must either converge or diverge to $+\infty$. Since they diverge by part(a), they must diverge to $+\infty$, as required. The proof for negative terms is similar. □

Examples 12.3.

A. $\displaystyle\sum_{k=1}^{\infty} k\sqrt{k}$ diverges. Why?

B. $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{k^2-1}$

C. Does this shed any light on the harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$?

To end this section we have another theorem:

Theorem 12.4. *Sums, Differences, and Constant Multiples*

(a) If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge to S_a and S_b respectively, then $\sum_{k=1}^{\infty} a_k + b_k$ converges to $S_a + S_b$.

(b) If $\sum_{k=1}^{\infty} a_k$ converges to S_a, then $\sum_{k=1}^{\infty} \lambda a_k$ converges to λS_a for any constant λ.

(c) If $\sum_{k=1}^{\infty} a_k$ converges, then so does $\sum_{k=r}^{\infty} a_k$ for any $r \geq 1$.

Exercise Set 12.

1. Which of the following are (eventually) geometric series? For those that are geometric, determine whether they converge, and to what.

a. $1 + \sqrt{2} + 2 + 2\sqrt{2} + \ldots$

b. $600 - \frac{1}{5} + 65 + \frac{6}{3} - \frac{12}{9} + \frac{24}{27} - \ldots$

c. $1 - \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} - \frac{3}{\sqrt{2}} + \ldots$

d. $1 + .6660 + .00006660 + .000000006660 + \ldots$

e. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{3^{2k-1}}$

f. $\displaystyle\sum_{k=0}^{\infty} \frac{2 \cdot 2^{2k}}{3^{k-1}}$

g. $\displaystyle\sum_{k=0}^{\infty} \frac{4 \cdot (-1)^{k+1} 3^{k+1}}{2^{2k-1}}$

h. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k + 3^{2k-1}}$

2. Determine the values of x that make the following series converge, and give the resulting sum when it does.

a. $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{3^{k-1}}$

b. $\displaystyle\sum_{k=0}^{\infty} \frac{(x-1)^k}{2^k}$

3. Determine which of the following series converge, diverge to infinity, or just plain diverge. Justify your answers. For those series that converge, give the infinite sum.

a. $1+1+1+\dots$

b. $\dfrac{1}{2}+\dfrac{1}{4}+\dots+\dfrac{1}{2n}+\dots$

c. $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k^2-3k+2}$

d. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{k}-\frac{1}{k+2}\right)$

e. $\dfrac{1}{1\cdot 3}+\dfrac{1}{2\cdot 4}+\dfrac{1}{3\cdot 5}+\dots$

f. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3-n}$

4. If the series $\sum a_n$ converges to s, then recall that this means that $\lim_{n\to\infty} s_n = s$, where s_n are the partial sums. Therefore, $\lim_{n\to\infty} |s_n - s| = 0$.

a. Compute $|s_n - s| = \displaystyle\sum_{k=n}^{\infty} ar^n$ for the geometric series $a + ar + ar^2 + \dots$.

b. Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$. Show that its n^{th} partial sum can be written as

$$s_n = \left[\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^n}\right]+\left[\frac{1}{4}+\dots+\frac{1}{2^n}\right]+ \dots + \left[\frac{1}{2^{n-1}}+\frac{1}{2^n}\right]+\left[\frac{1}{2^n}\right]$$

c. Think of each of the bracketed terms B_r; $r = 1, 2, \dots, n$ as a partial sum of a geometric series. Let L_r be the sum of the corresponding infinite series. Use part (a) to show that

$$|s_n - [L_1 + L_2 + \dots + L_n]| \le \frac{n}{2^n}$$

d. Now observe that $L_1 + L_2 + \dots + L_n + \dots$ is geometric and find its sum.

e. Deduce that $\sum_{k=1}^{\infty} \frac{k}{2^k}$ converges to 2.

f. Now look at the non-rigorous proof in Anton p. 672 # 36. See if you can see what this has to do with that.

Some Answers for Section 12

1. a. Geometric; diverges ($r = \sqrt{2}$). **b.** Eventually geometric; converges to 666. **c.** Not geometric **d.** Geometric; converges to $1 + \frac{6660}{9999}$ **e.** Geometric; converges to $\frac{6}{7}$ **f.** Geometric; diverges ($r = \frac{4}{3}$) **g.** Geometric; converges to $\frac{-96}{7}$ **h.** Not geometric **2. a.** $|x| < 3$; $\frac{9}{3-x}$. **b.** $-1 < x < 3$; $\frac{2}{3-x}$ **3. a.** Diverges to $+\infty$ by the no-hope test. **b.** Diverges to $+\infty$. Reason: the partial sums are $\frac{1}{2}$ the partial sums associated with the harmonic series. **c.** Telescoping series; converges to 1. **d.** Telescoping series; converges to $\frac{3}{2}$. **e.** Telescoping series; converges to $\frac{3}{4}$. **f.** Telescoping series; converges to $\frac{1}{4}$. **4 a.** $\frac{ar^n}{1-r}$

13. Tests for Convergence

Integral Test *A Test for Series with Positive Terms*
S'pose that f is a positive continuous eventually decreasing function defined on $[1,\infty)$. Then $\int_1^\infty f(x)\,dx$ and $\sum_{k=1}^\infty f(k)$ converge or diverge together.

Proof of integral test in class.

Examples 13.1.

A. ***p*-series:** $\sum_{k=1}^\infty \frac{1}{k^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

B. $\sum_{k=2}^\infty \frac{1}{k \ln k}$ – a kind of borderline case.

C. $\sum_{k=8}^\infty \frac{1}{k \ln k \ln(\ln k)}$ – even more of a borderline case.

D. $\sum_{k=1}^\infty \frac{1}{(4k-3)^{1.2}}$

Comparison Test *Another Test for Series with Positive Terms*
S'pose that $0 \leq a_k \leq b_k$ for $k \geq$ some N. Then:

$$\sum b_k \text{ converges } \Rightarrow \sum a_k \text{ converges;}$$
$$\sum a_k \text{ diverges } \Rightarrow \sum b_k \text{ diverges.}$$

Examples 13.2.

A. $\sum_{k=1}^\infty \frac{1}{k^k}$

B. $\sum_{k=5}^\infty \frac{1}{k-4}$

C. $\sum_{k=1}^\infty \frac{1}{k+4}$

D. $\sum_{k=1}^\infty \frac{\sin^2 k}{k^2}$

E. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k!}$

> **Limit Comparison Test** *Also for Series with Positive Terms*
> Let $\theta = \displaystyle\lim_{k\to\infty} \frac{a_k}{b_k}$ (assuming this limit exists). Then:
> (1) If θ is a finite nonzero number, Σa_k and Σb_k both converge or diverge together.
> (2) If $\theta = 0$ and Σb_k converges, then Σa_k converges too.
> (3) If $\theta = \pm\infty$ and Σb_k diverges, then so does Σa_k

Examples 13.3.
Go through all the above Examples 13.2 and see which of them are amenable to the Limit Comparison Test.

