Business Statistics I: QM 1

Lecture Notes
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BUSINESS STATISTCS I: QM 001

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LECTURE NOTES BY STEFAN WANER

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Note: Throughout these notes, all references to the "book" refer to the class text:
"Statistics for Business and Economics" 8th Ed.
by Anderson, Sweeney, Williams (South-Western/Thomson Learning, 2002)

Topic 0 Introduction

Q: What is statistics?
A: Basically, statistics is the "science of data." There are three main tasks in statistics: (A) collection and organization, (B) analysis, and (C) interpretation of data.

(A) Collection and organization of data: We will see several methods of organizing data: graphically (through the use of charts and graphs) and numerically (through the use of tables of data). The type of organization we do depends on the type of *analysis* we wish to perform.
Quick Example Let us collect the status (freshman, sophomore, junior, senior) of a group of 20 students in this class. We could then organize the data in any of the above ways.

(B) Analysis of data: Once the data is organized, we can go ahead and compute various quantities (called *statistics* or *parameters*) associated with the data.
Quick Example Assign 0 to freshmen, 1 to sophomores etc. and compute the mean.

(C) Interpretation of data: Once we have performed the analysis, we can use the information to make assertions about the real world (e.g. the average student in this class has completed x years of college).

Descriptive and Inferential Statistics
In **descriptive statistics**, we use our analysis of data in order to *describe* a the situation from which it is drawn (such as the above example), that is, to summarize the information we have found in a set of data, and to interpret it or present it clearly. In **inferential** statistics, we are interested in using the analysis of data (the "sample") in order to make predictions, generalizations, or other inferences about a larger set of data (the "population"). For example, we might want to ask how confidently we can infer that the average QM1 student at Hofstra has completed x years of college.

In QM1 we begin with descriptive statistics, and then use our knowledge to introduce inferential statistics.

Topic 1
Describing Data Graphically

(Based on Sections 2.1, 2.2 in text)

An **experiment** is an occurrence we observe whose result is uncertain. We observe some specific aspect of the occurrence, and there will be several possible results, or **outcomes**. The set of all possible outcomes is called the **sample space** for the experiment.

(a) Qualitative (Categorical) Data
In an experiment, the outcomes may be non-numerical, so we speak of **qualitative** data.

Example Choose a highly paid CEO and record the highest degree the CEO has received. Here is a set of fictitious data:

Highest Degree	None	Bachelors	Masters	Doctorate	Totals
Number (Frequency)	2	11	7	5	25
Relative Frequency (f)	.08	.44	.28	.20	1

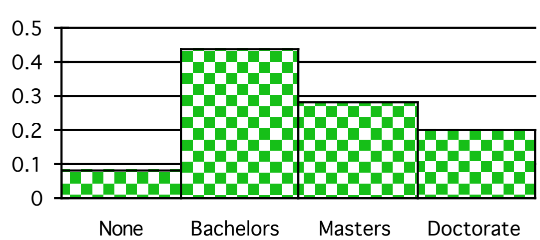
The four categories are called **classes**, and the relative frequencies are the fraction in each class:

$$\text{Relative Frequency of a class} = \frac{\text{frequency}}{\text{total}}.$$

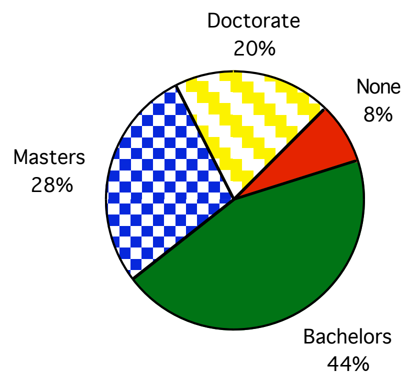
Question What does the relative frequency tell us?
Answer $f(\text{Bachelors}) = 0.44$ means that 44% of highly paid CEOs have bachelors degrees.

Note The relative frequencies add up to 1.

Graphical Representation
1. Bar graph
To get the graph, just select all the data and go to the Chart Wizard.



2. Pie chart

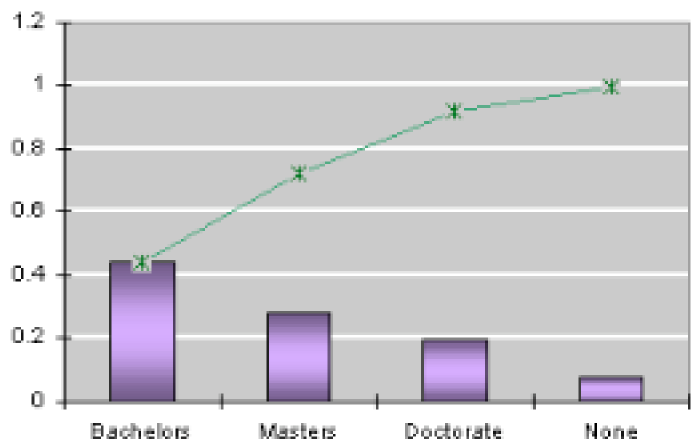


3. Cumulative Distributions

To get these, we sort the categories by frequency (largest to smallest) and then graph relative frequency as well as **cumulative** frequency:

Highest Degree	Bachelors	Masters	Doctorate	None
Relative Frequency (f)	.44	.28	.20	.08
Cumulative Frequency	.44	.72	.92	1.00

To get the graph in Excel, go to "Custom Types" and select "Line-Column"



This shows that, for instance, that more than 90% of all CEOs have some degree, and that 72% have either a Bachelors or Masters degree.

(b) Quantitative Data

In an experiment, the outcomes may be *numbers,* so we speak of **quantitative** data.

Example 1 Choose a lawyer in a population sample of 1,000 lawyers (the experiment) and record his or her income. Since there are so many lawyers, it is usually convenient to divide the outcome into **measurement classes** (or "brackets").

Suppose that the following table gives the number of lawyers in each of several income brackets.

Income Bracket	\$20,000 - \$29,999	\$30,000 - \$39,999	\$40,000 - \$49,999	\$50,000 - \$59,999	\$60,000 - \$69,999	\$70,000 - \$79,999	\$80,000 - \$89,999
Frequency	20	80	230	400	170	70	30

Let X be the number that is the midpoint of an income bracket. Find the frequency distribution of X.

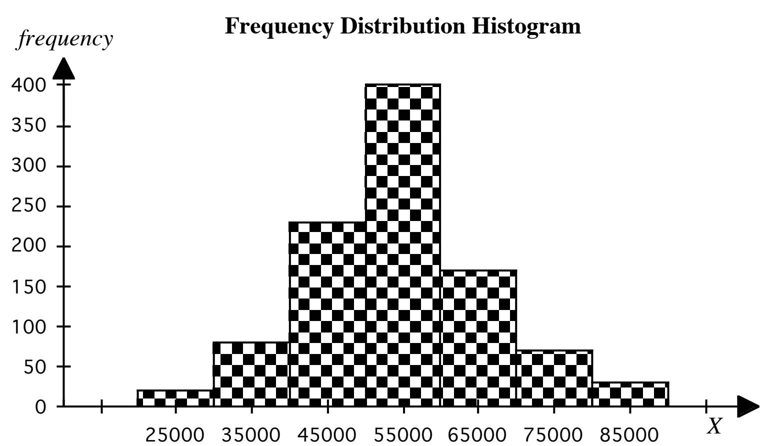
Solution Since the first bracket contains incomes that are at least \$20,000, but less than \$30,000, its midpoint is \$25,000. Similarly the second bracket has midpoint \$35,000, and so on. We can rewrite the table with the midpoints, as follows.

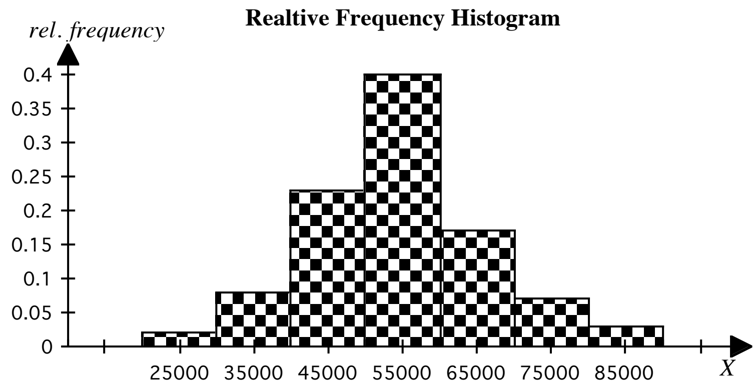
x	25,000	35,000	45,000	55,000	65,000	75,000	85,000
Frequency	20	80	230	400	170	70	30

Here is the resulting relative frequency table.

x	25,000	35,000	45,000	55,000	65,000	75,000	85,000
$f(X = x)$	0.02	0.08	0.23	0.40	0.17	0.07	0.03

In Figure 2 we see the histogram of the frequency distribution and the histogram of the probability distribution. The only difference between the two graphs is in the scale of the vertical axis (why?).



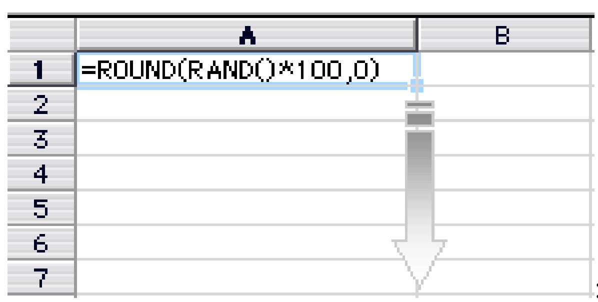


Note We shall often be given a distribution involving categories with *ranges* of values (such as salary brackets), rather than individual values. When this happens, we shall always take X to be the *midpoint* of a category, as we did above. This is a reasonable thing to do, particularly when we have no information about how the scores were distributed within each range.

Note Refining the categories leads to a smoother curve—illustration in class.

Arranging Data into Histograms
In class, we do the following Example

Example 2
We use the Data Analysis Toolpac to make a histogram for the some random whole numbers between 0 and 100:



:

Then we use "Bins" to sort the data into measurement classes. Each bin entry denotes the *upper* boundary of a measurement class; for instance, to get the ranges 0-99, 100–199, etc, use bin values of 99, 199, 299, etc. Here is what we can get for the current experiment:

	A	B	C	D	E	F
1	x	BINS			*Bin*	*Frequency*
2	27	9	means <= 9		9	1
3	10	19	> 9 and <=19		19	4
4	93	29	>19 and <= 20		29	4
5	29	39			39	0
6	83	49			49	1
7	13	59			59	0
8	2	69			69	0
9	17	79			79	0
10	28	89			89	1
11	98	99			99	2
12	20				More	0
13	10					
14	44					

Homework
p. 28 #5, 6, 10
p. 36 #16 (Table 2.9 appears on the next page.)

Topic 2
Measures of Central Tendency and Variability

(based on Section 3.2, 3.3, 3.4 in text)

The **central tendency** of a set of measurements is its tendency to cluster around one or more values. Its **variability** s its tendency to spread out.

Measures of Central Tendency

The **sample mean** of a variable X is the sum of the X-scores for a **sample** of the population divided by the sample size:

$$\bar{x} = \frac{\Sigma x_i}{n} = \frac{\text{sum of x-values}}{\text{sample size}}.$$

The **population mean** is the mean of the scores for the entire population (rather than just a sample) and we denote it by μ rather than $\bar{x}$.

Note In statistics, we use the sample mean to make an inference about the population mean.

Example 1 Calculate the mean of the sample scores {5, 3, 8, 5, 6} (in class)

Example 2 You are the manager of a corporate department with a staff of 50 employees whose salaries are given in the following frequency table.

Annual Salary	\$15,000	\$20,000	\$25,000	\$30,000	\$35,000	\$40,000	\$45,000
Number of Employees	10	9	3	8	12	7	1

What is the mean salary earned by an employee in your department?

Solution To find the average salary we first need to find the sum of the salaries earned by your employees.

10 employees at \$15,000:	$10\times 15{,}000 =$	150,000
9 employees at \$20,000:	$9\times 20{,}000 =$	180,000
3 employees at \$25,000:	$3\times 25{,}000 =$	75,000
8 employees at \$30,000:	$8\times 30{,}000 =$	240,000
12 employees at \$35,000:	$12\times 35{,}000 =$	420,000
7 employees at \$40,000:	$7\times 40{,}000 =$	280,000
1 employee at \$45,000:	$1\times 45{,}000 =$	45,000
	Total =	\$1,390,000

Thus, the average annual salary is $\mu = \dfrac{1{,}390{,}000}{50} = \$27{,}800.$

The sample **median** is the middle number when the scores are arranged in ascending order.

To find the median, arrange the scores in ascending order. If n is odd, m is the middle number, otherwise, it is the average of the two middle numbers. Alternatively, we can use the following formula:

$$m = \frac{n+1}{2} \text{ -th score.}$$

(If the answer is not a whole number, take the average of the scores on either side.)

Example 3 Calculate the median of {5, 7, 4, 5, 20, 6, 2} and {5, 7, 4, 5, 20, 6}

Example 4 The median in the employee example above is $30,000.

The **mode** is the score (or scores) that occur most frequently in the sample. The **modal class** is the measurement class containing the mode.

Example 5 Find the mode in {8, 7, 9, 6, 8, 10, 9, 9, 5, 7}.

Illustration of all three concepts on a graphical distribution.

Measures of Variability

Percentiles

When we say "the 30th percentile for the first quiz is 43" we mean that at least 30% of the student got a score ≤ 43 and at least 70% got a score ≥ 43. (We can't always find a score such that **exactly** 30% got less and exactly 70% got more, as happens in the first example below.)

In general, th**e *p*th percentile** is a number such that at least p% of the scores are ≤ that number and at least $(100-p)$% of the socres are ≥ that number. To compute it, arrange the scores in order, calculate

$$i = \left(\frac{p}{100}\right)n$$

If i is a whole number, take the average of the ith score and the next one above it (the $(i+1)$st score). If i is not a whole number, take the $(i+1)$st score.

Example 6 Find the 30th percentile for the scores {10, 10, 10, 10, 10, 80, 80, 80, 80, 80}.

Quartiles

Quartiles are just certain percentiles. The **first quartile** Q_1 is 25th percentile. the **second quartile** Q_2 is the 50% percentile (which is also the median) and the **third quartlie** Q_3 is the 75th percentile.

To get the quartile in Excel, use

`=QUARTILE(`Cell Range`,q)`

where q = 1 or 3 returns Q_1 and Q_3 respectively, q = 2 returns the median, and q = 0 or 1 return the minimum and maximum respectively.

Example 7 Compute all the quartiles of {8, 7, 9, 6, 8, 10, 9, 9, 5, 7}.

Range
This is just $X_{\max} - X_{\min}$, and measures the total spread of the data.

Variance and Standard Deviation
If a set of scores in a sample are $x_1, x_2, \ldots, x_n$ and their average is $\bar{x}$, we are interested in the distribution of the differences $x_i - \bar{x}$ from the mean. We could compute the *average* of these differences, but this average will always be 0 (why?). It is really the *sizes* of these differences that interests us, so we might try computing the average of the absolute values of the differences. This idea is reasonable, but leads to technical difficulties avoided by a slightly different approach. We shall compute an estimate of the average of the *squares* of the differences. This average is called the **sample variance**. Its square root is called the **sample standard deviation**. It is common to write s for the sample standard deviation and then to write s^2 for the sample variance.

Sample Variance and Sample Standard Deviation
Given a set of scores $x_1, x_2, \ldots, x_n$ with average $\bar{x}$, the **sample variance** is

$$s^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n-1}$$

$$= \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

and the **sample standard deviation** is

$$s = \sqrt{s^2}\ .$$

Shortcut Formula:

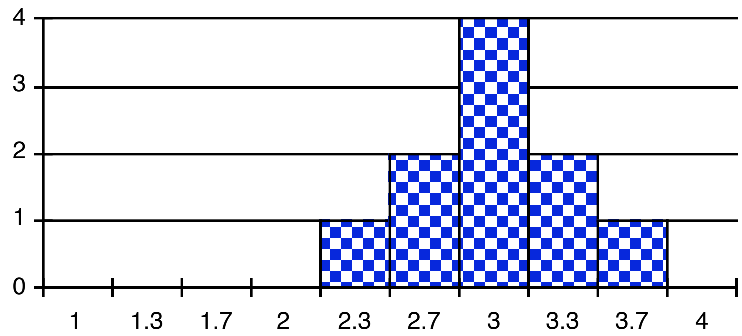
$$s^2 = \frac{\Sigma(x_i^2) - (\Sigma x_i)^2/n}{n-1}\ .$$

Excel:

Variance:	`=VAR`(Range)
St. Deviation:	`=STDEV`(Range)

Example 8 Calculate the sample variance and sample standard deviation for the data set
{3.7, 3.3, 3.3, 3.0, 3.0, 3.0, 3.0, 2.7, 2.7, 2.3}.

Here is a frequency histogram.