Examples 13.4.

A. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k - k}$

B. Ratios of powers of polynomial functions – "Erase from the Right"

Exercise Set 13.

1. Determine which of the following series converge or diverge. Justify your answers.

a. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

b. $\displaystyle\sum_{k=1}^{\infty} k^{-4/3}$

c. $\displaystyle\sum_{k=2}^{\infty} \left(1 + \frac{1}{k}\right)^k$

d. $\displaystyle\sum_{k=2}^{\infty} \frac{k}{e^k}$

e. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{2 + 4k^2}$

f. $\displaystyle\sum_{k=2}^{\infty} \sqrt[k]{e}$

g. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{(k \ln k)^2}$

h. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{(k+1)\ln(k+1)}$

i. $\displaystyle\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^{1.1}}$

j. $\displaystyle\sum_{k=10}^{\infty} \frac{1}{k(\ln k)^{1.1}}$

k. $\displaystyle\sum_{k=10}^{\infty} \frac{1}{k\ln k(\ln\ln k)^{1.1}}$

l. $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{k^k}$

Some Answers for Section 13

1. a. Diverges (p-series ($p = 1/2$)) **b.** Converges (p-series ($p = 4/3$)) **c.** Diverges by the No-Hope Test (terms approach $e \neq 0$) **d.** Converges by Limit Comparison Test with the geometric series $\sum \frac{1}{e^k}$ **e.** Converges by Comparison Test with the p-series $\sum \frac{1}{4k^4}$ **f.** Diverges by the No-Hope Test (terms approach 1) **g.** Converges by comparison with $\sum \frac{1}{k^2}$ **h.** Diverges by the Integral Test **i.** Converges by comparison with $\sum \frac{1}{k^{1.1}}$ **j.** Converges by the Integral Test **k.** Converges by the Integral Test **l.** Converges by comparison with $\sum \frac{2}{k^2}$

14. Alternating Series and Absolute Convergence

Definition 14.1. An **alternating series** is a series of the form

$$a_1 - a_2 + \cdots + (-1)^{n+1}a_n + \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

or

$$-a_1 + a_2 - \cdots + (-1)^{n}a_n + \cdots = \sum_{n=1}^{\infty}(-1)^{n}a_n$$

where $a_n \geq 0$ for each n. Simply put, the signs of its terms alternate.
A series is **eventually alternating** if its signs of its terms alternate from some point on.

Examples 14.2.

A. Alternating harmonic series:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots = \ln 2 \qquad \text{[Consider } \int_0^1 \frac{1}{1+x}\,dx \text{]}$$

B. Odd alternating half-harmonic series:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4} \qquad \text{[Consider } \int_0^1 \frac{1}{1+x^2}\,dx\text{.]}$$

C. Even alternating half-harmonic series:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \cdots = \ln\sqrt{2} \qquad \text{[This is A/2. Also, see Exercise 1.]}$$

Theorem 14.3. *Alternating Series Test*
If $a_n \searrow 0$, then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ and $\sum_{n=1}^{\infty}(-1)^{n}a_n$ converge.

Outline of proof: Consider the first kind of alternating series: $a_1 - a_2 + a_3 + \cdots + (-1)^{n+1}a_n$. The odd partial sums are positive and decreasing, and hence approach a limit by the theorem on monotone sequences (Theorem 11.5.). Since the odd and even terms differ only by a term that is approaching zero, the even sums also approach this same limit.

The second kind of series is treated similarly (also, see the exercises). □

If you stare hard at an alternating series, you also see the following:

Theorem 14.4. *Error Term for Alternating Series*
If $a_n \searrow 0$, and if s and s_n are, respectively, the infinite and partial sums of $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ and $\sum_{n=1}^{\infty}(-1)^{n}a_n$, then

$$|s_n - s| \leq a_{n+1}.$$

In other words, if we stop summing at the n^{th} term, then the error is no bigger than the following term.

Examples 14.5.

A. How far should you expand the alternating harmonic series to obtain $\ln 2$ accurate to 4 decimal places?

B. Evaluate $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k!}$ correct to 3 decimal places.

C. How far must the expansion of $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}$ be carried out to give obtain $\pi/4$ accurate to 5 decimal places?

Definition 14.6. The series $\sum a_k$ is **absolutely convergent** if the associated series $\sum |a_k|$ converges. If $\sum a_k$ converges but $\sum |a_k|$ diverges, we say that $\sum a_k$ is **conditionally convergent.**

Examples 14.7.

A. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k\sqrt{k}}$ is absolutely convergent.

B. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ is conditionally convergent.

C. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(k^2+1)}{10k^2+k}$ is neither.

D. What about $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(k\pi/2)}{k^{1.2}}$?

E. nd $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(k\pi/2)}{k^{0.9}}$?

Theorem 14.8. *Absolute Convergence Implies Convergence*
If a series is absolutely convergent, then it is convergent to begin with.

Proof. S'pose $\sum a_k$ is absolutely convergent. Then the inequality $-|a_k| \leq a_k \leq |a_k|$ gives:

$$0 \leq a_k + |a_k| \leq 2|a_k|$$

The result now follows from the comparison test and a certain homework problem (if the sum of two series is convergent, and one of them is, then so is the other.) □

Examples 14.9. Examples A and D above.

Theorem 14.10. *Ratio Test for Absolute Convergence*
Let $L = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$, *if it exists. Then:*

$$L < 1 \Rightarrow \text{ The series converges absolutely;}$$
$$L > 1 \Rightarrow \text{ The series diverges.}$$

Note: $L = 1$ tells you nothing at all, so the test is useless in this case.

Examples 14.11.

A. $\sum_{k=1}^{\infty} \frac{2^k}{k!}$

B. $\sum_{k=1}^{\infty} \frac{(-1)^k k^3}{3^k}$

C. $\sum_{k=1}^{\infty} \frac{k}{k!}$

D. $\sum_{k=1}^{\infty} \frac{k^k}{k!}$

Definition 14.12. A **rearrangement** of the series $a_1 + a_2 + \ldots$ is a series of the form $a_{n_1} + a_{n_2} + \ldots$ where $n_1, n_2, \ldots$ are all distinct integers ≥ 1, and include every integer ≥ 1.

Thus, for each n, all the terms $a_1, a_2, \ldots, a_n$ occur in some partial sum of the a_{n_r}.

Theorem 14.13 (Theorem on Rearrangements)**.**

a. *If* $\sum a_n$ *is absolutely convergent with sum* s*, then so is every rearrangement.*

b. *If* $\sum a_n$ *is conditionally convergent and if* L *is any real number, then there are rearrangements of* $\sum a_n$ *converging to* L*, there are rearrangements that diverge to* $+\infty$*, some that diverges to* $-\infty$*, and some whose partial sums have no limit at all.*

Sketch of Proof: For part (b) think of the given series as the sum of two series: $\sum \max\{a_n, 0\}$ and $\sum \min\{a_n, 0\}$. The fact that the original series conditionally converges implies that the terms in each of these two series approach zero. The fact that the given series is not absolutely convergent implies that one of these two series — and hence both of them — must diverge (see the exercise set). These facts allow us to rearrange the series to do anything we like.

Exercise Set 14.

1. By considering $\displaystyle\int_0^1 \frac{x}{1+x^2}\,dx$, show directly that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \cdots = \ln \sqrt{2}.$$

2. Prove the second case in Theorem 14.3 by reducing it to the first case: If $a_n \searrow 0$, then $-a_1 + a_2 - \cdots + (-1)^n a_n + \cdots = \sum_{n=1}^{\infty}(-1)^n a_n$ converges.

3. Prove that if the sum of two series is convergent, and one of them is, then so is the other. [Actually, you could have proved this one some time ago.]

4. Prove that if $\sum a_k$; $(a_k \neq 0)$ converges, then $\sum \frac{1}{a_k}$ diverges.

5. Determine which of the following series converge absolutely, converge conditionally, or diverge. Justify your answers.

a. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$

b. $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k\sqrt{k+1}}{\sqrt{k-1}}$

c. $\displaystyle\sum_{k=2}^{\infty} (-1)^k \ln\left(\frac{1}{k}\right)$

d. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k\sqrt{k}}$

e. $-\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} - \frac{1}{9} + \ldots$

f. $\frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} - \cdots - \frac{1}{16} + \ldots$

6. Consider the two series $\displaystyle\sum_{k=1}^{\infty} \frac{k^k}{e^k\,k!}$ and $\displaystyle\sum_{k=1}^{\infty} \frac{e^k\,k!}{k^k}$ Does either of these converge? Do they both converge? [The ratio test is useless here. Why?]

Some Answers for Section 14

5. a. Conditionally convergent (alternating series test and p-series test) b. Divergent (No-Hope Theorem) c. Divergent (No-Hope Theorem) d. Absolutely convergent (p-series test) e. Divergent. (The partial sums have the form $-\frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} .. - \frac{1}{3n-1} - (\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + ... + \frac{1}{3n})$. The first is alternating and convergent, while the second diverges.) f. Divergent (Groups of positive and negative terms are $\geq \frac{1}{2}$.)

15. Power Series

Definition 15.1. A **power series about** 0 is a series of the form

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k + \dots.$$

The c_k are called the **coefficients** of the series.

Example 15.2. If all the coefficients $c_k = 1$, then we have the geometric series $1+x+x^2+\dots$ This series converges for $|x| < 1$and diverges otherwise.

Definition 15.3. A **power series about the point** a is a series of the form

$$\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_k(x-a)^k + \dots.$$

Examples 15.4. For which x do the following power series converge?

A. $\sum_{k=0}^{\infty} k^2 x^k$

B. $\sum_{k=0}^{\infty} \frac{(x-3)^k}{k}$

Theorem 15.5 (Theorem on Power Series)**.**
Given any power series $\sum_{k=0}^{\infty} b_k(x-a)^k$, there is an associated **interval of convergence** *centered at $x = a$ such that the series converges for values of x in that interval and diverges outside it. The interval can be any kind of interval whatsoever, but is always centered at a. The half-width of this interval is called the* **radius of convergence.**

The proof of the theorem hinges on the following lemma:

Lemma 15.6. *If $\sum_{k=0}^{\infty} b_k(x-a)^k$ converges for any x with $|x-a| = r$, then it converges absolutely for* every *x with $|x-a| < r$.*

To prove the lemma, use the fact that the terms in a convergent series approach zero and therefore are eventually smaller than 1:

$$|b_n||x-a|^n < 1,$$
$$\text{so that } |b_n| < \frac{1}{|x-a|^n}.$$

This inequality allows you to prove quickly that the series converges for all x with $|x-a| < r$.

Examples 15.7. Find the intervals of convergence and radius of convergence of the following power series:

A. $\sum_{k=0}^{\infty} k^2 x^k$

B. $\sum_{k=0}^{\infty} \frac{x^k}{k!}$

C. $\sum_{k=0}^{\infty} \frac{(-1)^k (x-3)^k}{k}$

D. $\sum_{k=0}^{\infty} \frac{(x+3)^k}{k^2 2^k}$

Exercise Set 15.

1. Find the intervals and radius of convergence of the following power series:

a. $\sum_{k=0}^{\infty} x^k$

b. $\sum_{k=0}^{\infty} \frac{2^k (x-2)^k}{k^2}$

c. $\sum_{k=0}^{\infty} \frac{(-1)^k (x+5)^k}{\ln k}$

d. $\sum_{k=0}^{\infty} \frac{3^k (x-3)^k}{k 2^k}$

e. $\sum_{k=0}^{\infty} k!(x-10)^k$

f. $\sum_{k=0}^{\infty} \frac{(x-\pi)^{2k}}{(2k)!}$

2. Find power series with the following intervals of convergence:

a. $\{4\}$
b. $(-1, 2]$
c. $[-1, 2)$
d. $(a, b); \quad (b > a)$
e. $[a, b); \quad (b > a)$
f. $(a, b]; \quad (b > a)$

Some Answers for Section 15

1. a. $(-1, 1); \quad r = 1$ b. $[1\frac{1}{2}, 2\frac{1}{2}]; \quad r = 1/2$ c. $(-6, -5]; \quad r = 1$ d.

$[1\frac{1}{3}, 3\frac{2}{3})$; $r = 2/3$ e. $\{10\}$; $r = 0$ f. $\mathbb{R}$; $r = \infty$ **2. a.** $\sum k!(x-4)^k$ b. $\sum \frac{(-2)^k(x-\frac{1}{2})^k}{k3^k}$ c. $\sum \frac{2^k(x-\frac{1}{2})^k}{k3^k}$ d. $\sum \frac{(x-\frac{a+b}{2})^k}{(\frac{b-a}{2})^k}$ e. $\sum \frac{(x-\frac{a+b}{2})^k}{k(\frac{b-a}{2})^k}$ f. $\sum \frac{(-1)^k(x-\frac{a+b}{2})^k}{k(\frac{b-a}{2})^k}$

16. Taylor's Theorem

Theorem 16.1. (Taylor/McClaurin)
S'pose that f is $(n+1)$ times continuously differentiable on an open interval containing the points a and x. Then

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) = \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R_n(x),$$

where

$$R_n(x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\ dt.$$

The portion

$$T_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) = \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a)$$

is called the ***degree*** n ***Taylor polynomial*** *or* ***Taylor expansion of*** f ***about*** a**.**

Proof. We go back to Example 3.2 in Section 3 to discover that we have already proved the theorem! □

Note that the degree n taylor polynomial (the terms up to the remainder) constitute the n^{th} partial sum of the following power series.