Solution Organize the calculations in a table.

	x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
	3.7	0.7	0.49
	3.3	0.3	0.09
	3.3	0.3	0.09
	3.0	0	0
	3.0	0	0
	3.0	0	0
	3.0	0	0
	2.7	−0.3	0.09
	2.7	−0.3	0.09
	2.3	−0.7	0.49
Totals	30.0	0	1.34

The second column, $x_i - \bar{x}$, is obtained by subtracting the average, $\bar{x} = 3.0$, from each of the race times in the first column. The entries in the last column are the squares of the entries in the second column.

The sample variance, s^2, is the sum of the entries in the right-hand column, divided by $n-1 = 9$:

$$s^2 = \frac{1.34}{9} = 0.14888....$$

The sample standard deviation, s, is the square root of s^2.

$$s = \sqrt{0.14888..} \approx 0.38586.$$

Note For the **population** variance, we take the actual average of the $(x_i - \bar{x})^2$. That is, we divide by n instead of $n-1$, and we call this σ^2 instead of s^2.

Excel:

Pop Variance:	`=VARP(Range)`
Pop. St. Deviation:	`=STDEVP(Range)`

Homework

p. 79, #8, 12

p. 88 #18 (The **coefficient of variation** means the size of the standard deviation as a percentage of the size of the mean, given by $s/\bar{x}\times 100$, and can be used to compare the variability of samples with totally different means, like the variability of the lengths of rivers as compared with the variability of the number of stocks in a portfolio.) , #20 (The **interquartlie range** is the difference between Q_3 and Q_1 and is yet another measure of variability.)

Topic 3
Interpreting the Standard Deviation: Chebyshev's Rule & The Empirical Rule
(Section 2.6 in book)

Question Suppose we have a set of data with mean $\bar{x} = 10$ and standard deviation $s = 2$. How do we interpret this information?
Answer This is given by the following rules

Chebyshev's Rule
Applies to all distributions, regardless of shape.
1. At least 3/4 of the scores fall within 2 standard deviations of the mean; that is, in the interval $(\bar{x}-2s, \bar{x}+2s)$ for samples, or $(\mu-2\sigma, \mu+2\sigma)$ for populations.
2. At least 8/9 of the scores fall within 3 standard deviations of the mean; that is, in the interval $(\bar{x}-3s, \bar{x}+3s)$ for samples, or $(\mu-3\sigma, \mu+3\sigma)$ for populations.
3. In general, for $k > 1$, at least $1-1/k^2$ of the scores fall within k standard deviations of the mean; that is, in the interval $(\bar{x}-ks, \bar{x}+ks)$ for samples, or $(\mu-k\sigma, \mu+k\sigma)$ for populations.

We can refine this rule for a mound-shaped and symmetric distribution:

Empirical Rule
Applies to mound-shaped, symmetric distributions
1. Approximately 68% of the scores fall within 1 standard deviation of the mean; that is, in the interval $(\bar{x}-s, \bar{x}+s)$ for samples, or $(\mu-\sigma, \mu+\sigma)$ for populations.
2. Approximately 95% of the scores fall within 2 standard deviations of the mean; that is, in the interval $(\bar{x}-2s, \bar{x}+2s)$ for samples, or $(\mu-2\sigma, \mu+2\sigma)$ for populations.
3. Approximately 99.7% of the scores fall within 3 standard deviations of the mean; that is, in the interval $(\bar{x}-3s, \bar{x}+3s)$ for samples, or $(\mu-3\sigma, \mu+3\sigma)$ for populations.

Example 1 A survey of the percentage of company's revenues spent of R&D gives a distribution with mean 8.49 and standard deviation 1.98.
(a) In what interval can we find at least 15/16 (93.95%) of the scores?
(b) In what interval can we find at least 95% of the scores?

Answer
(a) $15/16 = 1 - 1/16 = 1 - 1/4^2$, so we take $k = 4$. By Chebyshev, the interval is

$$(\bar{x}-4s, \bar{x}+4s) = (8.49-4(1.98),\ 8.49+4(1.98)\ ) = (0.57,\ 16.41)$$

(b) We want $1 - \dfrac{1}{k^2}$ at least 0.98

Try various values of k:

	$k = 2$:	$1 - \dfrac{1}{2^2} = .75$	too small
	$k = 3$:	$1 - \dfrac{1}{3^2} = .888$	too small
	$k = 4$:	$1 - \dfrac{1}{4^2} = .9395$	too small
	$k = 5$:	$1 - \dfrac{1}{5^2} = .96$	big enough

Thus, we can take $k = 6$, and obtain

$$(\bar{x}-6s, \bar{x}+6s) = (8.49-6(1.98),\ 8.49+6(1.98)\ ) = (-3.39,\ 28.37)$$

So, we can use (0, 28.37), since no scores can be negative in this experiment.

Note Almost all (8/9 or 99.7% for nice distributions) will fall within 3 standard deviations of the mean, so the entire range of scores should not exceed approximately 3 standard deviations. This gives us a "guestimate" of whether our calculation of the standard deviation is reasonable.

Example 2 (Battery life)
Suppose a manufacturer claims that the mean lifespan of a battery is 60 months, with a standard deviation of 10 months, and suppose also that the distribution is mound-shaped and symmetric. You buy a battery and find that is fails prior to 40 months. How much confidence do you have in the manufacturer's claim?

Answer 40 months is two standard deviations from the mean. By the empirical rule, the chance of a battery falling within $(\mu-2\sigma, \mu+2\sigma)$ is 95%. Thus approximately only 5% fall outside that range. Half of those fall to the left, the rest to the right, so only about 2.5% of batteries should fail before 40 months. Thus, you have reason to doubt the claim, or else you were extremely unlucky to be in the bad 2.5%.

If you bought, say, 10 batteries and discovered that their mean lifespan was less than 40 months, you would be pretty confident that the manufacturer was wrong. How confident? We'll see towards the end of the course.

z-Scores and Outliers

The ***z*-score** of a specific datum x is given by

$$z = \frac{x - \bar{x}}{s} \quad \text{for samples}$$

or

$$z = \frac{x - \mu}{\sigma} \quad \text{for populations}$$

The z-score measures the number of standard deviations a specific value $x$is away from the mean. So, if a data value has $z = -1.5$, it means that it is 1.5 standard deviations below the mean. An **outlier** is a data value that has a $|z| > 3$. We need to carefully review outliers to check whether they belong there, or are due to measurement errors.

Note we can rewrite Chebyschev's rule and the empirical rule in terms of z-scores.

Example 3 (Battery life)
Find the z-score for a battery that lasts 32 months.

Homework

www.FiniteMath.com Student Web Site Chapter Review Exercises Statistics
2, 3, 4, 9
p. 93 #32, 34, 36

Topic 4
Introduction to Probability
(Based on 4.1, 4.2 in book)

Sample Spaces
Let's start with a familiar situation: If you toss a coin and observe which side lands up, there are two possible results: heads (H) and tails (T). These are the *only* possible results, ignoring the (remote) possibility that the coin lands on its edge. The act of tossing a coin is an example of an **experiment**. The two possible results H and T are the possible **outcomes** of the experiment, and the set $S = \{H, T\}$ of all possible outcomes is the **sample space** for the experiment.

Experiments, Outcomes, and Sample Spaces
An **experiment** is an occurrence whose result, or **outcome**, is uncertain. The set of all possible outcomes is called the **sample space** for the experiment.

Quick Examples

1. **Experiment**: Flip a coin and observe the side facing up.
   Outcomes: H, T
   Sample Space: $S = \{H, T\}$
2. **Experiment:** Select a student in your class.
   Outcomes: The students in your class
   Sample Space: The set of students in your class.
3. **Experiment:** Select a student in your class and observe the color of his or her hair
   Outcomes: red, black, brown, blond, green, ...
   Sample Space: { red, black, brown, blond, green, ...}
4. **Experiment**: Cast a die and observe the number facing up.
   Outcomes: 1, 2, 3, 4, 5, 6
   Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$
5. **Experiment:** Cast two distinguishable dice and observe the numbers facing up.
   Outcomes: (1,1), (1,2), ... , (6,6) (36 outcomes)
   Sample Space: $$S = \left\{\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix}\right\}$$
   $n(S) = 36$
6. **Experiment:** Cast two indistinguishable dice and observe the numbers facing up.
   Outcomes: (1,1), (1,2), ... , (6,6) (21 outcomes)
   Sample Space: $$S = \left\{\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ & & (3,3) & (3,4) & (3,5) & (3,6) \\ & & & (4,4) & (4,5) & (4,6) \\ & & & & (5,5) & (5,6) \\ & & & & & (6,6) \end{matrix}\right\};$$
   $n(S) = 21$
7. **Experiment**: Cast two dice and observe the *sum* of the numbers facing up.

Outcomes: 2. 3. 4. 5. 6. 7. 8. 9. 10. 11. 12
Sample Space: $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

8. **Experiment:** Choose 2 cars (without regard to order) at random from a fleet of 10.
   Outcomes: Collections of 2 cars chosen from 10.
   Sample Space: The set of all collections of 2 cars chosen from 10;
   $n(S) = C(10, 2) = 45$

Events

Looking at the last example, suppose that we are interested in the outcomes in which the factory worker was covered by some form of medical insurance. In mathematical language, we are interested in the *subset* consisting of all outcomes in which the worker was covered.

Events

Given a sample space S, an **event** E is a subset of S. The outcomes in E are called the **favorable** outcomes. We say that E **occurs** in a particular experiment if the outcome of that experiment is one of the elements of E; that is, if the outcome of the experiment is favorable.

Quick Examples

1. **Experiment:** Roll a die and observe the number facing up.
   $S = \{1, 2, 3, 4, 5, 6\}$
   Event: E: The number observed is odd.
   $E = \{1, 3, 5\}$
2. **Experiment:** Roll two distinguishable dice and observe the numbers facing up.
   $S = \{(1,1), (1,2) \dots , (6,6)\}$
   Event: F: The dice show the same number.
   $F = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$
3. **Experiment:** Roll two distinguishable dice and observe the numbers facing up.
   $S = \{(1,1), (1,2), \dots , (6,6)\}$
   Event: G: The sum of the numbers is 1.
   $G = \emptyset$ There are no favorable outcomes
4. **Experiment:** Select a city beginning with "J."
   Event: E: The city is Johannesburg.
   $E = \{\text{Johannesburg}\}$ An event can consists of a single outcome
5. **Experiment:** Roll a die and observe the number facing up
   Event: E: The number observed is either even or odd
   $E = S = \{1, 2, 3, 4, 5, 6\}$ An event can consist of all possible outcomes
6. **Experiment:** Select a student in your class.
   Event: E: The student has red hair.
   $E = \{\text{red-haired students in your class}\}$
7. **Experiment:** Draw a hand of two cards from a deck of 52.
   Event: H: Both cards are diamonds.

H is the set of all hands of 2 cards chosen from 52 such that both cards are diamonds.

Example 1 Let *S* be the sample space of Example 2.
(a) Describe the event *E* that a factory worker was covered by some form of medical insurance.
(b) Describe the event *F* that a factory worker was not covered by an individual medical plan.
(c) Describe the event *G* that a factory worker was covered by a government medical plan.

Example 2 You roll a red die and a green die and observe the numbers facing up. Describe the following events as subsets of the sample space.
(a) *E*: Both dice show the same number.
(b) *F*: The sum of the numbers showing is 6.
(c) *G*: The sum of the numbers showing is 2.

Probability Distribution
(1) A **probability distribution** is an assignment of a number $P(s_i)$ to each outcome s_i in a sample space $\{s_1, s_2, \ldots, s_n\}$, so that

(a) $0 \leq P(s_i) \leq 1$ and
(b) $P(s_1) + P(s_2) + \ldots + P(s_n) = 1$.

In words, the probability of each outcome must be a number between 0 and 1, and the probabilities of all the outcomes must add up to 1.

(2) Given a probability distribution, we can obtain the probability of an event *E* by adding up the probabilities of the outcomes in *E*.

Example 3 *Weighted Dice*In order to impress you friends with your dice-throwing skills, you have surreptitiously weighted your die in such a way that 6 is three times as likely to come up as any one of the other numbers. Find the probability distribution, and use it to calculate the probability of an even number coming up.

Example 4
A fair die is tossed, and the up face is observed. If it is even, you win $1. Otherwise, you lose $1. What is the probability that you win. (First obtain the event, then the probability.)

Note
Since the probability of an outcome can be zero, we are also allowing the possibility that $P(E) = 0$ for an event *E*. If $P(E) = 0$, we call *E* an **impossible event**. The event Ø is always impossible, since *something* must happen.

Example 5
Your broker recommends four companies. Unbeknownst to you, two of the four happen to be duds. You invest in two of them. Find the probability that:

(a) you have chosen the two losers
(b) you have chosen the two winners
(c) you have chosen one of each

Sometimes, the outcomes in an experiment are **equally likely**.

Equally Likely Outcomes
In an experiment in which all outcomes are equally likely, the probability of an event E is given by

$$P(E) = \frac{number\ of\ favorable\ outcomes}{total\ number\ of\ outcomes} = \frac{n(E)}{n(S)}\ .$$

To find $n(E)$ and $n(S)$, we sometimes need *combinatorial mathematics*:

You walk into an ice cream place and find that you can choose between ice cream, of which there are 15 flavors, and frozen yogurt, of which there are 5 flavors. How many different selections can you make? Clearly, you have $15 + 5 = 20$ different desserts to choose from. Mathematically, this is an example of the formula for the cardinality of a disjoint union: If we let A be the set of ice creams you can choose from, and B the set of frozen yogurts, then $A \cap B = \emptyset$ and we want $n(A \cup B)$. But, the formula for the cardinality of a disjoint union is $n(A \cup B) = n(A) + n(B)$, which gives $15 + 5 = 20$ in this case.

This example illustrates a very useful general principle.

Addition Principle
When choosing among r disjoint alternatives, if
alternative 1 has n_1 possible outcomes,
alternative 2 has n_2 possible outcomes,
…
alternative r has n_r possible outcomes,
then you have a total of $n_1 + n_2 + \ldots + n_r$ possible outcomes.

Quick Example
At a restaurant you can choose among 8 chicken dishes, 10 beef dishes, 4 seafood dishes, and 12 vegetarian dishes. This gives a total of $8 + 10 + 4 + 12 = 34$ different dishes to choose from.

Here is another simple example. In that ice cream place, not only can you choose from 15 flavors of ice cream, but you can also choose from 3 different sizes of cone. How many different ice cream cones can you select from? If we let A again be the set of ice cream flavors and now let C be the set of cone sizes, we want to pick a flavor *and* a size. That is, we want to pick an element of $A \times C$, the Cartesian product. To find the number of choices we have, we use the formula for the cardinality of a Cartesian product: $n(A \times C) = n(A)n(C)$. In this case, we get $15\times 3 = 45$ different ice cream cones we can select.

This example illustrates another general principle.

Multiplication Principle

When making a sequence of choices with r steps, if

step 1 has n_1 possible outcomes

step 2 has n_2 possible outcomes

...

step r has n_r possible outcomes

then you have a total of $n_1 \times n_2 \times \ldots \times n_r$ possible outcomes.

Quick Example

At a restaurant you can choose among 5 appetizers, 34 main dishes, and 10 desserts. This gives a total of $5 \times 34 \times 10 = 1700$ different meals (each including one appetizer, one main dish, and one dessert) you can choose from.

Things get more interesting when we have to use the addition and multiplication principles in tandem.

Example 6 ***Desserts***

You walk into an ice cream place and find that you can choose between ice cream, of which there are 15 flavors, and frozen yogurt, of which there are 5 flavors. In addition, you can choose among 3 different sizes of cones for your ice cream or 2 different sizes of cups for your yogurt. How many different desserts can you choose from?

Combinations

Question How many groups of 4 marbles can be selected from a bag containing 12?

Answer $\binom{12}{4} = \dfrac{12\cdot 11\cdot 10\cdot 9}{4\cdot 3\cdot 2\cdot 1} = 495$

Question How many groups of r marbles can be selected from a bag containing n?

Answer $\binom{n}{r} = \dfrac{n\cdot(n-1)\cdot\ldots\cdot(n-r+1)}{r\cdot(r-1)\cdot\cdots\cdot 1}$

Example 7 ***Poker Hands***

In the card game poker, a hand consists of a set of five cards from a standard deck of 52. A **full house** is a hand consisting of three cards of one denomination ("three of a kind"—e.g. three 10s) and two of another ("two of a kind"—e.g. two Queens). Here is an example of a full house: 10♣ , 10♦, 10♠, Q♥, Q♣.

(a) How many different poker hands are there?

(b) How many different full houses are there that contain three 10s and two Queens?

(c) How many different full houses are there altogether?

Solution

(a) Since the order of the cards doesn't matter, we simply need to know the number of ways of choosing a set of 5 cards out of 52, which is

$$C(52, 5) = 2{,}598{,}960 \text{ hands.}$$

(b) Here is a decision algorithm for choosing a full house with three 10s and two Queens.

Step 1: Choose three 10s. Since there are four 10s to choose from we have $C(4, 3) = 4$ choices.

Step 2: Choose 2 Queens; $C(4, 2) = 6$ choices.

Thus, there are $4 \times 6 = 24$ possible full houses with three 10s and two Queens.