Definition 16.2. Taylor Series of f about $x = a$
The Taylor Series of f about the point $x = a$ is given by

$$T(x) = \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} f^{(k)}(a).$$

The n^{th} **Taylor sum of f about the point** $x = a$ is given by

$$T_n(x) = n^{\text{th}} \text{ partial sum of Taylor Series } = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a).$$

Examples 16.3. Calculate the Taylor Series of the following functions about the indicated point:

A. $f(x) = e^x; \quad a = 0$
B. $f(x) = \ln x; \quad a = 1$
C. $f(x) = \cos x; \quad a = 0$
D. $f(x) = 3x^2 - 4x + 1; \quad a = -1$
E. $f(x) = \frac{1}{1-x}; \quad a = 0$

Theorem 16.4 (If it Looks like the Taylor Series it is the Taylor Series)**.**
If $f(x)$ is so kind as to be expressible as a power series about the point $x = a$; viz. $f(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$, then we must have $a_k = \frac{f^{(k)}(a)}{k!}$. That is, the series must be the Taylor Series about $x = a$.

16.5. Note What this nice little theorem is telling us is that, if we can somehow cook up a power series expansion for f(x)-even without using the above formula for the coefficients-then we automatically have the Taylor Series.

Examples 16.6. Use the above examples or other trickery to obtain the following new ones *without actually calculating the terms.*

A. $\dfrac{1}{1-x}; \quad a = 0$

B. $\ln(1-x); \quad a = 0$

C. $e^{-x^2}; \quad a = 0$

D. $\dfrac{1}{x}; \quad a = 1$

E. $\ln x; \quad a = 1$

F. $\arctan x; \quad a = 0$

G. $\ln(x+1); \quad a = 0$

H. $\dfrac{1}{(1-x)^2}; \quad a = 0$

I. $\sinh x; \quad a = 0$

Exercise Set 16.

1. Use the explicit formula to compute the following Taylor Series, and give the interval of convergence in each case.
   - **a.** $f(x) = \cos x; \quad a = 0$
   - **b.** $f(x) = \sin x; \quad a = \pi$
   - **c.** $\ln(1+x); \quad a = 0$
   - **d.** $x^3 + 2x^2 - 3x + 1; \quad a = 0$
   - **e.** $x^3 + 2x^2 - 3x + 1 \quad a = 1$
   - **f.** $f(x) = \cosh x; \quad a = 0$
   - **g.** $f(x) = \ln x; \quad a = 2$
2. Now produce all of the the series in **1 a–f** *without* using the explicit formulas.
3. Let

$$f(x) = \begin{cases} 0 & \text{If } x = 0 \\ e^{-\frac{1}{x^2}} & \text{If } x \neq 0 \end{cases}$$

Show that f has derivatives of all orders at $x = 0$ and that its Taylor Series is identically zero.

Some Answers for Section 16

1. a. $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}$; $\mathbb{R}$ b. $-(x-\pi)+\frac{(x-\pi)^2}{3!}-\cdots=\sum_{k=0}^{\infty}(-1)^{k+1}\frac{(x-\pi)^{2k+1}}{(2k+1)!}$; $\mathbb{R}$ c. $x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x}{k}$; $(-1,1]$ d. x^3+2x^2-3x+1; $\mathbb{R}$ e. $1+4(x-1)+5(x-1)^2+(x-1)^3=x^3+2x^2-3x+1$; $\mathbb{R}$ f. $1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots=\sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}$; $\mathbb{R}$ g. $\ln 2+\frac{(x-2)}{2}-\frac{(x-2)^2}{4\cdot 2}+\frac{(x-2)^3}{8\cdot 3}-\cdots=\ln 2+\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(x-2)^k}{2^k k}$; $(0,4]$

17. Approximation by Taylor Polynomials

Since we can't ever write down an actual power series fully-welcome to the "real" world—we must be content to write down a bunch of terms and hope that the answer is "accurate enough." So here are two questions to ponder:

Question 1 First of all, when is a function equal to its Taylor series. That is, given $f(x)$, when is it true that $f(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$? (Note that Exercise 3 in Section 2 should give us pause.)

Question 2 If we replace $f(x)$ by the Taylor polynomial

$$T_n(x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a)$$

(stopping at the n^{th} term) how accurate an approximation to $f(x)$ is this?

Answer to Question 1
First, Taylor's Theorem tells us that

$$\begin{aligned} f(x) &= T_n(x) + R_n(x), && \text{so that} \\ f(x) - T_n(x) &= R_n(x) && \ldots \text{(I)} \end{aligned}$$

If it happens that $R_n \to 0$ as $n \to \infty$, then so does $f(x) - T_n(x)$, whence $T_n(x) \to f(x)$ as $n \to \infty$. In other words,

$$\lim_{n\to\infty} T_n = f(x), \quad \text{that is,}$$
$$\sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} f^{(k)}(x) = f(x)$$

meaning that $f(x)$ equals its Taylor Series. Conversely, Equation (I) tells us that $f(x)$ cannot equal its Taylor series unless the R_ns go to zero. This little discussion thus provides the answer to Question 1:

> In order for a function to equal its Taylor series, it is necessary and sufficient that the remainder terms approach 0 as $n \to \infty$.

Answer to Question 2
The magnitude of the difference between $f(x)$ and $T_n(x)$ is

$$|f(x) - T_n(x)| = |R_n(x)|,$$

so that the error term is $|Rn|$, and we must estimate how large it is. Well, (and let's assume for the sake of simplicity that $x \geq a$. If not, then you will need make a few adjustments as you go) ...

$$\begin{aligned}
|R_n| &= \Big|\int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\ dt\Big| \\
&\le \int_a^x \Big|\frac{(x-t)^n}{n!} f^{(n+1)}(t)\Big|\ dt && \text{because } |\textstyle\int f(x)\ dx| \le \int |f(x)|\ dx \\
&= \int_a^x \frac{(x-t)^n}{n!} |f^{(n+1)}(t)|\ dt && \text{because } (x-t) \text{ is already } \ge 0
\end{aligned}$$

Now let M be an upper bound for the continuous function $f^{(n+1)}(t)$ as t ranges over the interval $[a, x]$. Then the last term is

$$\begin{aligned}
\int_a^x \frac{(x-t)^n}{n!} |f^{(n+1)}(t)|\ dt &\le \int_a^x \frac{(x-t)^n}{n!} M\ dt \\
&= M \int_a^x \frac{(x-t)^n}{n!}\ dt \\
&= M \frac{(x-a)^{n+1}}{(n+1)!} \\
&= M \frac{|x-a|^{n+1}}{(n+1)!}
\end{aligned}$$

(the last step being by the assumption that $a \le x$). Thus:

Estimate of the Remainder in Taylor's Theorem
Magnitude of the Error when $f(x)$ is replaced by its Taylor Series of order n is

$$|R_n(x)| \le M \frac{|x-a|^{n+1}}{(n+1)!}$$

where M is an upper bound of $|f^{(n+1)}(t)|$ as t ranges over the interval $[a, x]$.

Notes 17.1. The term $M\frac{|x-a|^{n+1}}{(n+1)!}$ is an "estimation of the error"—in fact the inequality makes it an *overestimation,* the point being that if we know that this quantity is, say, 1/1000 or less, then the actual error $R_n(x)$ is certainly also no larger than 1/1000. (Who cares if we know the size of the actual error—if we did, then we would know the exact function to begin with, so there would be no need to approximate.)

Examples 17.2.

A. First we check that e^x, $\sin x$, $\cos x$ and so forth are in fact equal to their Taylor series.
B. Use the fourth Taylor polynomial about 0 to approximate e. How accurate is this approximation?
C. Which Taylor polynomial should we use to approximate e to six decimal places?.

D. Find the maximum possible error when $\cos x$ is approximated by $1 - \frac{x^2}{2}$ with values of x between 0 and 1.

E. Approximate $\ln 1.1$ to ten decimal places using the approximation

$$\ln x = x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots,$$

Example 17.3. Binomial Series
First compute the TS for $(1+x)^r$ (r any real number whatsoever) as

$$1 + rx + \frac{r(r-1)}{2!}x^2 + \frac{r(r-1)(r-2)}{3!}x^3 + \dots$$

Q. Does it converge to $(1+x)^r$?
A. The inequality we have been using for the other Taylor series won't give us the desired result (try it and see...) so we need to look at the original integral:

$$\begin{aligned} R_n &= \int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\ dt \\ &= \int_0^x \frac{(x-t)^n}{n!} r(r-1)(r-2)\dots(r-n)(1-t)^{r-n-1}\ dt \end{aligned}$$

Now group the variable terms together to get

$$\frac{r(r-1)(r-2)\dots(r-n)}{n!} \int_0^x \left(\frac{x-t}{1-t}\right)^n (1-t)^{r-1}\ dt.$$

Since we are interested only in its magnitude, we look at how big the integrand can get as t varies between 0 and x. The magnitude of the first term there is

$$\left|\frac{x-t}{1-t}\right|^n = \pm\left(\frac{t-x}{t-1}\right)^n$$

If $x \neq 1$ the term in parentheses has a nowhere zero derivative with respect to t, so its maximum occurs at an endpoint $t = 0$ or $t = x$. The latter case gives zero, so the maximum must be at $t = 0$, giving maximum value of $|x|^n$. Thus

$$|R_n| \le \left|\frac{r(r-1)(r-2)\dots(r-n)}{n!}\right| |x|^n \int_0^x (1-t)^{r-1}\ dt.$$

Notice now that the integral does not depend on n, so all we need to find is the limit of the term outside the integral as $n \to +\infty$. We do this by the ratio test, and find that the limit of thee ratios is $|x|$ which is < 1 when $|x| < 1$ meaning that the *associated series* converges for $|x| < 1$ so certainly the terms approach zero! (Whew!)

Thus we have shown:

Binomial Series If $|x| < 1$, and r is any real number, then

$$(1+x)^r = 1 + rx + \frac{r(r-1)}{2!}x^2 + \frac{r(r-1)(r-2)}{3!}x^3 + \dots$$

Extra Credit: Proof that e is irrational

Step 1: S'pose, for the sake of argument, e was rational—that is, $e = p/q$, where p and q are integers (and we may as well assume that q is at least 2.) Use the TS expansion of e^x with $x = 1$ to write:

$$\frac{p}{q} = e = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!} + R_q.$$

Step 2: Show directly that (assuming that $e = p/q$)

$$q!\Big[e - 1 - \frac{1}{2!} - \frac{1}{3!} - \cdots - \frac{1}{q!}\Big]$$

is a positive integer. [Hint to show it's not zero: If it was zero, then e would equal its partial sum, even though the sequence of partial sums is increasing. . .]

Step 3. Establish that

$$q!\Big[e - 1 - \frac{1}{2!} - \frac{1}{3!} - \cdots - \frac{1}{q!}\Big] = q!R_q \leq \frac{e}{q+1} < 1.$$

So here we have a *positive integer which is strictly smaller than 1.* But this is absurd!

Exercise Set 17.

1. In each of the following, find a Taylor polynomial around a approximating $f(x)$ with an error no larger than ± 0.0001. Calculate the approximation given by the Taylor polynomial, and compare to the answer given by a calculator.
  - **a.** $f(x) = e^x$ around $a = 0$, approximate $f(\frac{1}{2})$
  - **b.** $f(x) = \frac{1}{x}$ around $a = 1$, approximate $f(1.1)$
  - **c.** $f(x) = \sqrt{x}$ around $a = 100$, approximate $f(99)$

2. Find the 5th Taylor polynomial for $f(x) = \ln x$ around 1, and estimate the error term if we use this polynomial to approximate $\ln 1.1$.

3. How many terms would we need in order to compute $\ln 1.1$ to 10 decimal places?

4. How can we use the Taylor Series to obtain the equation of the tangent line to the graph of f at the point $(a, f(a))$?

5. Show that the n^{th} Taylor sum T_n about $x = a$ has the same first through n^{th} derivatives as f when evaluated at $x = a$.

6. Use the binomial series to approximate $\sqrt[n]{1.5}$ to 3 decimal places. [Use the formula for R_n we developed in the notes.]

7. The Amex Gold BUGS[4] Index was at 150 points in January 2003, decreasing at a rate of 14.5 points/month, and accelerating at 3.6 points/month^2.
  - **a.** Taking t as time in months since January 2003, obtain the first and second Taylor polynomials of the BUGS index b as a function of t around $t = 0$.

[4]BUGS stands for "basket of unhedged gold stocks."

b. Use the second order Taylor polynomial to approximate the value of the index in January 2004, and compare the predicted value with the (approximate) actual value, which you can gelan from the Internet.

Some Answers for Section 17

1. a. $n = 5:\ 1+1/2+1/8+1/(8\cdot 3!)+1/(16\cdot 4!)+1/(32\cdot 5!) = 1.6487,\ e^{1/2} = 1.64872127\ldots$ **b**. $n = 3:\ 1-(0.1)+(0.1)^2-(0.1)^3 = 0.909,\ 1/1.1 = 0.90909\ldots$ **c**. $n = 2:\ 10-1/20-1/8000 = 9.94987,\ \sqrt{99} = 9.9498743\ldots$ **2**. $R_n \leq 0.000000166$ **3**. $n = 9; \ln 1.1 \cong 0.0953101798$ **4**. The tangent line is the graph of the first (linear) Taylor approximation. **6**. $n = 11$ **7. a**. First Taylor Polynomial: $b(t) \cong 150 - 14.5t$; Second Taylor Polynomial: $b(t) \cong 150 - 14.5t + 1.8t^2$ **b**. $b(12) \cong 235.2$ points. The actual value was around 220 points.

18. Polar Coordinates

First, we learn how to locate a point in the plane via the polar coordinates (r, θ).

Examples 18.1.
Locate: $P(0, \frac{\pi}{2}),\ Q(1, \frac{\pi}{4}),\ R(-1, \frac{\pi}{3}),\ S(2, 666\pi)$

Conversion Formulas

$$x = r\cos\theta; \qquad y = r\sin\theta$$

$$r = \sqrt{x^2+y^2}; \qquad \theta = \begin{cases} \arctan\frac{y}{x} & \text{if } x > 0 \\ \pi + \arctan\frac{y}{x} & \text{if } x < 0 \end{cases}$$

Examples 18.2.