(c) Here is a decision algorithm for choosing a full house.

Step 1: Choose a denomination for the three of a kind; 13 choices.

Step 2: Choose 3 cards of that denomination. Since there are 4 cards of each denomination (one for each suit), we get $C(4, 3) = 4$ choices.

Step 3: Choose a different denomination for the two of a kind. There are only 12 denominations left, so we have 12 choices.

Step 4: Choose 2 of that denomination; $C(4, 2) = 6$ choices.

Thus, by the multiplication principle, there are a total of $13 \times 4 \times 12 \times 6 = 3744$ possible full houses.

Homework

In Exercises 1–3, describe the sample space S of the experiment and list the elements of the given event. (Assume that the coins are distinguishable and that what is observed are the faces or numbers that face up.)

1. Two coins are tossed; the result is at most one tail.

2. Two indistinguishable dice are rolled; the numbers add to 5.

3. You are deciding whether to enroll for Psychology 1, Psychology 2, Economics 1, General Economics, or Math for Poets; you decide to avoid economics.

4. A packet of gummy candy contains 4 strawberry gums, 4 lime gums, 2 black currant gums, and 2 orange gums. April May sticks her hand in and selects 4 at random. Complete the following sentences:

(a) The sample space is the set of ...

(b) April is particularly fond of combinations of 2 strawberry and 2 black currant gums. The event that April will get the combination she desires is the set of ...

5. Complete the following. An event is a ____.

6. True or False? Every set S is the sample space for some experiment. Explain.

7. True or false: every sample space S is a finite set. Explain.

8. The probability of an event E is the number of outcomes in E divided by the total number of outcomes, right?

9. ***Motor Vehicle Safety*** The following table shows crashworthiness ratings for 10 small SUVs.[1] (3=Good, 2=Acceptable, 1=Marginal, 0=Poor)

Frontal Crash Test Rating	3	2	1	0
Frequency	1	4	4	1

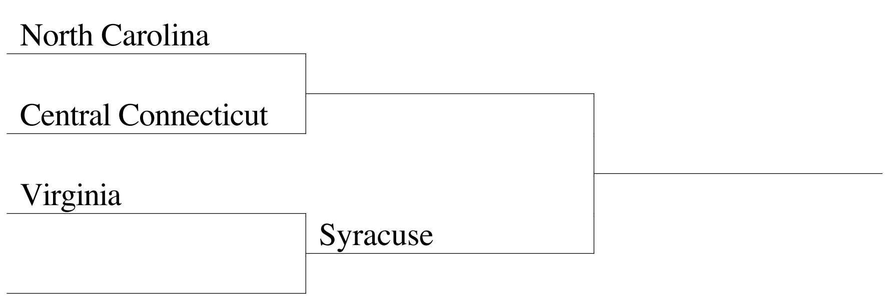
(a) Find the estimated probability distribution for the experiment of choosing a small SUV at random and determining its frontal crash rating.
(b) What is the estimated probability that a randomly selected small SUV will have a crash test rating of "Acceptable" or better?

10. It is said that lightning never strikes twice in the same spot. Assuming this to be the case, what is the estimated probability that lightning will strike your favorite dining spot during a thunderstorm? Explain.

11. ***Zip™ Disks*** Zip™ disks come in two sizes (100MB and 250MB), packaged singly, in boxes of five, or in boxes of ten. When purchasing singly, you can choose from five colors; when purchasing in boxes of five or ten you have two choices, black or an assortment of colors. If you are purchasing Zip disks, how many possibilities do you have to choose from?

12. ***Tests*** A test requires that you answer either Part A or Part B. Part A consists of 8 true-false questions, and Part B consists of 5 multiple-choice questions with 1 correct answer out of 5. How many different completed answer sheets are possible?

13. ***Tournaments*** How many ways are there of filling in the blanks for the following (fictitious) soccer tournament?



[1] Ratings by the Insurance Institute for Highway Safety. Sources: Oak Ridge National Laboratory: "An Analysis of the Impact of Sport Utility Vehicles in the United States" Stacy C. Davis, Lorena F. Truett, (August 2000)/Insurance Institute for Highway Safety
http://www-cta.ornl.gov/Publications/Final SUV report.pdf http://www.highwaysafety.org/vehicle_ratings/

14. ***HTML*** Colors in HTML (the language in which many web pages are written) can be represented by 6-digit hexadecimal codes: sequences of six integers ranging from 0 to 15 (represented as 0, ..., 9, A, B, .., F).

(a) How many different colors can be represented?

(b) Some monitors can only display colors encoded with pairs of repeating digits (such as 44DD88). How many colors can these monitors display?

(c) Grayscale shades are represented by sequences *xyxyxy*. consisting of a repeated pair of digits. How many grayscale shades are possible?

(d) The pure colors are pure red: *xy*0000; pure green: 00*xy*00; and pure blue: 0000*xy*. ($xy = FF$ gives the brightest pure color, while $xy = 00$ gives the darkest: black). How many pure colors are possible?

Poker Hands A poker hand consists of five cards from a standard deck of 52. (See the chart preceding Example 7.) In Exercises 15–18, find the number of different poker hands of the specified type.

15. Two pairs (two of one denomination, two of another denomination, and one of a third)

16. Three of a kind (three of one denomination, one of another denomination, and one of a third)

17. Two of a kind (two of one denomination and three of different denominations)

18. Four of a kind (all four of one denomination and one of another)

Answers

1. $S = \{\text{HH, HT, TH, TT}\}$; $E = \{\text{HH, HT, TH}\}$

2. $S = \left\{\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix}\right\}$ $E = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$

3. S = {Psychology 1, Psychology 2, Economics 1, General Economics, Math for Poets}; E = {Psychology 1, Psychology 2, Math for Poets} **4. (a)** all sets of 4 gummy bears chosen from the packet of 12. **(b)** all sets of 4 gummy bears in which two are strawberry and two are blackcurrant.

5. Subset of the sample space **6.** True; Consider the following experiment: Select an element of the set S at random. **7.** False; for instance, consider the following experiment: Flip a coin until you get heads, and observe the number of times you flipped the coin.

8. Only when all the outcomes are equally likely.

9. (a)

Test Rating	3	2	1	0
Probability	0.1	0.4	0.4	0.1

(b) 0.5

10. Zero; according to the assumption, no matter how many thunderstorms occur, lightning cannot only strike your favorite spot more than once, and so, after n trials the estimated probability will never exceed $1/n$, and so will approach zero as the number of trials gets large. **11.** $(2\times5) + (2\times2\times2) = 18$ **12.** $2^8 + 5^5 = 3{,}381$ **13.** 4 **14. (a)** $16^6 =$

16,777,216 **(b)** $16^3 = 4096$ **(c)** $16^2 = 256$ **(d)** $3\times16^2 - 2 = 766$ **15.** $C(13,2)C(4,2)C(4,2)\times44 = 123,552$ **16.** $13\times4\times C(12,2)\times4\times4 = 54,912$ **17.** $13\times C(4,2)C(12,3)\times4\times4\times4 = 1,098,240$ **18.** $13\times48 = 624$

Topic 5
Unions, Intersections, and Complements

(Based on 4.3 in book)

Events may often be described in terms of other events, using set operations. An example is the **negation** of an event E, the event that E does not occur. If in a particular experiment E does not occur, then the outcome of that experiment is not in E, so is in its *complement* (in S). It is called E^c and its probability is given by

$$P(E^c) = 1 - P(E).$$

Example 1 You roll a red die and a green die and observe the two numbers facing up. Describe the event that the sum of the numbers is not 6. What is its probability?

Question If E and F are events, how can we describe the event $E\cup F$?
Answer Consider a simple example: the experiment of throwing a die. Let E be the event that the outcome is a 5, and let F be the event that the outcome is an even number. Thus,

$E = \{5\}$, $F = \{2, 4, 6\}$.

So, $E\cup F = \{5, 2, 4, 6\}$.

In other words, $E\cup F$ is the event that the outcome is *either* a 5 *or* an even number. In general we can say the following.

Question If E and F are events, how can we describe the event $E\cap F$?
Answer in class
Example 2 The following table shows sales of recreational boats in the U.S. during the period 1999–2001.[2]

	Motor boats	Jet skis	Sailboats	Total
1999	330,000	100,000	20,000	450,000
2000	340,000	100,000	20,000	460,000
2001	310,000	90,000	30,000	430,000
Total	980,000	290,000	70,000	1,340,000

Consider the experiment in which a recreational boat is selected at random from those in the table. Let E be the event that the boat was a motor boat, let F be the event that the boat was purchased in 2001, and let G be the event that the boat was a sailboat. Find the probabilities of the following events:

[2] Figures are approximate, and represent new recreational boats sold. ("Jet skis" includes similar vehicles, such as "wave runners".) Source: National Marine Manufacturers Association/*New York Times,* January 10. 2002, p. C1.

(a) E **(b)** F **(c)** $E \cap F$ **(d)** G' **(e)** $E \cup F'$.

If A and B are events, then A and B are said to be **disjoint** or **mutually exclusive** if $A \cap B$ is empty.

Example 3 A coin is tossed three times and the sequence of heads and tails is recorded. Decide whether the following pairs of events are mutually exclusive.
(a) A: the first toss shows a head, B: the second toss shows a tail.
(b) A: all three tosses land the same way up, B: one toss shows heads and the other two show tails.

Complement of an Event
The complement E^c of an event E is the event that E does not occur.

$$P(E^c) = 1 - P(E).$$

Union of Events
If E and F are events, then $E \cup F$ is the event that either E occurs or F occurs (or both).

$$P(E \cup F) = P(E) + P(F) - P(E \cap F) \quad \text{(if not mutually exclusive)}$$
$$P(E \cup F) = P(E) + P(F) \quad \text{(if mutually exclusive)}$$

Intersection of Events
If E and F are events, then $E \cap F$ is the event that both E and F occur.

$$P(E \cap F) = P(E)P(F) \quad \text{(if independent)}$$

Example 4 *Astrology* The astrology software package Turbo Kismet works by first generating random number sequences, and then interpreting them numerologically. When I ran it yesterday, it informed me that there was a $1/3$ probability that I would meet a tall dark stranger this month, a $2/3$ probability that I would travel within the next month, and a $1/6$ probability that I would meet a tall dark stranger on my travels this month. What is the probability that I will either meet a tall dark stranger or that I will travel this month?

Example 5 *Salaries* Your company's statistics show that 30% of your employees earn between \$20,000 and \$39,999, while 20% earn between \$30,000 and \$59,999. Given that 40% of the employees earn between \$20,000 and \$59,999,
(a) what percentage earn between \$30,000 and \$39,999?
(b) what percentage earn between \$20,000 and \$29,999?

Homework
Suppose two dice (one red, one green) are rolled. Consider the following events: A: the red die shows 1; B: the numbers add to 4; C: at least one of the numbers is 1; and D: the numbers do not add to 11. In Exercises 1–4, express the given event in symbols and say how many elements it contains.

1. The red die shows 1 and the numbers add to 4.
2. The numbers do not add to 4 but they do add to 11.
3. Either the numbers add to 11 or the red die shows a 1.
4. At least one of the numbers is 1 or the numbers add to 4.

Let W be the event that you will use the web site tonight, let I be the event that your math grade will improve, and let E be the event that you will use the web site every night. In Exercises 5–8, express the given event in symbols.

5. You will use the web site tonight and your math grade will improve.

6. Either you will use the web site every night, or your math grade will not improve.

7. Your math grade will not improve even though you use the web site every night.

8. You will either use the web site tonight with no grade improvement, or every night with grade improvement.

9. Complete the following. Two events E and F are mutually exclusive if their intersection is ______.

10. If E and F are events, then $(E \cap F)'$ is the event that ____ .

Publishing Exercises 11–15 are based on the following table, which shows the results of a survey of 100 authors by a publishing company.

	New Authors	Established Authors	Total
Successful	5	25	30
Unsuccessful	15	55	70
Total	20	80	100

Compute the following estimated probabilities in of the given events.

11. An author is established and successful

12. An author is a new author.

13. An author is unsuccessful.

14. An unsuccessful author is established.

15. A new author is unsuccessful.

16. ***Steroids Testing*** A pharmaceutical company is running trials on a new test for anabolic steroids. The company uses the test on 400 athletes known to be using steroids and 200 athletes known not to be using steroids. Of those using steroids, the new test is positive for 390 and negative for 10. Of those not using steroids, the test is positive for 10 and negative for 190. What is the estimated probability of a **false negative** result (the probability that an athlete using steroids will test negative)? What is the estimated probability of a **false positive** result (the probability that an athlete not using steroids will test positive)?

17. Tony has had a "losing streak" at the casino—the chances of winning the game he is playing are 40%, but he has lost 5 times in a row. Tony argues that, since he should have won 2 times, the game must obviously be "rigged." Comment on his reasoning.

18. ***Computer Sales*** In 1999 (one year after the *iMac* was first launched by Apple), a retail or mail-order purchase of a personal computer was approximately 7 times as likely to be a non-Apple PC as an Apple PC.[3] What is the probability that a randomly chosen personal computer purchase was an Apple?

In Exercises 19–26, use the given information to find the indicated probability.

19. $P(A) = 0.1$, $P(B) = 0.6$, $P(A \cap B) = 0.05$. Find $P(A \cup B)$.

20. $A \cap B = \emptyset$, $P(A) = 0.3$, $P(A \cup B) = 0.4$. Find $P(B)$.

21. $A \cap B = \emptyset$, $P(A) = 0.3$, $P(B) = 0.4$. Find $P(A \cup B)$.

22. $P(A \cup B) = 0.9$, $P(B) = 0.6$, $P(A \cap B) = 0.1$. Find $P(A)$.

23. $P(A) = 0.22$. Find $P(A')$.

24. A, B and C are mutually exclusive. $P(A) = 0.2$, $P(B) = 0.6$, $P(C) = 0.1$. Find $P(A \cup B \cup C)$.

25. A and B are mutually exclusive. $P(A) = 0.4$, $P(B) = 0.4$. Find $P((A \cup B)')$.

26. $P(A \cup B) = 0.3$ and $P(A \cap B) = 0.1$. Find $P(A) + P(B)$.

In Exercises 27–29, determine whether the information shown is consistent with a probability distribution. If not, say why.

27.

Outcome	a	b	c	d	e
Probability	0	0	0.65	0.3	0.05

28. $P(A) = 0.2$, $P(B) = 0.1$; $P(A \cup B) = 0.4$

29. $P(A) = 0.2$, $P(B) = 0.4$; $P(A \cap B) = 0.3$.

30. $P(A) = 0.1$, $P(B) = 0$; $P(A \cap B) = 0$.

31. ***Holiday Shopping*** In 1999, the probability that a consumer would shop for holiday gifts at a discount department store was .80, and the probability that a consumer would shop for holiday gifts from catalogs was .42.[4] Assuming that 90% of consumers shopped from one or the other, what percentage of them did both?

32. ***Online Households*** In 2001, 6.1% of all U.S. households were connected to the Internet via cable, while 2.7% of them were connected to the internet through DSL. What percentage of U.S. households did not have high-speed (cable or DSL) connection to the Internet? (Assume that the percentage of households with both cable and DSL access is negligible.)

33. ***Fast-Food Stores*** In 2000 the top 100 chain restaurants in the U.S. owned a total of approximately 130,000 outlets. Of these, the three largest (in numbers of outlets) were McDonalds, Subway, and Burger King, owning between them 26% of all of the outlets.[5] The two hamburger companies, McDonalds and Burger King, together owned approximately 16% of all outlets, while the two largest, McDonalds and Subway,

[3] Figure is approximate. Source: PC Data/*The New York Times,* April 26. 1999, p. C1.

[4] Sources: Commerce Department, Deloitte & Touche Survey/*The New York Times,* November 24, 1999, p. C1.

[5] Source: *Technomic 2001 Top 100 Report*, Technomic, Inc. Information obtained from their web site, www.technomic.com.

together owned 19% of the outlets. What was the probability that a randomly chosen restaurant was a McDonalds?

34. ***Auto Sales*** in 1999, automobile sales in Europe equaled combined sales in NAFTA (North American Free Trade Agreement) countries and Asia. Further, sales in Europe were 70% more than sales in NAFTA countries.[6]
(a) Write down the associated probability distribution.
(b) A total of 34 million automobiles were sold in these three regions. How many were sold in Europe?

Answers:

1. $A \cap B$; $n(A \cap B) = 1$ **2.** $B' \cap D'$; $n(B' \cap D') = 2$ **3.** $D' \cup A$ $n(D' \cup A) = 8$ **4.** $C \cup B$; $n(C \cup B) = 12$ **5.** $W \cap I$ **6.** $E \cup I'$ **7.** $I' \cap E$ **8.** $(W \cap I') \cup (E \cap I)$ **9.** Empty **10.** E and F do not both occur. **11.** 0.25 **12.** 0.2 **13.** 0.7 **14.** 1/14 **15.** 0.75 **16.** P(false negative) = 10/400 = 0.025, P(false positive) = 10/200 = 0.05 **17.** He is wrong. It is possible to have a run of losses of any length.. Tony may have grounds to *suspect* that the game is rigged, but no proof. **18.** 0.125 **19.** 0.65 **20.** 0.1 **21.** 0.7 **22.** 0.4 **23.** 0.78 **24.** 0.9 **25.** 0.2 **26.** 0.4 **27.** Yes **28.** No; $P(A \cup B)$ should be $\leq P(A)+P(B)$. **29.** No; $P(A \cap B)$ should be $\leq P(A)$ **30.** Yes **31.** 32% **32.** 91.2% **33.** 09

34. (a)

Outcome	NATFA	Asia	Europe	
Probability		5/17	7/34	1/2

(b) 17 million.

[6] Source: Economist Intelligence Unit (EIU), March 15, 2002.
http://www.autoindustry.co.uk/statistics/sales/world.html

Topic 6
Conditional Probability & Independent Events
(Section 4.4 in the book)

Q Who cares about conditional probability? What is its relevance in the business world?
A Let's consider the following scenario: Cyber Video Games, Inc., has been running a television ad for its latest game, “Ultimate Hockey.” As Cyber Video's director of marketing, you would like to assess the ad’s effectiveness, so you ask your market research team to make a survey of video game players. The results of their survey of 50,000 video game players are summarized in the following chart.

	Saw Ad	Did Not See Ad
Purchased Game	1,200	2,000
Did Not Purchase Game	3,800	43,000

The market research team concludes in their report that the ad campaign is highly effective.

Question But wait! How could the campaign possibly have been effective? only 1,200 people who saw the ad purchased the game, while 2,000 people purchased the game without seeing the ad! It looks as though potential customers are being *put off* by the ad.
Answer Let us analyze these figures a little more carefully. First, we can look at the event E that a randomly chosen video game player purchased Ultimate Hockey. In the “Purchased Game” row we see that a total of 3,200 people purchased the game. Thus, the experimental probability of E is

$$P(E) = \frac{fr(E)}{N} = \frac{3,200}{50,000} = 0.064.$$

To test the effectiveness of the television ad, let's compare this figure with the experimental probability that *a video game player who saw the ad* purchased Ultimate Hockey. This means that we restrict attention to the “Saw Ad” column. This is the fraction

$$\frac{\text{Number of people who saw the ad and purchased the game}}{\text{Total number of people who saw the ad}} = \frac{1,200}{5,000} = 0.24.$$

In other words, 24% of those surveyed who saw the ad bought Ultimate Hockey, while overall, only 6.4% of those surveyed bought it. Thus, it appears that the ad campaign *was* highly successful.

Let us first introduce some terminology. In this example there were two related events of importance,

E, the event that a video game player purchased Ultimate Hockey, and
F, the event that a video game player saw the ad.

The two probabilities we compared were the experimental probability $P(E)$ and the experimental probability that a video game player purchased Ultimate Hockey *given that* he or she saw the ad. We call the latter probability the (experimental) **probability of *E*, given *F*,** and we write it as $P(E|F)$. We call $P(E|F)$ a **conditional probability**—it is the probability of E under the condition that F occurred.

Q How do we calculate conditional probabilities?
A In the example above we used the ratio

$$P(E|F) = \frac{\text{Number of people who saw the ad and bought the game}}{\text{Total number of people who saw the ad}}$$

$$= \frac{\text{Number of favorable outcomes in } F}{\text{Total number of outcomes in } F}.$$

The numerator is the frequency of $E \cap F$, while the denominator is the frequency of F. Thus, we can say the following.

Conditional Probability
If E and F are events, then

$$P(E|F) = \frac{fr(E \cap F)}{fr(F)}$$

We can write this formula in another way.

$$P(E|F) = \frac{n(E \cap F)}{n(F)} = \frac{n(E \cap F)/n(S)}{n(F)/n(S)} = \frac{P(E \cap F)}{P(F)}$$

Example (Based on p. 146, Example 3.15 of *Statistics for Business and Economics 8th Ed* by McClave, Benson, and Sicich, Prentice Hall, 2001) A manufacturer of an electric kitchen utensil conducted a survey of consumer complaints. The results are summarized in the following table:

	Reason for Complaint			
	Electrical	**Mechanical**	**Appearance**	**Totals**
During Guarantee Period	18%	13%	32%	63%
After Guarantee Period	12%	22%	3%	37%
Totals	30%	35%	35%	100%

(a) Calculate the probability that a customer complains about appearance (dents, scratches, etc.) given that the complaint occurred during the guarantee time.
(b) Calculate the probability that a customer complains about appearance.

Independence
We saw that the formula

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

could be used to calculate $P(E \cap F)$ if we rewrite the formula in the following form, known as the **multiplication principle.**

Multiplication Principle
If E and F are events, then

$$P(E \cap F) = P(F)P(E|F).$$

Example 4 An experiment consists of tossing two coins. The first coin is fair, while the second coin is twice as likely to land with heads facing up as it is with tails facing up. Draw a tree diagram to illustrate all the possible outcomes, and use the multiplication principle to compute the probabilities of all the outcomes.

Let us go back to Cyber Video Games, Inc., and their ad campaign. We would like to assess the ad's effectiveness. As before, we consider

E, the event that a video game player purchased Ultimate Hockey, and
F, the event that a video game player saw the ad.

As we saw, we could use survey data to calculate

$P(E)$, the probability that a video game player purchased Ultimate Hockey, and
$P(E|F)$, the probability that a video game player *who saw the ad* purchased Ultimate Hockey.

When these probabilities are compared, one of three things can happen.

Case 1 $P(E|F) > P(E)$: This is what the survey data actually showed: a video game player was more likely to purchase Ultimate Hockey if he or she saw the ad. This indicates that the ad is effective—seeing the ad had a positive effect on a player's decision to purchase the game.
Case 2 $P(E|F) < P(E)$: If this happens, then a video game owner is *less* likely to purchase Ultimate Hockey if he or she saw the ad. This would indicate that the ad has "backfired:" it has, for some reason, put potential customers off. In this case, just as in the first case, the event F has an effect—a negative one—on the event E.
Case 3 $P(E|F) = P(E)$: In this case seeing the ad had absolutely no effect on a potential customer's buying Ultimate Hockey. Put another way, the event *F had no effect at all on the event E*. We would say that the events E and F are **independent.**

In general, we say that two events E and F are independent if $P(E|F) = P(E)$. When this happens, we have

$$P(E) = P(E|F) = \frac{P(E\cap F)}{P(F)},$$

so $$P(E\cap F) = P(E)P(F).$$

Conversely, if $P(E\cap F) = P(E)P(F)$, then, assuming $P(F) \neq 0$,† $P(E) = P(E\cap F)/P(F) = P(E|F)$.

Independent Events
The events E and F are **independent** if

$$P(E|F) = P(E)$$

or, equivalently,

$$P(E\cap F) = P(E)P(F)$$

If two events E and F are not independent, then they are **dependent.**

Notes□
(1) The formula $P(E\cap F) = P(E)P(F)$ also says that $P(F|E) = P(F)$. Thus, if F has no effect on E, then likewise E has no effect on F.
(2) Sometimes it is obviously the case that two events, by their nature, are independent. For example, the event that a die you roll comes up 1 is clearly independent of whether or not a coin you toss comes up heads. In some cases, though, we need to check for independence by comparing $P(E\cap F)$ to $P(E)P(F)$. If they are equal then E and F are independent, but if they are unequal then E and F are dependent.

Example According to a computer store's records, 80% of previous PC customers purchased clones, and 20% purchased IBM's.
(a) What is the probability that the next 2 customers will purchase clones?
(b) What is the probability that the next 10 customers will purchase clones?

Homework
p. 158 #30, 32, 34, 38
Also:

Publishing Exercises 1–6 are based on the following table, which shows the results of a survey of 100 authors by a publishing company.

	New Authors	Established Authors	Total
Successful	5	25	30
Unsuccessful	15	55	70
Total	20	80	100

Compute the following conditional probabilities:

† We shall only discuss the independence of two events in cases where their probabilities are both non-zero.

1. That an author is established, given that she is successful
2. That an author is successful, given that he is established
3. That an author is unsuccessful, given that she is established
4. That an author is established, given that he is unsuccessful
5. That an unsuccessful author is established
6. That an established author is successful

[Answers: **1.** 5/6 **2.** 5/16 **3.** 11/16 **4.** 11/14 **5.** 11/14 **6.** 5/16]

Topic 7
Discrete Random Variables & Their Probability Distributions

(Based on Section 5.1, 5.2, 5.3)

In many experiments, the outcomes can be assigned numerical values. For instance, if you roll a die, then each outcome has the numerical values 1 through 6. If you select a lawyer and ascertain her annual income, then the outcome is again a number. We call a rule that assigns a numerical value to each outcome of an experiment a **random variable**.

A **random variable** X is a rule that assigns a numerical value to each outcome in the sample space of an experiment.

A random variable may have only finitely many values, such as the outcome of a roll of a die. Or, its possible values may be infinite but **discrete**, such as the number of times it takes you to roll a 6 if you keep rolling until you get one. Or, the variable may be **continuous**, as we shall see in the last section of this chapter.

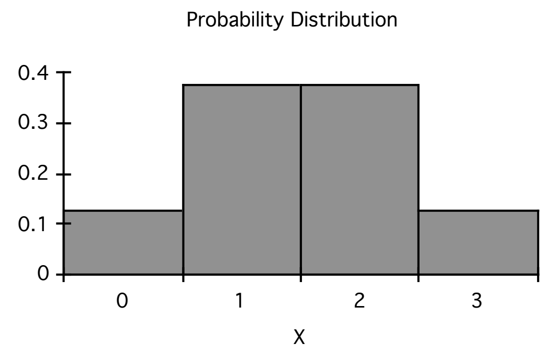
Examples 1
(A) *(discrete finite)* Let X be the number of heads that comes up when a coin is tossed three times. List the value of X for each possible outcome. What are the possible values of X?
(B) *(discrete, infinite)* Book, p. 163) The EPA inspects a factory's pesticide discharge in to a lake once a month by measuring the amount of pesticide in a sample of lake water. If it exceeds the legal maximum, the company is held in violation and fined. Let X be the number of months since the last violation. Also, let Y be the amount of pesticide found in a sample of lake water.
(C) *(discrete finite)* You have purchased \$10,000 worth of stock in a biotech company whose newest arthritis drug is awaiting approval by the FDA If the drug is approved this month, the value of the stock will double by the end of the month. If the drug is rejected this month, the stock's value will decline by 80%, and if no decision is reached this month, its value will decline by 10%. Let X be the value of your investment at the end of this month. List the value of X for each possible outcome.
(D) *(discrete finite)* Survey a group of 50 high school graduates for their SAT scores and let X be the score obtained. When we are given a collection of values of a random variable X we refer to the values as ***X*-scores**. We also call such data **raw data**, as these are the original values on which we often perform statistical analysis. One important purpose of statistics is to interpret the raw data from the sample to get information about the entire population.
(E) Sampling *(continuous)* Survey a group of 50 high school graduates for their SAT scores. Let $\bar{X}$ be the mean score of the sample of 50; let Y be the median. We call $\bar{X}$ and Y **statistics** of the raw scores.

Probability Distribution of a Discrete Random Variable
The **probability distribution** of a discrete random variable is a function which assigns to each possible value x of X the probability (of the event) that $X = x$.

Example 2 Let X being the number of heads that come up when a coin is tossed three times—we obtain

the event that $X = 0$ is $\{TTT\}$	$P(x{=}0) = 1/8 = 0.125$
the event that $X = 1$ is $\{HTT, THT, TTH\}$	$P(x{=}1) = 3/8 = 0.375$
the event that $X = 2$ is $\{HHT, HTH, THH\}$	$P(x{=}2) = 3/8 = 0.375$
the event that $X = 3$ is $\{HHH\}$	$P(x{=}3) = 1/8 = 0.125$
the event that $X = 4$ is Ø	$P(x{=}4) = 0$

Representing the Probability Distribution The most common way to represent the distribution is via a histogram, such as the following for the above example.



Note The probabilities must add to 1 as usual, and be non-negative:

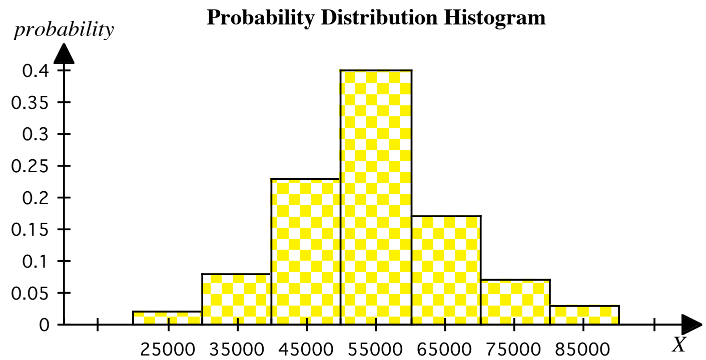
$$P(X = x) \geq 0, \text{ and } \Sigma_x P(X = x) = 1.$$

Example: Sampling Distribution The experiment consists of repeatedly sampling groups of 10 lawyers, and X represents the sample mean income range (if, say, X is between \$30,000 and \$40,000, we take $X = 35{,}000$). Although the actual sampling random variable is continuous, using classes (income brackets) allows us to approximate it by a discrete random variable. Here is a fictitious table, showing the result of 100 surveys.

x	25,000	35,000	45,000	55,000	65,000	75,000	85,000
Frequency (number of groups surveyed)	2	8	23	40	17	7	3

What is the (approximate) sampling distribution? Graph it.

Note In the actual sampling distribution, we think of an arbitrarily large number of groups (of 10 in this case) being surveyed; not just 100.



Mean (Expected Value) Median, and Mode of a Random Variable

We know what the mean of a bunch of x-scores means (and also the median, standard deviation, etc.). If we think of the x-scores as the values of a random variable X, we can also obtain the mean value of X. There are two approaches to measuring this mean:

Method 1 (As before: using the raw x-scores) Measure X a large number of times and take the mean of your set of measurements. For example, look at the lawyer salary example:

x	25,000	35,000	45,000	55,000	65,000	75,000	85,000
Frequency (number of groups surveyed)	2	8	23	40	17	7	3

To get the mean, we add the x-scores as follows:

2 lawyers @ \$25,000 .... \$50,000
8 lawyers @ \$35,000 .... \$280,000
23 lawyers @ \$45,000 .... \$1,035,000
40 lawyers @ \$55,000 .... \$2,200,000
17 lawyers @ \$65,000 .... \$1,105,000
7 lawyers @ \$75,000 .... \$525,000
3 lawyers @ \$85,000 .... \$255,000

\$5,450,000

$$\bar{x} = \frac{\Sigma x_i}{n} = \frac{5,450,000}{100} = \$54,000$$

Method 2 (Using the probability distribution) Since we have multiplied each x-value by its frequency and then divided by the total number, we might as well have just multiplied each value of x by its probability, and then added. This would result in the same answer:

Expected Value, etc. of a Random Variable
If X is a finite random variable taking on values $x_1, x_2, \ldots, x_n$, the **expected value** of X, written μ or $E(X)$, is

$$\mu = E(X) = x_1 \cdot P(X = x_1) + x_2 \cdot P(X = x_2) + \ldots + x_n \cdot P(X = x_n)$$

$$= \Sigma_{all\ x} xP(x) \qquad \text{(book's way of writing this is } P(x)\text{)}$$

The **variance** of the random variable X is

$$\sigma^2 = E((X-\mu)^2) = \Sigma_{all\ x}(x-\mu)^2 P(x)$$

The **standard deviation** of X is then

$$\sigma = \sqrt{\sigma^2}\ .$$

Note We can use Chebyshev's Rule and the Empirical Rule to make inferences about the values of X.

Example In class, we expand the above table to compute σ for the lawyers, and answer the following question: Using Chebyshev, complete the statement: at most 12.5% of lawyers earn less than _____.

Homework
p. 179 #2, 3
p. 182 # 7, 8
p. 186 # 16, 18, 24
Also:
www.FiniteMath.com → Student Web Site → Chapter Review Exercises → Statistics
5, 6, 7

Topic 8
Binomial Random Variable

(Based on Section 5.4)

Bernoulli Trial, Binomial Random Variable

A **Bernoulli**[7] **trial** is an experiment with two possible outcomes, called **success** and **failure**. If the probability of success is p then the probability of failure is $q = 1 - p$.

Tossing a coin three times is an example of a **sequence of independent Bernoulli trials**: a sequence of Bernoulli trials in which the outcomes in any one trial are independent (in the sense of the preceding chapter) of those in any other trial.

A **binomial random variable** is one that counts the number of successes in a sequence of independent Bernoulli trials.

Quick Examples: Binomial Random Variables

1. Roll a die 10 times and let X be the number of times you roll a six.
2. Provide a property with flood insurance for 20 years; let X be the number of years, during the 20-year period, during which the property is flooded[8].
3. 60% of all bond funds will depreciate next year, and you randomly select 4 from a very large number of possible choices; X is the number of bond funds you hold that will depreciate next year. (X is approximately binomial.[9])

Example 1 *Probability Distribution of a Binomial Random Variable*

Suppose that we have a possibly unfair coin, whose probability of heads is p and whose probability of tails is $q = 1-p$.