A. Convert $(2, \pi/3)$ to Cartesian coordinates.
B. Convert $\left(-\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$ to polar coordinates.
C. Represent the circle radius 2 center $(0, 0)$ in polar coordinates.
D. Represent the circle radius R center (a, b) in polar coordinates. Can you solve for r as a function of θ?
E. What is the Cartesian form of the curve $r = 2\cos\theta$? [Multiply by r & use above formulas]

Curves in Polar Coordinates
These have the form $r = f(\theta)$.

Examples 18.3.

A. Cardioid: $r = a(1-\cos\theta)$
B. Rose: $r = \sin 3\theta$
C. Spiral: $r = \theta$.
D. Hyperbolic spiral: $r = 1/\theta$

Features of Polar Curves

A. Slope of Tangent: By the chain rule:

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{\sin\theta\frac{dr}{d\theta} + r\cos\theta}{\cos\theta\frac{dr}{d\theta} - r\sin\theta}$$

B. Vertical tangent:

$$\cos\theta\frac{dr}{d\theta} - r\sin\theta = 0$$
$$\sin\theta\frac{dr}{d\theta} + r\cos\theta \neq 0$$

C. Horizontal tangent:

$$\sin\theta\frac{dr}{d\theta} + r\cos\theta = 0$$
$$\cos\theta\frac{dr}{d\theta} - r\sin\theta \neq 0$$

D. If both are zero, must use l'Hospital's rule.

Lengths and Areas of Polar Things
In class, we derive the following formulas:

Area Swept Out by the Curve $r = f(\theta)$ for $a \leq \theta \leq b$

$$A = \tfrac{1}{2}\int_a^b f(\theta)^2 \, d\theta$$

Arclength of the Curve $r = f(\theta)$ for $a \leq \theta \leq b$

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

Examples 18.4. Find the areas swept out by the following curves:
A. Cardioid $r = 1 + \cos\theta$; $0 \leq \theta \leq 2\pi$
B. Rose $r = \cos 2\theta$; (entire curve)

Examples 18.5. Find the arc lengths of the following curves:
A. Cardioid $r = 1 + \cos\theta$; $0 \leq \theta \leq 2\pi$
B. Exponential Spiral $r = e^{\theta}$; $0 \leq \theta \leq \pi$

Exercise Set 18. Stewart, p. 706 #5, 9, 11, 15, 17, 19, 31, 35, 39, 43, 45, 55, 57, 59, 61, 63, 69

19. Parametric Curves

Imagine a particle moving around on the xy-plane. Then its x- and y-coordinates will vary with time t, meaning that they are **functions** of t :

$$x = x(t)$$
$$y = y(t)$$

The curve traced out as t varies is called a **parametric curve.** Specifically, let's assume that t lies in a certain range:

$$t \in \text{some (possibly infinite) interval } I.$$

So, mathematically, a parametric curve is a set of the form

$$\{(x(t), y(t)) \mid t \in I\}$$

for some interval I.

Examples 19.1.

A. Straight line segment from $P(-2,3)$ to $Q(4,2)$
B. Straight line segment from $P(x_1, y_1)$ to $Q(x_2, y_2)$
C. Straight line through $P(x_1, y_1)$ to $Q(x_2, y_2)$
D. Any cartesian curve: Graph of $f : I \to \mathbb{R}$
E. The circle center (x_0, y_0) radius r
F. The ellipse $\dfrac{(x-x_0)^2}{a^2} + \dfrac{(y-y_0)^2}{b^2} = 1$
G. Any polar curve: graph of $r = r(\theta)$

Features of Parametric Curves

A. Slope of Tangent: By the chain rule:

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

B. Vertical tangent:

$$\frac{dx}{dt} = 0, \qquad \frac{dy}{dt} \neq 0$$

C. Horizontal tangent:

$$\frac{dy}{dt} = 0, \qquad \frac{dx}{dt} \neq 0$$

D. If both are zero, we must take the limit at that value of t.
E. Arc length of the curve for $a \le t \le b$:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

F. Arc Length as a function of t:

$$s(t) = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du$$

Examples 19.2.

A. Cycloid: The curve traced out by a point on the rim of a wheel radius a as it rolls along a straight line without slipping.
To parameterize it, pretend that the circle is rolling forward at one radian per second. The forward speed of the center is then a units per second, so the x-coordinate of the center after t seconds is at and the y-coordinate is just a. Also assume that, at time $t = 0$, the point on the circumference is at the bottom of the wheel. Then its position relative to the center at time t is $a(-\sin t, -\cos t)$, as it is rotating clockwise. So, the coordinates are given by

$$x = a(t - \sin t)$$
$$y = a(1 - \cos t).$$

Find the arc length of one arch of the cycloid.

B. Curtate and Prolate Cycloid: Like a cycloid, but with the point tracing out the curve inside the circle, at a radius of $b < a$ (curtate) or $b > a$ (prolate)

$$x = at - b\sin t$$
$$y = a - b\cos t.$$

Consider the prolate cycloid. At which values of t is the tangent vertical? For which t is the point moving backward?

Exercise Set 19. Stewart, p. 685 #5, 7, 9, 13, 17, 19, 21, 41 p. 695 #5, 17, 19, 27, 41, 43, 45

20. Appendix: Limit Forms and L'Hospital's Rule

Limit Forms

When we are trying to understand limits of functions that are made up of simpler component functions, it is often useful to look at the "limit form? of the function. That is, to look at the limit of the simpler component functions.

For example, the function $\frac{x-1}{x^2-1}$ is a quotient of two simpler functions: $x-1$ and x^2-1. When I were to try to understand the limit as x goes to 2 of that quotient, it can useful to see what the limit as x goes to 2 of $x-1$, which is 1, and the limit as x goes to 2 of x^2-1, which is 3, and fit them together. The original limit has the limit form "$\frac{1}{3}$" because the numerator goes to 1 and the denominator goes to 3. In this case, that limit form is **determined**. If we know the limit form is "$\frac{1}{3}$", then we know the limit is $\frac{1}{3}$. In fact, whenever the limit form is simply a real number, it is a determined limit form. That is, it is *always* the case that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{1}{3}$$

if

$$\lim_{x\to a} f(x) = 1 \qquad \text{and} \qquad \lim_{x\to a} g(x) = 3.$$

However, if I then try to analyze the limit as x goes to 1 of the same function, we will see that it has the limit form "$\frac{0}{0}$" because both the numerator and the denominator go to zero. In fact, we can't tell anything about the limit by looking solely at its limit form in this case. This means that "$\frac{0}{0}$" is an **indeterminate** limit form.

Warning!
When we say a limit is of the form "$\frac{0}{0}$," we are <u>not</u> saying the limit <u>is equal to</u> $\frac{0}{0}$, which isn't a number or any analogous concept (like infinity). We use quotes around limit forms to emphasize that they are limit forms, not numbers. Limit forms are simply tools for analyzing limits.