(a) Let X be the number of heads you get in a sequence of 5 tosses. Find $P(X = 2)$.

(b) Let X be the number of heads you get in a sequence of n tosses. Find $P(X = x)$.

Solution

(a) We are looking for the probability of getting exactly 2 heads in a sequence of 5 tosses. Let's start with a simpler question.

Question What is the probability that we will get the sequence HHTTT?

Answer The probability that the first toss will come up heads is p.
The probability that the second toss will come up heads is also p.
The probability that the third toss will come up tails is q.
The probability that the fourth toss will come up tails is q.
The probability that the fifth toss will come up tails is q.

---

[7] Jakob Bernoulli (1654–1705); one of the pioneers of probability theory.

[8] Assuming that the probability of flooding one year is independent of whether there was flooding in earlier years.

[9] Since the number of bond funds is extremely large, choosing a "loser" (a fund that will depreciate next year) does not significantly deplete the pool of "losers," and so the probability that the next fund you choose will be a "loser," is hardly affected. Hence we can think of X as being a binomial variable.

The probability that the first toss will be heads *and* the second will be heads *and* the third will be tails *and* the fourth will be tails *and* the fifth will be tails equals the probability of the *intersection* of these five events. Since these are independent events, the probability of the intersection is the product of the probabilities, which is

$$p \times p \times q \times q \times q = p^2 q^3.$$

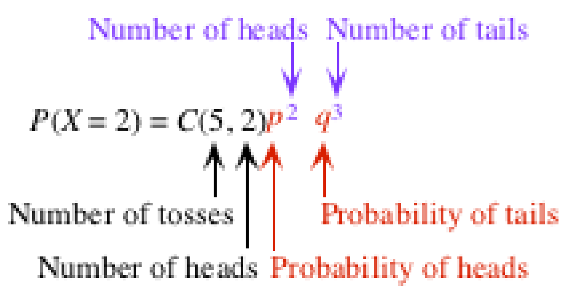
Now HHTTT is only one of several outcomes with two heads and three tails. Two others are HTHTT and TTTHH.

Question How many such outcomes are there all together?
Answer This is the number of “words” with two H's and three T's, and we know from the preceding chapter that the answer is $C(5,2) = 10$.

Each of these 10 outcomes has the same probability: p^2q^3 (why?). Thus, the probability of getting one of these 10 outcomes is the probability of the union of all these (mutually exclusive) events, and we saw in the preceding chapter that this is just the sum of the probabilities. In other words, the probability we are after is

$$\begin{aligned} P(X = 2) &= p^2q^3 + p^2q^3 + \dots + p^2q^3 \qquad C(5,2) \text{ times} \\ &= C(5,2)p^2q^3 \end{aligned}$$

The structure of this formula is as follows.



Notice that we can replace $C(5,2)$ (where 2 is the number of heads), by $C(5,3)$ (where 3 is the number of tails), since $C(5,2) = C(5,3)$.

(b) There is nothing special about 2 in part (a). To get $P(X = x)$ rather than $P(X = 2)$, replace 2 with x:

$$P(X=x) = C(5,x)p^x q^{5-x}.$$

Again, there is nothing special about 5. The general formula for n tosses is

$$P(X=x) = C(n,x)p^x q^{n-x}.$$

Probability Distribution of Binomial Random Variable

If X is the number of successes in a sequence of n independent Bernoulli trials, then

$$P(X = x) = C(n,x)p^x q^{n-x},$$

where

n = number of trials,
p = probability of success, and
q = probability of failure $= 1-p$.

Quick Example

If you roll a fair die 5 times, the probability of throwing exactly 2 sixes is

$$P(X = 2) = C(5,2)\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 = 10\times\frac{1}{36}\times\frac{125}{216} \approx 0.1608.$$

Here, we used $n = 5$ and $p = 1/6$, the probability of rolling a six on one roll of the die.

Examples 2

(Example 4.7 (b)) 100 customers must select a preference among three sodas: your company's new Hyper Cola and the two competitors (you know what they are...). Success, of course, means selecting Hyper Cola. Is this binomial?

(Example 4.7 (a)) You select 3 bonds from 10 recommended ones. Unbeknownst to you, 8 of them will go up, and three are stones. x is the number of winners you select. Is this binomial?

(An Extra One) You select 3 bonds from a large number of recommended ones. Unbeknownst to you, 80% of them will go up, and 30% are stones. x is the number of winners you select. Is this binomial?

Example 3 ***Will You Still Need Me When I'm 64?***

The probability that a randomly chosen person in the US is 65 or older[10] is approximately 0.2.

(a) What is the probability that, in a randomly selected sample of 6 people, exactly 4 of them are 65 or older?

(b) If X is the number of people of age 65 or older in a sample of 6, construct the probability distribution of X and plot its histogram.

(c) Compute $P(X \le 2)$.

(d) Compute $P(X \ge 2)$.

Solution

(a) The experiment is a sequence of Bernoulli trials; in each trial we select a person and ascertain his age. If we take "success" to mean selection of a person 65 or older, the probability distribution is

[10] Source: Carnegie Center, Moscow/*The New York Times,* March 15, 1998, p. 10.

$$P(X = x) = C(n,x)p^x q^{n-x},$$

where n = number of trials = 6,
p = probability of success = 0.2, and
q = probability of failure = 0.8.

So,

$$\begin{aligned} P(X = 4) &= C(6,4)(0.2)^4(0.8)^2 \\ &= 15 \times 0.0016 \times 0.64 = 0.01536 \end{aligned}$$

(b) We have already computed $P(X = 4)$. Here are all the calculations.

$$\begin{aligned} P(X = 0) &= C(6,0)(0.2)^0(0.8)^6 \\ &= 1\times1\times0.262144 = 0.262144 \\ P(X = 1) &= C(6,1)(0.2)^1(0.8)^5 \\ &= 6\times0.2\times0.32768 = 0.393216 \\ P(X = 2) &= C(6,2)(0.2)^2(0.8)^4 \\ &= 15\times0.04\times0.4096 = 0.24576 \\ P(X = 3) &= C(6,3)(0.2)^3(0.8)^3 \\ &= 20\times.008\times0.512 = 0.08192 \\ P(X = 4) &= C(6,4)(0.2)^4(0.8)^2 \\ &= 15\times0.0016\times0.64 = 0.01536 \\ P(X = 5) &= C(6,5)(0.2)^5(0.8)^1 \\ &= 6\times0.00032\times0.8 = 0.001536 \\ P(X = 6) &= C(6,6)(0.2)^6(0.8)^0 \\ &= 1\times0.000064\times1 = 0.000064 \end{aligned}$$

The probability distribution is therefore the following.

x	0	1	2	3	4	5	6
$P(X=x)$	0.262144	0.393216	0.24576	0.08192	0.01536	0.001536	0.000064

Figure 1 shows its histogram.

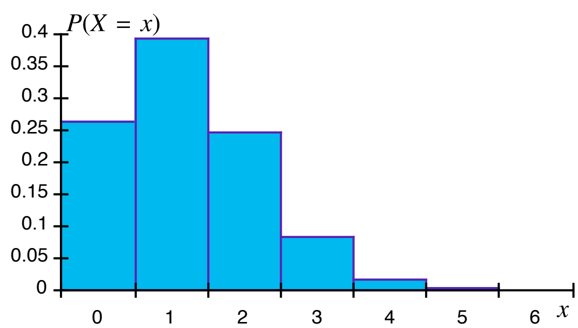


Figure 1

(c) $P(X \le 2)$, the probability that the number of people selected who are at least 65 years old is either 0, 1, or 2, is the union of these events, and is thus the sum of the three probabilities,

$$\begin{aligned} P(X \le 2) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &= 0.262144 + 0.393216 + 0.24576 \\ &= 0.90112. \end{aligned}$$

(d) To compute $P(X \ge 2)$, we *could* compute the sum

$$P(X \ge 2) = P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6),$$

but it is far easier to compute the probability of the complement of the event,

$$P(X < 2) = P(X = 0) + P(X = 1) = 0.262144 + 0.393216 = 0.65536,$$

and then subtract the answer from 1:

$$P(X \ge 2) = 1 - P(X < 2) = 1 - 0.65536 = 0.34464.$$

Spreadsheet

You can generate the binomial distribution as follows in Excel.

	A	B	C	D	E	F	G
1	`0`	`1`	`2`	`3`	`4`	`5`	`6`
2	`=BINOMDIST(A1,6,0.2,0)`	□	□	□	□	□	□

The values of X are shown in Row 1, and the probabilities are computed in Row 2. The arguments of the BINOMDIST function are as follows:

BINOMDIST(x, n, p, Cumulative (0 = no, 1 = yes))

Setting the last coordinate to 0 (as shown) gives $P(X = x)$. Setting it to 1 gives $P(X \le x)$.

Web Site

Follow the path

Web site □ Everything for Finite Math □ Chapter 8
□ Binomial Distribution Utility,

where you can obtain the distribution and also graph the histogram.

Question OK Now, what are the mean and standard deviation of the binomial distribution?

Answer in the box

Mean, Variance, and Standard Deviation of Binomial Random Variable

Mean = $\mu = np$
Variance = $\sigma^2 = npq$
St. Deviation = $\sigma = \sqrt{npq}$

Example 4 (Similar to Example 4.9 and 4.10 in book)
(a) Your manufacturing plant produces 10% defective airbags. If the next 5 airbags are tested, find the probability that three of them are defective.
(b) Compute the probability distribution (that is, find $p(0)$, $p(1)$, ..., $p(5)$), graph them, and locate μ and σ on the graph.
(c) What fraction of the outcomes will fall within 2 standard deviations of the mean?

Answer to part (b)

$$P(X=x) = \binom{5}{x}(0.1)^x(0.9)^{5-x}.$$

Thus: $P(X=0) = \binom{5}{0}(0.1)^0(0.9)^{5-0} = 0.59049$

$$P(X=1) = \binom{5}{1}(0.1)^1(0.9)^{5-1} = 0.32805$$

$$P(X=2) = \binom{5}{2}(0.1)^2(0.9)^{5-2} = 0.07290$$

$$P(X=3) = \binom{5}{3}(0.1)^3(0.9)^{5-3} = 0.0081$$

$$P(X=4) = \binom{5}{4}(0.1)^4(0.9)^{5-4} = 0.00045$$

$$P(X=5) = \binom{5}{5}(0.1)^5(0.9)^{5-5} = 0.00001$$

Answer to (c) We calculate $\mu = 0.5$, and $\sigma = 0.67$. Thus, the interval is
$[\mu - 2\sigma, \mu + 2\sigma] = [-0.84, 1.84]$
These are values of x, and the interval includes $x = 0$ *and* 1. Since
$P(X=0 \text{ or } X=1) = 0.59049 + 0.32805 = 0.9185$, we conclude that at least 91.85% of the outcomes will be within 2 standard deviations of the mean.

Example 5 Use Excel (cumulative probabilities if necessary)
60% of a company's employees favor unionization, and a poll of 20 employees is taken. Use the tables for each of the following.
(a) Find $P(X < 10)$
(b) Find $P(X > 12)$
(c) Find $P(X = 11)$

Homework
p. 197 #25, 30, 32, 34
Also
www.FiniteMath.com → Student Web Site → Chapter Review Exercises → Statistics
1

Topic 9
The Poisson and Hypergeometric Random Variables
(Sections 5.5 & 5.6 in book)

Poisson Random Variable
The discrete random variable X is **Poisson** if X measures the number of successes that occur in a fixed interval of time, and satisfies:

(1) The expected number of successes per unit time does not depend on the time interval.
(2) The event of success in any one interval of time is independent of that in any other interval.

For example, X could be the number of people arriving at a store in a fixed period of time over the lunch-hour, or the number of leaks in 200 miles of pipeline, or the number of cars arriving at a carwash in a given hour. If X is Poisson, we compute $P(X = x)$ as follows:

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

where λ is the expected number of successes for the time interval we are interested in

Example In a bank, people arrive per minute on average. Find the probability that, in a given minute, exactly 2 people will arrive. Also generate the entire probability distribution for X.
Solution
$\lambda = 3$ (given). Thus,

$$P(X = 2) = \frac{e^{-3}3^{2}}{2!} \approx 0.2240$$

To get the entire table we use Excel and obtain, using the formula

```
=EXP(-3)*3^x,FACT(X)
```

x	0	1	2	3	4	5	6	7	8	9	10	11
P(X=x)	0.0498	0.1494	0.224	0.224	0.168	0.1008	0.0504	0.0216	0.0081	0.0027	0.0008	0.0002

Hypergeometric Random Variable
This is similar to the binomial random variable, excpet that, instead of of performing trials with replacement (eg. select a lightbulb, determine whether it is defective, then replace it and repeat the experiment) we do not replace it. This makes the success more likely after a string of failures.

For example, we know that 30 of the 100 workers at the Petit Mall visit your diner for lunch. You choose 10 workers at random; X is the number of workers who visit your diner. (Note that the problem is becomes the same as the binomial distribution for a large propulation, where we can ignore the issue of replacement). If X is hypergeometric, we compute $P(X = x)$ as follows:

If N is the population size,
n is the number of trials,
and r is the total number of successes possible

,then

$$P(X = x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

Example The Gods of Chaos have promised you that you will win on exactly 40 of the next 100 bets at the Happy Hour Casino. However, your luck has not been too good up to this point: you have bet 50 times and have lost 46 times. What are your chances of winning both of the next two bets?

Solution Here N = number of bets left = 100–50 = 50, n = number of trials = 2 and r = number of successes possible = 40–4 = 36 (you have used up 4 of your guaranteed 40 wins). So we can now compute $P(X = 2)$ using the formula.

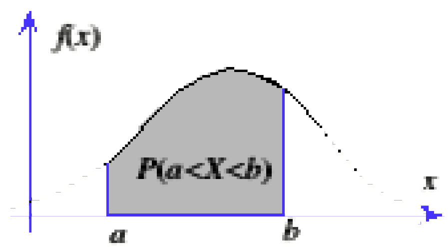
Homework
p. 201 #40, 42
p. 204, #48, 50

Topic 10
Continuous Random Variables: Uniform and Normal
(Based on Sections 6.1-6.2 in the book)

When a random variable is continuous, we use the following to describe the associated probabilities. Note that, in this case, $P(X = x) = 0$. So instead, we will look at probabilities in a range: $P(a < X < b)$.

A **probability density function** (or **probability distribution**) for a continuous random variable X is a function $f(x)$ so that $P(a < X < b)$ is the area under the curve between a and b. Further, we require:

- $f(x) \geq 0$ for every x
- The area under the entire curve = 1 (why?)

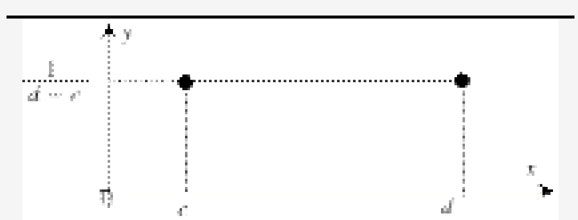


An Example:

The Uniform Distribution The **uniform density function on the interval** $[c,d]$ is given by

$$f(x) = \frac{1}{d-c}.$$

Its graph is a horizontal line (see the figure)



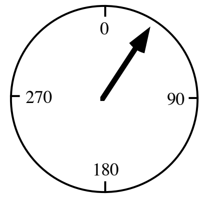
Probabilities are calculated by

$$P(a < X < b) = \frac{b-a}{d-c}.$$

The mean and standard deviation of a uniformly distributed random variable is given by

$$\mu = \frac{c+d}{2} \qquad \sigma = \frac{d-c}{\sqrt{12}}.$$

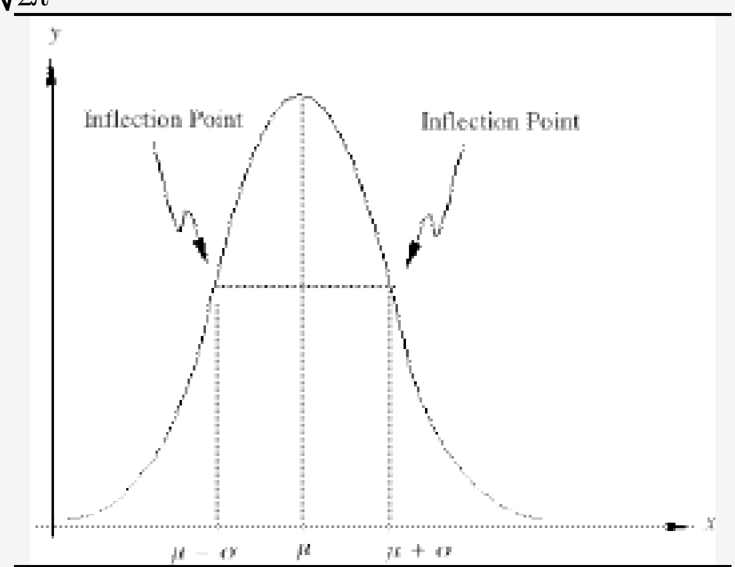
Example 1 *Spinning a Dial* Suppose that you spin the dial shown below so that it comes to rest at a random position. Model this with a suitable distribution, and use it to find the probability that the dial will land somewhere between 5˚ and 300˚.



The Normal Distribution

A **normal density function** is a function of the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$



μ = Mean
σ = standard deviation

The **standard normal distribution** has $\mu = 0$ and $\sigma = 1$. We use Z rather than X to refer to the associated random variable.

Tables The following tables give the probabilities $P(Z \le z)$.

Note Excel formula for this is `=NORMSDIST(z)`

Negative z

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-0.0	0.50000	0.49601	0.49202	0.48803	0.48405	0.48006	0.47608	0.47210	0.46812	0.46414
-0.1	0.46017	0.45620	0.45224	0.44828	0.44433	0.44038	0.43644	0.43251	0.42858	0.42465
-0.2	0.42074	0.41683	0.41294	0.40905	0.40517	0.40129	0.39743	0.39358	0.38974	0.38591
-0.3	0.38209	0.37828	0.37448	0.37070	0.36693	0.36317	0.35942	0.35569	0.35197	0.34827
-0.4	0.34458	0.34090	0.33724	0.33360	0.32997	0.32636	0.32276	0.31918	0.31561	0.31207
-0.5	0.30854	0.30503	0.30153	0.29806	0.29460	0.29116	0.28774	0.28434	0.28096	0.27760
-0.6	0.27425	0.27093	0.26763	0.26435	0.26109	0.25785	0.25463	0.25143	0.24825	0.24510
-0.7	0.24196	0.23885	0.23576	0.23270	0.22965	0.22663	0.22363	0.22065	0.21770	0.21476
-0.8	0.21186	0.20897	0.20611	0.20327	0.20045	0.19766	0.19489	0.19215	0.18943	0.18673
-0.9	0.18406	0.18141	0.17879	0.17619	0.17361	0.17106	0.16853	0.16602	0.16354	0.16109
-1	0.15866	0.15625	0.15386	0.15151	0.14917	0.14686	0.14457	0.14231	0.14007	0.13786
-1.1	0.13567	0.13350	0.13136	0.12924	0.12714	0.12507	0.12302	0.12100	0.11900	0.11702
-1.2	0.11507	0.11314	0.11123	0.10935	0.10749	0.10565	0.10383	0.10204	0.10027	0.09853
-1.3	0.09680	0.09510	0.09342	0.09176	0.09012	0.08851	0.08692	0.08534	0.08379	0.08226
-1.4	0.08076	0.07927	0.07780	0.07636	0.07493	0.07353	0.07215	0.07078	0.06944	0.06811
-1.5	0.06681	0.06552	0.06426	0.06301	0.06178	0.06057	0.05938	0.05821	0.05705	0.05592
-1.6	0.05480	0.05370	0.05262	0.05155	0.05050	0.04947	0.04846	0.04746	0.04648	0.04551
-1.7	0.04457	0.04363	0.04272	0.04182	0.04093	0.04006	0.03920	0.03836	0.03754	0.03673
-1.8	0.03593	0.03515	0.03438	0.03362	0.03288	0.03216	0.03144	0.03074	0.03005	0.02938
-1.9	0.02872	0.02807	0.02743	0.02680	0.02619	0.02559	0.02500	0.02442	0.02385	0.02330
-2	0.02275	0.02222	0.02169	0.02118	0.02068	0.02018	0.01970	0.01923	0.01876	0.01831
-2.1	0.01786	0.01743	0.01700	0.01659	0.01618	0.01578	0.01539	0.01500	0.01463	0.01426
-2.2	0.01390	0.01355	0.01321	0.01287	0.01255	0.01222	0.01191	0.01160	0.01130	0.01101
-2.3	0.01072	0.01044	0.01017	0.00990	0.00964	0.00939	0.00914	0.00889	0.00866	0.00842
-2.4	0.00820	0.00798	0.00776	0.00755	0.00734	0.00714	0.00695	0.00676	0.00657	0.00639
-2.5	0.00621	0.00604	0.00587	0.00570	0.00554	0.00539	0.00523	0.00508	0.00494	0.00480
-2.6	0.00466	0.00453	0.00440	0.00427	0.00415	0.00402	0.00391	0.00379	0.00368	0.00357
-2.7	0.00347	0.00336	0.00326	0.00317	0.00307	0.00298	0.00289	0.00280	0.00272	0.00264
-2.8	0.00256	0.00248	0.00240	0.00233	0.00226	0.00219	0.00212	0.00205	0.00199	0.00193
-2.9	0.00187	0.00181	0.00175	0.00169	0.00164	0.00159	0.00154	0.00149	0.00144	0.00139
-3	0.00135	0.00131	0.00126	0.00122	0.00118	0.00114	0.00111	0.00107	0.00104	0.00100
-3.1	0.00097	0.00094	0.00090	0.00087	0.00084	0.00082	0.00079	0.00076	0.00074	0.00071
-3.2	0.00069	0.00066	0.00064	0.00062	0.00060	0.00058	0.00056	0.00054	0.00052	0.00050
-3.3	0.00048	0.00047	0.00045	0.00043	0.00042	0.00040	0.00039	0.00038	0.00036	0.00035
-3.4	0.00034	0.00032	0.00031	0.00030	0.00029	0.00028	0.00027	0.00026	0.00025	0.00024
-3.5	0.00023	0.00022	0.00022	0.00021	0.00020	0.00019	0.00019	0.00018	0.00017	0.00017
-3.6	0.00016	0.00015	0.00015	0.00014	0.00014	0.00013	0.00013	0.00012	0.00012	0.00011
-3.7	0.00011	0.00010	0.00010	0.00010	0.00009	0.00009	0.00008	0.00008	0.00008	0.00008
-3.8	0.00007	0.00007	0.00007	0.00006	0.00006	0.00006	0.00006	0.00005	0.00005	0.00005
-3.9	0.00005	0.00005	0.00004	0.00004	0.00004	0.00004	0.00004	0.00004	0.00003	0.00003

Positive z

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91308	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997

Example 2 Calculate the following probabilities using the table.

(a) $P(0 < Z < 1.34)$
(b) $P(-1.34 < Z < 1.34)$
(c) $P(-1.23 < Z < 0.44)$
(d) $P(Z > 0.22)$
(e) $P(Z < 0.32)$
(f) $P(|Z| > 1.96)$

Dealing with a Non-Standard Normal Distribution

We use the following important property of normal distributions (seen from the formula). If X is any old normally distributed random variable with mean μ and standard deviation σ, then the random variable Z given by the z-score,

$$z = \frac{x-\mu}{\sigma}$$

has a standard normal distribution..

Example 3 Pressure gauges manufactured by Precision Corp. must be checked for accuracy before being placed on the market. To test a pressure gauge, a worker uses it to measure the pressure of a sample of compressed air known to be at a pressure of exactly 50 pounds per square inch. If the gauge reading is off by more than 1% (0.5 pounds), the gauge is rejected. Assuming that the reading of a pressure gauge under these circumstances is a normal random variable with mean 50 and standard deviation 0.5, find the percentage of gauges rejected.

Solution We are seeking $P(49.5 < X < 50.5)$, where X is a normal random variable with $\mu = 50$ and $\sigma = 0.5$. To do this, convert all the values to z-values:

$$z_1 = \frac{49.5-50}{0.5} = -1,$$

$$z_2 = \frac{50.5-50}{0.5} = 1.$$

Hence the probability is $P(-1 < z < 1) = 0.84134 - 0.15866 = 0.68268$.

Note The following calculations are true for any normal random variable, are very useful to remember:

$P(\mu-\sigma \leq X \leq \mu+\sigma) \approx 0.68268$
$P(\mu-2\sigma \leq X \leq \mu+2\sigma) \approx 0.95450$
$P(\mu-3\sigma \leq X \leq \mu+3\sigma) \approx 0.99730$

Now you see where the empirical rule comes from!

Example 4 An automobile manufacturer advertises an average city gas mileage of 27 mpg, and claims that the standard deviation is 3 mpg. You purchase one of these cars, and get no more than 20 mpg. What can you conclude?

Solution In class, we calculate $P(X < 20) \approx 0.01$. Thus, either:

(a) The claim is correct, and you were in the unlucky 1% group that gets the duds
(b) The claim is wrong—perhaps the standard deviation should be bigger...

Using the Table Backwards: Finding *Z*

Example 5 (based on Example 5.10 in the book) Daily paint production at a manufacturing plant has a mean of 100,000 gals. with a standard deviation of 10,000 gals. Management wants to reward production crews that exceed the 90th percentile. How many gallons of paint does this represent?
Solution We want a value of x_0 such that 90% of production is below that level; that is,

$$P(X \le x_0) = 0.90.$$

First obtain the appropriate z−score:

$$P(Z \le z_0) = 0.90$$

From the table, we find $z_0 \approx 1.28$.

Excel Formula for this: `=NORMSINV(0.9) = 1.28155`

Next, convert this to an x-score:

$$z_0 = \frac{x_0 - \mu}{\sigma}$$

$$1.285 = \frac{x_0 - 100{,}000}{10{,}000}$$

so that $x_0 = 12{,}800 + 100{,}000 = 112{,}800$ gallons.

Technology Notes

Graphing Calculator

Many calculators permit you to calculate the area under the standard normal curve without using a table. On the TI-83, press `[2nd]` VARS to obtain the selection of distribution functions. The first function, `normalpdf`, gives the values of the normal density function (whose graph is the normal curve). The second, `normalcdf`, gives $P(a \le Z \le b)$. For example, to compute $P(0 \le Z \le 2.43)$, enter

```
normalcdf(0, 2.43).
```

To compute $P(-1.37 \le Z \le 2.43)$, enter

```
normalcdf(-1.37, 2.43).
```

Spreadsheet

Spreadsheet programs also come equipped with built-in statistical software that allows you to compute $P(a \le Z \le b)$. For example, to compute $P(0 \le Z \le 2.43)$ in Excel, enter

```
=NORMSDIST(2.43)-NORMSDIST(0)
```

in any vacant cell. To compute $P(-1.37 \le Z \le 2.43)$ directly, enter

```
=NORMSDIST(2.43)-NORMSDIST(-1.37).
```

Web site

Follow the path
Web site → Everything for Finite Math → Chapter 8 → Normal Distribution Utility
where you will find an on-line utility that computes areas under the normal curve to a high accuracy.

Approximating a Binomial Distribution by a Normal Distribution
You might have noticed that the histograms of binomial distributions that we have drawn (for example, those in Figure 1) have a very rough bell shape. In fact, in many cases it is possible to draw a normal curve that closely approximates a given binomial distribution.

If X is the number of successes in a sequence of n independent Bernoulli trials with probability p of success in each trial, and if the range of values of X within three standard deviations of the mean lies entirely within the range 0 to n (the possible values of X), then

$$P(a \le X \le b) \approx P(a-0.5 \le Y \le b+0.5)$$

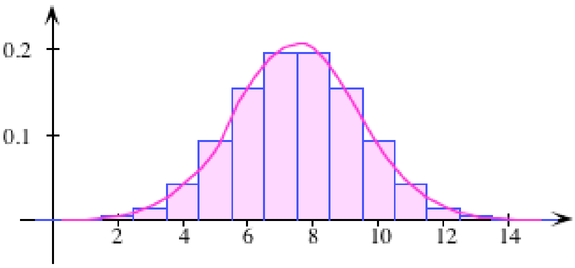
where Y has a normal distribution with the same mean and standard deviation as X; that is,

$$\mu = np \text{ and } \sigma = \sqrt{np(1-p)}\ .$$

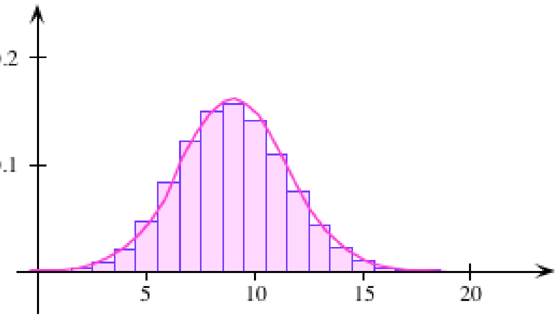
Notes

1. The condition that $0 \le \mu-3\sigma < \mu+3\sigma \le n$ is satisfied if n is sufficiently large and p is not too close to 0 or 1, and ensures that almost all of the normal curve lies in the range 0 to n.
2. In the formula $P(a \le X \le b) \approx P(a-0.5 \le Y \le b+0.5)$ we assume that a and b are integers. The use of $a-0.5$ and $b+0.5$ is called the **continuity correction**. To see that it is necessary, consider what would happen if you wanted to approximate, say, $P(X = 2) = P(2 \le X \le 2)$.

Here are some binomial distributions with their normal approximations superimposed.



$n = 15,\ p = 0.5$



$n = 30,\ p = 0.3$

Example 7 ***Coin Flips***

(a) If you flip a fair coin 100 times, what is the probability of getting more than 55 heads or fewer than 45 heads?

(b) What number of heads (out of 100) would make you suspect that the coin is not fair?

Solution

(a) We are asking for

$$P(X<45 \text{ or } X>55) = 1 - P(45 \le X \le 55).$$

We *could* compute this by calculating

$$C(100,45)(0.5)^{45}(0.5)^{55} + C(100,46)(0.5)^{46}(0.5)^{54} + \cdots,$$

but we can much more easily *approximate* it by looking at a normal distribution with mean $\mu = 50$ and standard deviation $\sigma = \sqrt{100 \cdot 0.5 \cdot 0.5} = 5$. (Notice that three standard deviations above and below the mean is the range 35 to 65, which is well within the range of possible values for X, which is 0 to 100, so the approximation should be a good one.) Let Y have this normal distribution. Then

$$\begin{aligned} P(45 \le X \le 55) &\approx P(44.5 \le Y \le 55.5) \\ &= P(-1.1 \le Z \le 1.1) \\ &= .86433 - .13567 = .72866. \end{aligned}$$

Therefore,

$$P(X<45 \text{ or } X>55) \approx 1 - .72866 = 0.27134.$$

(b) This is a deep question, touching on the question of **statistical significance**: what evidence is strong enough to overturn a reasonable assumption (the assumption that the coin is fair)? Statisticians have developed various sophisticated ways of answering this question, but we can look at one simple test now. Suppose that we tossed a coin 100 times and threw 66 heads. If the coin were fair, then $P(X > 65) \approx P(Y > 65.5) = P(Z > 3.1) \approx 0.001$. This is small enough to raise a reasonable doubt that the coin is fair. However, we should not be too surprised if we threw 56 heads, since we can calculate that $P(X > 55) \approx 0.1357$, which is not such a small probability. As we said, the actual tests of statistical significance are more sophisticated than this, but we shall not go into them.

Homework

p. 217, #6

p. 229, #10, 12, 14, 18, 24

Also:
1. If you roll a die 100 times, what is the probability that you will roll between 15 and 20 ones? (Round your answer to 2 decimal places.)
[0.57]

2. ***Aviation*** The probability of a plane's crashing on a single trip in 1989 was 0.00000165.[11] Find the probability that, in 100,000,000 flights, there will be fewer than 180 crashes.
[0.871]

3. ***Polls*** In a certain political poll, each person polled has a 90% probability of telling his or her real preference. Suppose that 55% of the population really prefer candidate Goode, and 45% prefer candidate Slick. Find first the probability that a person polled will say that he or she prefers Goode, then find the probability that, if 1,000 people are polled, candidate Goode will get more than 52%. □
[Probability that a person will say Goode = 0.54. Probability that Goode polls more than 52% ≈ 0.892.]

[11] Source for this exercise and the following three: National Transportation Safety Board.

Topic 11
Sampling Distributions and Central Limit Theorem
(Based on Section 5.5 in the book)

Follow the path

Web site → Everything for Finite Math → Chapter 8 → Sampling Distributions
for an interactive on-line version of this section.

It is often impossible to measure the mean or standard deviation of an entire population unless the population is small, or we do a nationwide census. The population mean and standard deviation are examples of **population parameters**—descriptive measurements of the entire population. Given the impracticality of measuring population parameters, we instead measure **sample statistics**—descriptive measurements of a sample. Examples of sample statistics are the sample mean, sample median, and sample standard deviation.

Q. OK, so why not use the sample statistic as an estimate of the corresponding population parameter: for instance, why not use the sample mean as an estimate of the population mean?
A. This is exactly what we do to estimate population means and medians (with a slight modification in the case of the standard deviation). However, a sample statistic (such as the sample mean) may be "all over the place," so a further question is: how confident can we be in the sample statistic?

Q. Give me an example.
A. If we cast a fair die and take X to be the uppermost number, we know that the population mean is $\mu = 3.5$, and that the population median is also $m = 3.5$. But if we take a sample of, say, four throws, the mean may be far from 3.5. Here are the results of 5 such samples of 4 throws (we used a random number generator to obtain these samples):

	X_1	X_2	X_3	X_4	$\bar{X}$
Sample 1	6	2	5	6	4.75
Sample 2	2	3	1	6	3
Sample 3	1	1	4	6	3
Sample 4	6	2	2	1	2.75
Sample 5	1	5	1	3	2.5

Notice that none of the five samples gave us the correct mean, and that the mean of the first sample is far from the actual mean.