That is, since "$\frac{0}{0}$" is an indeterminate form, we really can't say *anything* about

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

if we only know that

$$\lim_{x\to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to a} g(x) = 0.$$

The limit may be 0, it may be ∞, it may be $\frac{1}{3}$ and it may not even exist. We need more information to evaluate the limit if it is in an indeterminate form. We state the (informal) definition of an indeterminate form below.

Definition 20.1. We say that the limit

$$\lim_{x\to a} H(x),$$

where is made up of component functions $f(x)$ and $g(x)$, is in an **indeterminate** form if you cannot determine the limit (up to a sign on ∞) simply by looking at the limits of $f(x)$ and $g(x)$ separately. Otherwise we say it is in a **determined** limit form.

Often, when faced with an indeterminate limit form, we will simplify the function to change the limit form.

Examples 20.2. Notice that the first four limits below start in the form "$\frac{0}{0}$" and each has a very different limit. This tells us that "$\frac{0}{0}$" is indeed an indeterminate limit form.

A. In this example we simplify a limit of the form "$\frac{0}{0}$" to the form "$\frac{1}{2}$," which is determined:

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}.$$

B. In this example we simplify a limit of the form "$\frac{0}{0}$" to the form "$\frac{0}{2}$," which is determined:

$$\lim_{x\to 1} \frac{(x-1)^2}{x^2-1} = \lim_{x\to 1} \frac{x-1}{x+1} = 0.$$

C. In this example we simplify a limit of the form "$\frac{0}{0}$" to the form "$\frac{2}{0}$," which is determined (up to the sign on ∞, which we can figure out quickly):

$$\lim_{x\to 1} \frac{x^2-1}{(x-1)^3} = \lim_{x\to 1} \frac{x+1}{(x-1)^2} = \infty.$$

D. You should have seen in Calc I that

$$\lim_{x\to 0} \frac{\sin x}{x} = 0.$$

This is also a limit in the indeterminate form "$\frac{0}{0}$". In Calc I your instructor probably used a geometric argument to derive this limit.

E. You should have seen in Calc I that

$$\lim_{x\to\infty} \frac{3x^2 - 2x + 1}{x^2 + 1} = 3.$$

This is a limit in the indeterminate form "$\frac{\infty}{\infty}$". In Calc I your instructor probably showed you why this works by noting first that forms of the form "$\frac{\text{non-infinite}}{\infty}$" go to zero and then rewriting the above in as determined form as follows

$$\lim_{x\to\infty} \frac{3x^2 - 2x + 1}{x^2 + 1} = \lim_{x\to\infty} \frac{3x^2 - 2x + 1}{x^2 + 1} \times \frac{1/x^2}{1/x^2}$$
$$= \lim_{x\to\infty} \frac{3 - 2/x + 1/x^2}{1 + 1/x^2} = \frac{3 - 0 + 0}{1 + 0} = 3.$$

Generally a limit form will be indeterminate if there is some 'competition' between the parts of the function. For example, if the function is a quotient of the form "$\frac{0}{0}$" then the numerator going to 0 tends to pull the function towards 0, while the denominator going to 0 tends to pull the function towards $\pm\infty$. Thus, we cannot tell the limit by looking at the form alone.

Examples 20.3. Below are examples of indeterminate limit forms

A. As we've seen above, "$\frac{0}{0}$" is indeterminate. That is, knowing $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ is not enough information to determine

$$\lim_{x\to a} \frac{f(x)}{g(x)}.$$

B. The limit form "$\frac{0}{\infty}$" is determined. That is, knowing $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ tells us that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 0.$$

In this case there is no'competition' between the functions $f(x)$ and $g(x)$ in this form since the numerator going to positive or 0 tends to pull the whole quotient towards 0, and the denominator going to ∞ also pulls the whole quotient towards 0.

C. The limit form "$\frac{\infty}{\infty}$" is indeterminate. That is, knowing $\lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x)$ is not enough information to determine

$$\lim_{x\to a} \frac{f(x)}{g(x)}.$$

Notice that the there is 'competition' between the functions $f(x)$ and $g(x)$ in this form since the numerator going to ∞ tends to pull the whole quotient towards positive or negative infinity, while the denominator going to ∞ tends to pull the whole quotient towards 0. See if you can come up with examples of limits in this form that have very different values.

Note that the sign on the ∞'s is irrelevant here. So not only is "$\frac{\infty}{\infty}$" indeterminate, in general "$\frac{\pm\infty}{\pm\infty}$" is indeterminate.

D. The limit form "$0\times\infty$" is indeterminate. That is, knowing $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ is not enough information to determine

$$\lim_{x\to a} f(x)g(x).$$

See if you can come up with examples of limits in this form that have very different values.

E. The limit form "∞^0" is indeterminate. That is, knowing $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ is not enough information to determine

$$\lim_{x\to a} f(x)^{g(x)}.$$

Notice that the there is 'competition' between the functions $f(x)$ and $g(x)$ in this form since the exponent going to 0 tends to pull the whole quotient towards infinity, while the base going to ∞ tends to pull the whole towards ∞. See if you can come up with examples of limits in this form that have very different values.

There are other indeterminate forms, as well. You can tell if a form is indeterminate by deciding if there is competition between the component functions.

L'Hospital's Rule

L'Hospital's Rule is a way for us to use the derivatives of the component functions find limits in the indeterminate forms "$\frac{0}{0}$" and "$\frac{\infty}{\infty}$" when we can't otherwise simplify the expression.

L'Hospital's Rule (LHR): Suppose f and g are both differentiable functions and that

$$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty.$$

Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right exists.

That is, if $\lim_{x\to a}\frac{f(x)}{g(x)}$ is in one of the indeterminate forms "$\frac{0}{0}$" or "$\frac{\pm\infty}{\pm\infty}$," then you can evaluate the limit by evaluating the derivatives of each of the component functions separately and then evaluating the limit.

Examples 20.4.

(A.) $\lim_{x\to 1}\frac{x-1}{\ln x}$

Since $\lim_{x\to 1}(x-1) = 0 = \lim_{x\to 1}\ln x$ the limit is of the form "$\frac{0}{0}$." It doesn't simplify, but we can use LHR.

$$\lim_{x\to 1}\frac{x-1}{\ln x} =_{\text{LHR}} \lim_{x\to 1}\frac{1}{1/x} = 1.$$

(B.) $\lim_{x\to 0^+}\frac{\ln x}{1/x}$

Since $\lim_{x\to 0^+}\ln x = -\infty$ and $\lim_{x\to 0^+} 1/x = \infty$ the limit is of the form "$\frac{-\infty}{\infty}$." It doesn't simplify, but we can use LHR.

$$\lim_{x\to 0^+}\frac{\ln x}{1/x} =_{\text{LHR}} \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+} x = 0.$$

(C.) $\lim_{x\to 0}\frac{x^2}{\cos x-1}$

Since $\lim_{x\to 0}\cos x - 1 = 0 = \lim_{x\to 0}x^2$ the limit is of the form "$\frac{0}{0}$." It doesn't simplify, but we can use LHR.

$$\lim_{x\to 0}\frac{x^2}{\cos x-1} =_{\text{LHR}} \lim_{x\to 0}\frac{2x}{-\sin x}.$$

According to the theorem, the equality above only holds if the second limit also exists. Notice that this second limit is also in the form "$\frac{0}{0}$" and so we can try LHR again:

$$\lim_{x\to 1}\frac{2x}{-\sin x}2x =_{\text{LHR}} \lim_{x\to 1}\frac{2}{-\cos x} = -2.$$

(We could have applied the fact that $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ to that second limit, too.)