Q. The table above is interesting: look at the values of the mean X. Their median is 3 and their mean is 3.2. Thus, although the mean of a particular sample may not be a good predictor of the population mean, we get better results if we take the mean of a whole bunch of sample means.

A. You have put your thumb on one of the most important concepts inferential statistics; the values of $\bar{X}$ are values of a random variable (take a sample of 5, and measure the mean), and its probability distribution is called the **sampling distribution** of the sample mean. The above table suggests that the expected value of the sampling distribution of the mean is the same as the population mean, and this turns out to be true.

Sampling Distribution
The **sampling distribution** of a statistic S for samples of size n is defined as follows. The experiment consists of choosing a sample of size n from the population and measuring the statistic S. The sampling distribution is the resulting probability distribution.

Example 1 ***Sampling Distribution***
An unfair coin has a 75% chance of landing heads-up. Let $X = 1$ if it lands heads-up, and $X = 0$ if it lands tails-up. Find the sampling distribution of the mean $\bar{X}$ for sample size 3.
Solution The experiment consists of tossing a coin 3 times and measuring the sample mean $\bar{X}$. The following table shows the collection of all possible outcomes (samples) and associated sample mean.

Outcome	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
Probability	$\frac{27}{64}$	$\frac{9}{64}$	$\frac{9}{64}$	$\frac{3}{64}$	$\frac{9}{64}$	$\frac{3}{64}$	$\frac{3}{64}$	$\frac{1}{64}$
$\bar{X}$	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0

The values of $\bar{X}$ are 0, 1/3, 2/3, and 1. The desired sampling distribution is its probability distribution, shown below.

$\bar{X}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$P(\bar{X} = \bar{x})$	$\frac{1}{64}$	$\frac{9}{64}$	$\frac{27}{64}$	$\frac{27}{64}$

Note For this small sample size, the distribution of the sample mean is a binomial distribution. The Central Limit Theorem will tell us that, for large sample sizes, it must look more and more like a normal distribution.

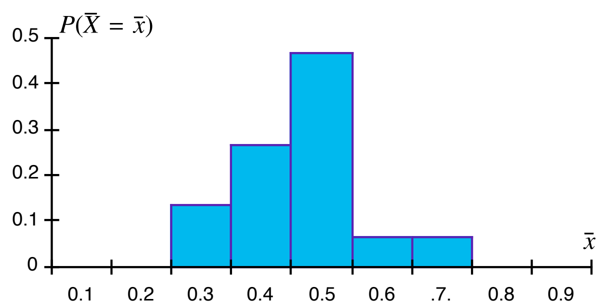
Example 2
Look at Example 6.1 on p. 242-243, which has a larger list of possible samples.

Example 3 ***Sampling from a Uniform Distribution***
The example with which we began this section involved taking five samples from a finite uniform random variable. Here is a sequence of 15 samples with $n = 6$ taken from a continuous uniform distribution with domain [0, 1]. Give the relative frequency histogram for $\bar{X}$ using measurement classes 0.05-0.14, 0.15-0.24, 0.25-0.34, 0.35-0.44, ...

	X_1	X_2	X_3	X_4	X_5	X_6	$\bar{X}$
Sample 1	0.136	0.397	0.278	0.029	0.810	0.496	0.358
Sample 2	0.918	0.455	0.482	0.148	0.494	0.440	0.49
Sample 3	0.076	0.868	0.626	0.104	0.902	0.425	0.5
Sample 4	0.374	0.772	0.748	0.415	0.043	0.612	0.494
Sample 5	0.855	0.005	0.203	0.950	0.526	0.246	0.464
Sample 6	0.147	0.579	0.790	0.906	0.766	0.998	0.698
Sample 7	0.303	0.159	0.990	0.055	0.031	0.715	0.376
Sample 8	0.143	0.362	0.093	0.047	0.767	0.769	0.364
Sample 9	0.523	0.232	0.296	0.096	0.983	0.423	0.426
Sample 10	0.566	0.598	0.253	0.943	0.757	0.588	0.618
Sample 11	0.096	0.375	0.062	0.230	0.437	0.434	0.272
Sample 12	0.887	0.952	0.019	0.242	0.637	0.358	0.516
Sample 13	0.208	0.099	0.802	0.157	0.956	0.818	0.507
Sample 14	0.096	0.375	0.062	0.230	0.725	0.434	0.32
Sample 15	0.887	0.952	0.019	0.242	0.393	0.358	0.475

Solution If we use the measurement classes above, we obtain the following frequency table (omitted classes have frequency 0) and histogram (using the center values of each class).

Class	0.25-0.34	0.35-0.44	0.45-0.54	0.55-0.64	0.65-0.74
Frequency	2	4	7	1	1
Relative Frequency	2/15	4/15	7/15	1/15	1/15



Note The histogram gives a "sample" of the actual sampling distribution; we can't produce the whole sampling distribution in the above manner, since there are, in principle, infinitely many possible samples.

Unbiased Estimates of Population Parameters

Suppose we want to estimate the population mean from a sample of 100. We could use the sample mean, or perhaps the sample median, as such an estimate. Such an estimate is called a **point estimator**. Suppose, for instance, that we want to use the sample median as a point estimator of the population mean. How accurate is it?

First of all, there are going to be lots of different medians corresponding to the different samples of 100. If we knew the sampling distribution of the sample median with $n = 100$, we could compute the *expected value* (mean) of this sampling distribution. That is, we can compute the expected value of the sample median. If it equals the population mean, we would say that the sample median is an **unbiased estimator** of the population mean. Otherwise, we say that it is a **biased estimator** with bias equal to the difference between the expected value of the estimator and the value of the population parameter.

Further, in order to obtain a more accurate estimate of the population parameter, we should use a sample statistic whose standard deviation (the standard deviation of its sampling distribution) is as small as possible. In this way, the statistic of a single sample is more likely to be close to the expected value.

Example 4 Refer to Example 1: X is the number of heads when we toss an unfair coin (with a 75% chance of heads coming up). That is, $X = 1$ if it's a head and $X = 0$ if it's a tail. Determine whether the sample mean is an unbiased estimator of the population mean.

Solution

We need to compare the population mean with the expected value of the sampling distribution of the sample means.

Step 1 *Compute the population mean* μ.

This means we must compute the average number of heads that comes up when a coin is tossed (not three times—that is the sample size we used—but once). But, the expected value of X is given by

$$\mu = \Sigma x P(X=x) = 0(0.25) + 1(0.75) = 0.75.$$

Step 2 *Compute the expected value of the sampling distribution of the sample mean.*

To do this, we need the sampling distribution of the sample mean, and we already calculated that: the sampling distribution of $\bar{X}$ was found to be

$\bar{X}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$P(\bar{X} = \bar{x})$	$\frac{1}{64}$	$\frac{9}{64}$	$\frac{27}{64}$	$\frac{27}{64}$

We now compute its expected value in the usual way:

$\bar{X}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$P(\bar{X} = \bar{x})$	$\frac{1}{64}$	$\frac{9}{64}$	$\frac{27}{64}$	$\frac{27}{64}$
$\bar{X}P(\bar{X} = \bar{x})$	0	$\frac{9}{192}$	$\frac{54}{192}$	$\frac{27}{64}$

Adding up the numbers in the bottom row gives the expected value of the sampling distribution:

$$E(\bar{X}) = \frac{144}{192} = \frac{3}{4} \; !$$

Since this is the same as the population mean, the estimator is unbiased.

Note The following results can be proved (but are apparently not mentioned in the text!)
1. The sample mean is *always* an unbiased estimator of the population mean, regardless of the distribution or the sample size!
2. The sample standard deviation (recall that it uses a different formula from the population standard deviation) is *always* an unbiased estimator of the population standard deviation, again regardless of the distribution of the sample size! That is why we used $n-1$ instead of n in the formula for sample standard deviation; if we used the same formula as for the population standard deviation, it would have been a biased estimator.

Properties of the Sampling Distribution
1. The mean of the sampling distribution = mean of the sampled population:

$$\mu_{\bar{X}} = \mu$$

2. The standard deviation of the sampling distribution[12]

$$= \frac{\text{Standard deviation of sampled population}}{\text{Square root of sample size}}$$

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} .$$ (See footnote.[13])

3. If the population distribution is normal, then so is the sampling distribution of $\bar{X}$.
4. The Central Limit Theorem If the population distribution is not necessarily normal, and has mean μ and standard deviation σ, then, for sufficiently large[14] n, the sampling distribution of $\bar{X}$ is approximately normal, with mean

$$\mu_{\bar{X}} = \mu$$

and standard deviation

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} .$$

(See the figure on p. 273 of the text.)

Example 5 (Based on Example 6.8 of *Statistics for Business and Economics 8th Ed* by McClave, Benson, and Sicich, Prentice Hall, 2001)
A battery manufacturer claims that the lifespan of the batteries produced has a mean of 54 months and a st. deviation of 6 months. Your consumer advocacy group tests 50 of them. What is the probability that it finds a mean lifetime of less than 52 months?

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[12] This is nothing to do with the sampling distribution of the standard deviation!
[13] Actually, this result assumes an infinie population (or a "very large" one. In general, for a population of finite size N, we must multiply the formula for $\sigma_{\bar{X}}$ by a factor $\sqrt{(N-n)/(N-1)}$.
[14] How large? In practive, if $n \geq 30$, provided the population distribution is not "extremely skewed."

Answer In symbols, we are seeking $P(\bar{X} \le 52)$. Now, $\bar{X}$ is approximately normally distributed by the CLT, and has a mean of $\mu = 54$ and a standard deviation of $\sigma_{\bar{X}} = 6/\sqrt{50}$ ≈ 0.85 months. To find the required probability, we need to convert to z-scores:

$$z = \frac{\bar{x} - \mu_{\bar{X}}}{\sigma_{\bar{X}}} = \frac{52 - 54}{0.85} \approx -2.35$$

Therefore,

$$P(\bar{X} \le 52) = P(Z \le -2.35) = .00939$$

Thus, the probability of this happening is 0.00939, or approximately 0.94%. Thus, we can be 99.06% certain that this won't happen (if they are right!).

Exercises

p. 263 # 19, 26, and also the on-line exercises at

Web site → Everything for Finite Math → Chapter 8 → Sampling Distributions

Topic 12
Confidence Interval for a Population Mean
(Based on Sections 8.1, 8.2, 8.3 in the book)

Large Samples
Suppose we have calculated the mean $\bar{x}$ of a large sample ($n \geq 30$) of a random variable X, and we get 120. We would like to say something like:

> "We can be 95.44% certain that the population mean is 120 ±___."

To make things easier, let us assume we know the population standard deviation σ. Then, since the sampling distribution is approximately normal with mean μ and standard deviation $\sigma/\sqrt{n}$. Looking at the standard normal tables, we find that

$$P(-2 \leq Z \leq 2) \approx 2(0.4772) = 0.9544.$$

In other words:

> **There is a 95.44% probability that $\bar{X}$ will be within 2 standard deviations of the (unknown) population mean.**

Put another way:

> **There is a 95.44% probability that the (unknown) population mean will be within 2 standard deviations of $\bar{X}$.**

In other words:

> **We can be 95.44% certain that μ is 120 ± $2\sigma_{\bar{X}}$.**

Question: What does being 99.54% certain mean?
Answer: It means that if we repeated this experiment many times: measure $\bar{X}$ and form the interval $\bar{x} \pm 2\sigma_{\bar{X}}$, then 95.44% of the time, the interval we formed would contain the true population mean μ. In other words, if we repeated this experiment 100 times (calculating the interval in this way) we would be right 95.44 times on average.

Question: What if we want to be, say, 90% certain instead?
Answer: Look at what we did above backwards, where this time, we don't know the range:

$$P(-z_{0.05} \leq Z \leq z_{0.05}) = 0.90,$$

Where $z_{0.05}$ is the unknown number of standard deviations. This becomes

$$P(0 \leq Z \leq z_{0.05}) = 0.45.$$

So, we look up the table and find $z_{0.05} \approx 1.645$. Thus, there is a 90% probability that μ lies within 0.45 standard deviations of the sample mean $\bar{X}$.

Question: Why did we call that value $z_{0.05}$.

Answer: It is the value of Z such that $P(0 \le Z \le z_{0.05}) = 0.45$. In other words, $z_{0.05}$ is the number such that

$$P(Z \ge z_{0.05}) = 0.05,$$

That is, it is the number (measured in standard deviations) such that the area of the upper half of the tail of the Z-distribution is 0.05.

Here is the usual convention. We let α be such that $(1-\alpha)$ is the desired confidence. For instance, here

$$1-\alpha = 0.9,$$

so $\alpha = 0.10$

for 90% confidence. Then, the z-value we want is $z_{0.05} = z_{\alpha/2}$.

Common Values of $z_{\alpha/2}$ (so we don't have to go to the tables every time)

Confidence Level: $(1-\alpha)$	α	$\alpha/2$	$z_{\alpha/2}$
0.90	0.10	0.05	1.645
0.95	0.05	0.025	1.96
0.99	0.01	0.005	2.575

Now we can put everything together and get a "method box:"

How to find the $(1-\alpha)$ Confidence Interval for the Population Mean μ

(1) If we know the population standard deviation σ, then we can be $100(1-\alpha)\%$ certain that μ is in the interval

$$\bar{x} \pm z_{\alpha/2}\sigma/\sqrt{n}\,.$$

(2) If σ is not known, we can estimate it as s, the sample standard deviation.

Example 1 (Computing a Confidence Interval)

You are an airline executive, and are trying to decide whether to increase the carrier size for a particular flight from LA to NY at a certain time.† 225 flight records are randomly selected, giving a sample mean of $\bar{x} = 11.6$ unoccupied seats with $s = 4.1$ seats. Estimate a 90% confidence interval for the mean number of unoccupied seats.

Solution A confidence level of 90% gives $\alpha/2 = 0.05$, and $z_{\alpha/2} = 1.645$ standard deviations. Thus, the 90% confidence interval is

$$\bar{x} \pm z_{\alpha/2}\sigma/\sqrt{n} = 11.6 \pm 1.645(4.1)/\sqrt{225} \approx 11.6 \pm 0.45,$$

or $[11.15, 12.05]$.

† Californian residents are fleeing to New York in droves now that Arnold Schwartzenegger has become governor.

Example 2 (Estimating the Sample Size)
Referring to the above example, if I wanted to estimate the average number of seats to within ±0.5 with a confidence of 99%, how large a sample would I need?
Solution This time, we know the confidence interval is

$$\pm \frac{z_{\alpha/2}\sigma}{\sqrt{n}} = \pm 0.5.$$

For 99%, $\alpha/2 = 0.005$, and $z_{\alpha/2} = 2.575$. Thus we have

$$\frac{2.575\times 4.1}{\sqrt{n}} = 0.5,$$

giving $\sqrt{n} = \dfrac{2.575\times 4.1}{0.5} \approx 21.115,$

so $\quad n \approx 445.8.$

Thus, we would require a sample of size at least 446 to ensure this interval with a 99% confidence.

Small Samples

When we use a small sample, there are two problems:
(1) We can no longer assume that $\bar{x}$ is normally distributed, since the Central Limit Theorem no longer applies.
(2) The estimate $\sigma \approx s$ is no longer reliable.

We address (1) by making the assumption that the population is approximately normal, so that we no longer need the Central Limit Theorem. For (2), there is still a problem since when we were calculating $z_{\alpha/2}$, we computed its value for large samples using the normal distribution of the statistic

$$z = \frac{\bar{x}-\mu}{\sigma_{\bar{x}}} = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}.$$

(recall that $\bar{x}$ is normally distributed here.) If we use the sample standard deviation instead, we get

$$t = \frac{\bar{x}-\mu}{s/\sqrt{n}}$$

instead, it is no longer normally distributed (even if the original population is; note that we are taking the quotient of two random variables here, and we cannot expect the result to be a normal variable). The sampling distribution of this statistic, called the ***t*-statistic for $(n-1)$ degrees of freedom**[15] is also bell-shaped, but a little broader than the normal distribution, and depends upon the value of n; the smaller n, the broader the distribution. Its values are given in the book (front inside cover).

[15] Discovered by W.S. Gosset in 1908

Summary: Dealing with Small Samples

1. If we know the population standard deviation, we can use the z-statistic as usual (making the assumption that the original distribution is approximately normal), and the confidence interval is

$$\bar{x} \pm z_{\alpha/2}\sigma/\sqrt{n}$$

as usual.

2. If all we know is the sample standard deviation, we must use the t-statistic for $n-1$ degrees of freedom ($\nu = n-1$), and the confidence interval is

$$\bar{x} \pm t_{\alpha/2}s/\sqrt{n}\,.$$

Assumption in both cases: The population distribution is approximately normal.

Example 3

The lifetime of an inkjet printer head (in millions of characters printed until failure) for 15 different inkjet heads is 1.24, with a sample standard deviation of 0.19.