Warnings!

(1) You can *only* use LHR on limits of the forms "$\frac{0}{0}$" or "$\frac{\pm\infty}{\pm\infty}$." You cannot use it on any determined forms or any of the other indeterminate forms.

(2) When you use LHR, you are taking the derivative of the denominator and the numerator *separately*. You are not the quotient rule to take the derivative of the whole function.

Limits of the form "$0 \times \pm\infty$"

As we noted above, "$0 \times \pm\infty$" is indeterminate and we cannot use LHR on it directly. Note that limits of the form "$\frac{\text{non-zero}}{0}$" always go to $\pm\infty$ if they exist and limits of the form "$\frac{\text{non-infinite}}{\pm\infty}$." We can use that fact to change limits of the form "$0 \times \pm\infty$" into one of the forms that you can use LHR on as follows.

Suppose $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$. Then the limit $\lim_{x\to 0} f(x)g(x)$ of the form "$0 \times \pm\infty$" can be rewritten as

$$\lim_{x\to 0} f(x)g(x) = \lim_{x\to 0} \frac{g(x)}{1/f(x)} = \lim_{x\to 0} \frac{f(x)}{1/g(x)}$$

The limit $\lim_{x\to 0} \frac{g(x)}{1/f(x)}$ is of the form "$\frac{\pm\infty}{\pm\infty}$" (if the limit of denominator exists) and the limit $\lim_{x\to 0} \frac{f(x)}{1/g(x)}$ is of the form "$\frac{0}{0}$."

Examples 20.5.

A. The limit $\lim_{x\to 0^+} x\ln x$ is of the form "$0 \times -\infty$".

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0} \frac{\ln x}{1/x} =_{\text{LHR}} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0.$$

Note that we changed it to a limit of the form "$\frac{\infty}{-\infty}$". You will find that in this case, changing it to the form "$\frac{0}{0}$" will not prove too useful because LHR will only make that one more complicated. You should try that on your own to see what happens.

B. The limit $\lim_{x\to -\infty} xe^x$ is of the form "$-\infty \times 0$".

$$\lim_{x\to -\infty} xe^x = \lim_{x\to -\infty} \frac{x}{e^{-x}} =_{\text{LHR}} \lim_{x\to -\infty} \frac{1}{-e^{-x}} = -\infty.$$

Exponential Indeterminate Forms

We saw in Example 20.3(E) that "∞^0" is an indeterminate form. There are a number of exponential indeterminate forms. You should convince yourself that all of the forms below are indeterminate.

"∞^0" "0^0" "1^∞" "$1^{-\infty}$"

These are possible forms that the limit $\lim_{x\to a} f(x)^{g(x)}$ can take. We can use the natural log to convert such limits to the form "$0 \times \pm\infty$".

Note that

$$\lim_{x\to a} f(x)^{g(x)} = e^{\ln\left(f(x)^{g(x)}\right)} = e^{g(x)\ln f(x)} = e^{\lim_{x\to a} g(x)\ln(f(x))}$$

If the first limit is in an exponential indeterminate form, then the last limit is in the form "$0 \times \pm\infty$".

Example 20.6. The limit below starts as a limit in the form "0^0".

$$\lim_{x\to 0^+} x^{\tan x} = e^{\lim_{x\to 0^+} \tan x \ln x}$$

So, considering the exponent

$$\lim_{x\to 0^+} \tan x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\cot x} =_{\text{LHR}} \lim_{x\to 0^+} \frac{1/x}{-\csc^2 x}$$

$$= \lim_{x\to 0^+} \frac{-\sin^2 x}{x} =_{\text{LHR}} \frac{-2\sin x\cos x}{1} = 0$$

Thus,

$$\lim_{x\to 0^+} x^{\tan x} = e^{\lim_{x\to 0^+} \tan x \ln x} = e^0 = 1$$

Limits of the Form "$\infty - \infty$"

Note that "$\infty - \infty$" is an indeterminate form since the first ∞ pulls the limit towards ∞ and the second towards $-\infty$. However, there is no trick that works with every limit of the form "$\infty - \infty$". Some creativity is needed for these limits. Below are some examples to give you ideas of how to deal with them.

Examples 20.7.

A. You can combine fractions.

$$\lim_{x\to 1^+} \left[\frac{x}{x-1} - \frac{2-2x}{(x-1)^2}\right] = \lim_{x\to 1^+} \frac{x^2+x-2}{(x-1)^2} = \lim_{x\to 1^+} \frac{x+2}{x-1} = \infty$$

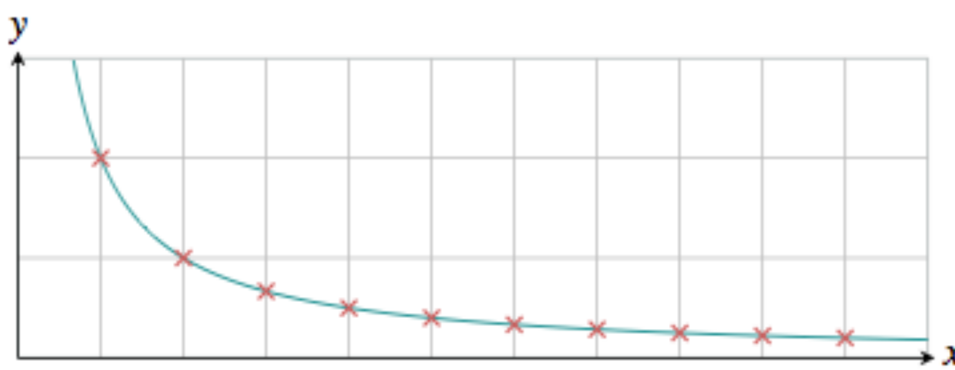
B.

$$\lim_{x\to 0} (\csc x - \cot x) = \lim_{x\to 0} \frac{1-\cos x}{\sin x} =_{\text{LHR}} \lim_{x\to 0} \frac{\sin x}{\cos x} = 0$$

C. You can use laws of logarithms.

$$\lim_{x\to 1^+} \left[-\ln(x^2-1) + \ln(x^2+x-2)\right] = \lim_{x\to 1^+} \ln \frac{x^2+x-2}{x^2-1}$$

$$= \lim_{x\to 1^+} \ln \frac{x+2}{x+1} = \ln \frac{3}{2}$$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