(a) Form the 99% confidence interval.

(b) If the population standard deviation is also 0.19, is the resulting confidence interval wider or narrower?

Solution

(a) Number of degrees of freedom is $\nu = n-1 = 14$

$$t_{0.005} = 2.977.$$

Thus, the interval is

$$\bar{x} \pm t_{\alpha/2}s/\sqrt{n} = 1.24 \pm \frac{2.977\times 0.19}{\sqrt{15}} \approx 1.24\pm 0.146,$$

or $[1.094, 1.386]$ million characters.

(b) If we used z instead, we would obtain

$$\bar{x} \pm z_{\alpha/2}\sigma/\sqrt{n} = 1.24 \pm \frac{2.575\times 0.19}{\sqrt{15}} \approx 1.24\pm 0.126,$$

a much narrower interval.

Example 4 (Estimating Sample Size Again)

If we wanted to estimate the lifetime of the above inkjet head to within ±0.1 million characters with 99% certainty, how large should our sample size be?

Solution It is impossible to solve for n in the t−distribution, since it also depends on both s and n. Instead, we go with the z-distribution, and hope for the best:

$$\frac{z_{\alpha/2}\sigma}{\sqrt{n}} = 0.1.$$

For 99%, $\alpha/2 = 0.005$, and $z_{\alpha/2} = 2.575$. Thus we have

$$\frac{2.575\times 0.19}{\sqrt{n}} = 0.1,$$

giving $\sqrt{n} = \dfrac{2.575\times 0.19}{0.1} \approx 4.8925,$

so $n \approx 24.$

Homework
p. 293 # 2, 8
p. 300, #18, 22
p. 304 # 30

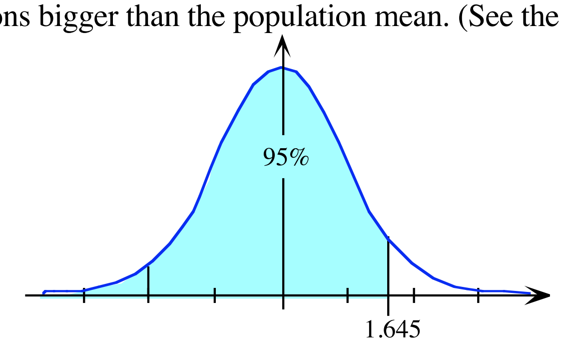
Topic 13
Introduction to Hypothesis Testing
(Based on Sections 9.1-9.3 in the book)

We have seen that the sample mean can be used to estimate the population mean, if the latter is unknown. More precisely, when we used confidence intervals, we were making an inference about the value of the population mean. In this section, we will *test a hypothesis about the value of the population mean.*

For example, you might want to test whether the vitamin tablets made by your company have more than 120 mg vitamin C. In such a scenario, you know that the population mean is supposed to be > 120, and the question you ask is this: Can I be "95% confident" (whatever that means) that the average vitamin C content in my pills is > 120 mg?

We use two hypotheses:
H_0: The hypothesis that the pills fail to meet the required standard; that is, $\mu \le 120$. This is called the **null hypothesis** (customarily taken to be the "status quo" hypothesis; we will assume that the pills have too little until we obtain enough evidence to reject this assumption.)
H_a: The **alternative, or research hypothesis**; that is, the hypothesis that the experiment was designed to establish: that $\mu > 120$.

Q How do we determine whether to reject the null hypothesis H_0? That is, how can I be confident that μ is above 120?
A To simplify things, let us talk in terms of the standard normal distribution and the number of standard deviations from the mean. We know that 95% of the sample means will be ≤ 1.645 standard deviations bigger than the population mean. (See the figure.)



Put another way, *if the population mean is 0*, then 95% of the readings will be less than 1.645. Thus, if the population mean is 0 (or less), the probability of getting a sample mean greater than 1.645 is < 5%. In terms of conditional probability,

$$P(\bar{z} > 1.645 \mid \mu \le 0) < 0.05.$$

i.e., $$P(\bar{z} > 1.645 \mid H_0 \text{ true}) < 0.05.$$

Now suppose I have the following decision rule:

Rule R: If $\bar{z}$ is greater than 1.645, I will reject the null hypothesis.

Then, the above formula translates to:

P(Rule R tells me to reject H_0 | H_0 is true) < 0.05.

Rejecting H_0 (using the rule) when in fact it is true is called a **Type I error.** (Accepting the null hypothesis when it is false is called a **Type II error**.) Thus,

P(Type 1 error) < 0.05.

Interpretation of the 95% Confidence Level for Hypothesis Testing **1.** The probability of rejecting the null hypothesis (using Rule R) *when it is true* is less than 5%. Equivalently, **2.** In 95% of the cases where the null hypothesis is true, our procedure will not result in our (wrongly) rejecting it. In other words, the 95% confidence is a confidence in the *procedure* (Rule R).

Note: This does not mean that, if we reject the null hypothesis, the probability that it is true is < 0.05. (In other words, we **cannot** be 95% certain that the null hypothesis is false; ie, that the vitamin C content is > 120 mg.) The probability that the null hypothesis is true is

P(H_0 is true | Rule R tells me to reject it)
$\neq P$(Rule R tells me to reject H_0 | H_0 is true) $= \alpha$

How confident *can* I be that H_0 is false if Rule R tells me to reject it? That's hard to say, as we would need to compute P(H_0 is false |Rule R tells me to reject H_0). What we *can* be 95% confident about is that we have not made a Type I error: that is we can be 95% certain that *if H_0 was true,* we would not reject it.

Here is an example: Suppose H_0 is "Football player Hugo Huge has not been taking steroids" and my steroids test has only a 5% false positive rate. That is, if Hugo is not using steroids, then there is only a 5% chance that the test will be positive. Now, suppose Hugo Huge's test comes up positive. If I am the coach, and my policy is to reject everyone who comes up positive (regardless of whether or not they are actually using steroids) then Hugo Huge will be rejected. In this context, the probability that he actually uses steroids need not be 95%. For instance, if only 1 in a million athletes actually used steroids, then the vast majority of those who, like Hugo, test positive (5%, or 50,000 in each million) are *not* using steroids!
Thus, *I cannot be 95% confident that H_0 is false* (i.e., that Hugo is using steroids) at all. All I can be sure of, is that, if Hugo was *not* using steroids, then there would only be a 5% chance that the test came up positive. In this context, a Type I error would be rejecting Hugo if he is not using steroids, and I can be 95% certain that I am not making a Type I error in rejecting Hugo (even though I can **not** be 95% certain that Hugo is using steroids.) Put another way, I can be 95% confident that my policy (Rule R) is reliable in the sense that I don't get a false positive, but I cannot be 95% certain that it is reliable *if it comes up positive*.

So, if $\bar{z} > 1.645$, I would therefore reject the null hypothesis, and I can be 95% confident that I am not making a Type I error. The possible values of $\bar{z}$ that would cause us to reject H_0 is the area to the right of the vertical line in the diagram above. We call this the **rejection region**.

Now we can go back to the vitamin *C* pills. To convert everything to the normal variable, we use the "test statistic"

$$z = \frac{\bar{x}-120}{\sigma_{\bar{x}}} = \frac{\bar{x}-120}{\sigma/\sqrt{n}},$$

where *n* is the sample size. Then, if $z > 1.645$, we reject H_0. Simple as that.

Example 1

(a) Your measurements on a sample of 35 vitamin C pills give an average of 120.4 mg with a sample standard deviation of 1.2. How can I be certain (with 95%, or $\alpha = 0.05$[16]) that the average dose in all my pills is > 120 mg?

Solution The test statistic is $z = \frac{\bar{x}-120}{\sigma_{\bar{x}}} = \frac{\bar{x}-120}{\sigma/\sqrt{n}} = \frac{120.4-120}{1.2/\sqrt{35}} = 1.9720.$

Since this is > 1.645, I reject the null hypothesis with a confidence of 95%.

(b) You realized that you had made a mistake; the mean was actually 120.3. Are you still confident that the mean is above 120?

Solution The test statistic is $z = \frac{\bar{x}-120}{\sigma/\sqrt{n}} = \frac{120.3-120}{1.2/\sqrt{35}} = 1.4790.$

Thus, I cannot reject the null hypothesis. In other words, the sample evidence is not sufficient to reject the null hypothesis.

Q Does that mean I should *accept* the null hypothesis (that is, reject the alternative hypothesis)?

A Suppose we invented a new rule:

Rule T: Accept the null hypothesis if $\bar{z} \le 1.645$.

Accepting the null hypothesis (using Rule T) when it is false would be called a **Type II error.** The probability of a Type II error is (going back to the standard distribution)

$\beta = $ P(Rule T tells me to reject H_a | H_a is true) = $P(\bar{z} \le 1.645 \mid \mu > 0)$.

In general, this probability is difficult to estimate, and it depends on exactly how big μ actually is. (You need to supply a value of μ in order to say anything—see Section 8.6 in the book.)

Summary:

- To decide what H_0 and H_a should be, follow the following guideline: H_a is the hypothesis you are deciding whether to accept. (You will never accept H_0.) This, H_a is

[16] α is the probability of making a Type I error. The probability of making a Type II error is called β.

the hypothesis you are testing, and H_0 is the "status quo:" the hypothesis that is assumed true until you have found evidence to the contrary.

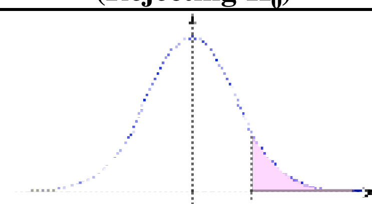
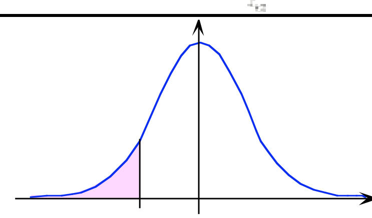
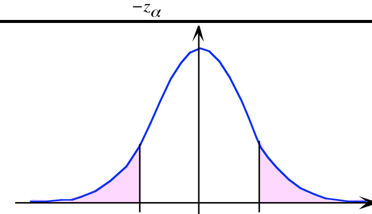
- To test a hypothesis with **level of significance α**, take the test statistic and compute the value of z_α for the rejection region.
- If your value of z is in the rejection region, you must, by Rule R, reject the null hypothesis.
- If your value of z is not in the rejection region, you cannot reject the null hypothesis (but that does not mean you must *accept* it!)

Applying Hypothesis Testing to Large Samples
So far, we have had H_a of the form "$\mu > 1200$."

Note Although the corresponding null hypothesis is H_0: $\mu \le 1200$, some textbooks take H_0 to be $\mu = 1200$, since if we reject the hypothesis that $\mu \le 1200$, then we can reject the hypothesis that $\mu = 1200$ as well.

In the case we looked at, the rejection region was to the right of z_α in the normal distribution. This is one of three possibilities:

Three Types of Hypothesis Testing

Type	Hypotheses	Rejection Region (Rejecting H_0)
One-tailed; upper	H_0: $\mu \le \mu_0$ H_a: $\mu > \mu_0$	
One-tailed; lower	H_0: $\mu \ge \mu_0$ H_a: $\mu < \mu_0$	
Two-tailed	H_0: $\mu = \mu_0$ H_a: $\mu \ne \mu_0$	

Significance Level:	α	$\alpha/2$	Z_α	$Z_{\alpha/2}$
90%	0.10	0.05	1.28	1.645
95%	0.05	0.025	1.645	1.96
99%	0.01	0.005	2.326	2.575

Example 1
You want to test whether the cereal boxes made by your plant conform to the requirement that they contain 12 oz cereal. You wish to test at the 99% significance level, and you sample 100 boxes, finding $\bar{x} = 11.85$, $s = 0.5$. Do your cereal boxes meet the standard?
Solution
Take H_0 to be $\mu = 12$ (two-tailed).
The test statistic is

$$z = \frac{\bar{x}-12}{s/\sqrt{n}} = \frac{11.85-12}{0.5/10} = \frac{-0.15}{0.05} = -3.$$

The value of $z_{\alpha/2}$ for the test is

$$z_{.005} = 2.575.$$

Referring to the diagram, we see that z is in the rejection region, so we reject H_0. In other words, your cereal *does not* conform to the requirement; the boxes are being under-filled.

Example 2 Your muffler factory claims to manufacture mufflers with a lifespan of more than 10,000 miles of usage. A consumer group tests this claim at the 95% significance level, and finds that a sample of 64 mufflers have a mean lifespan of 10,002 miles, with a standard deviation of 10 miles. Test the following alternate hypotheses using this data, and interpret the results:

(a) Manufacturer's hypothesis: H_a: $\mu > 10{,}000$
(b) Consumer group's hypothesis: H_a: $\mu < 10{,}000$
(c) If the manufacturer wanted to state that the survey proved their claim to be true, what should $\bar{x}$ have been?
(d) If the consumer group wanted to state that the survey proved the manufacturer's claim to be false, what should $\bar{x}$ have been?

Solution
The test statistic is

$$z = \frac{\bar{x}-10{,}000}{s/\sqrt{n}} = \frac{10{,}002-10{,}000}{10/8} = \frac{2}{1.25} = 1.6$$

$$z_\alpha = z_{.05} = 1.645$$

(a) The rejection region is the area to the right of 1.645. Since z is below this, we cannot reject H_0, so we cannot reject the hypothesis that $\mu \le 10{,}000$. Thus, the manufacturer cannot claim that the lifespan of the mufflers is above 10,000 miles.
(b) The rejection region is the area to the left of -1.645. Since z is positive, it is not in the rejection region. Thus, we cannot reject the hypothesis that $\mu \ge 10{,}000$. In other words, the consumer group cannot state that the manufacturer's claim is wrong.
(c) To validate the manufacturer's claim, z should have been in the rejection region. That is,

$$z = \frac{\bar{x}-10{,}000}{1.25} > 1.645.$$

This gives

$$\bar{x} - 10{,}000 > 2.05625,$$

so $\bar{x} > 10{,}002.06.$

(c) To validate the consumer group's claim, z would have to have been in their rejection region: to the left of -1.645. Thus,

$$z = \frac{\bar{x}-10{,}000}{1.25} < -1.645.$$

This gives

$$\bar{x} - 10{,}000 < -2.05625,$$

so $\quad \bar{x} < 9{,}997.94.$

Homework:
p. 327 #2, 4
p. 329 #5, 8
p. 337 # 10, 18
p. 345 # 25, 30

Topic 14
Observed Significance & Small Samples

Q Instead of selecting an α first and then testing a hypothesis, can we first test the hypothesis and then get a value for the appropriate α/ For instance, suppose you test H_0 with a right-tailed test (H_a: $\mu > \mu_0$) and you get a test statistic of $z = 2.12$. The question is, at what significance level can you reject H_0?

A Since the probability of getting 2.12 or above can be calculated to be $0.5-0.4830 = 0.0170$, you conclude that there is only a 1.7% chance of having gotten that score or higher. So we say that we can reject H_0 with an **observed significance level,** or ***p*-value** of

$$p\text{-value} = P(Z \geq 2.12) = 0.0170.$$

In other words, we can reject H_0 with a significance level of $p = 0.0170$, (or 98.3%). Since this value is small, we say that the test result is "statistically very significant."

Calculating the p-value

First, calculate the test statistic as usual.

1. If the test is one-tailed, take p to be the area under the standard normal curve beyond the observed value of z in the same direction as the alternative hypothesis.
2. If the test is two-tailed, the p-value is *twice* the area beyond the observed value of z.

Note

Some packages like Excel only give the p-value for the two-tailed test. Thus, to get the p-value for the associated one-tailed test, given the 2-tailed p-value, divide it by 2.

Example 1

(Based on Examples 8.1 and 8.2 in the book)

You want to test whether the cereal boxes made by your plant conform to the requirement that they contain 12 oz cereal. You sample 100 boxes, finding $\bar{x} = 11.85$, $s = 0.5$. At what level of significance do your cereal boxes meet the standard?

Solution

Take H_0 to be $\mu = 12$ (two-tailed).

The test statistic is

$$z = \frac{\bar{x}-12}{s/\sqrt{n}} = \frac{11.85-12}{0.5/10} = \frac{-0.15}{0.05} = -3.$$

Since this is two-tailed, we calculate twice the area beyond $z = -3$. This is found to be

$$2(.00135) = 0.00270.$$

Thus, $p = 0.0027$ (corresponding to 99.73%.

Thus, there is a large statistical significance that H_0 should be rejected.

Q Suppose I am given a significance level α to test beforehand. Then should I bother with the p-value at all?

A Yes. Calculate p anyway. If p is less than, or approximately equal to α, then you can safely reject H_0. If not, you cannot do so.

Example 2 (Cereal boxes again)
You want to test whether the cereal boxes made by your plant conform to the requirement that they contain 12 oz cereal. You sample another 100 boxes, this time finding $\bar{x} = 11.88,\ s = 0.5$. Do the boxes meet the 12 oz standard at the 99% level of confidence?

Answer This time, the test statistic is

$$z = \frac{\bar{x}-12}{s/\sqrt{n}} = \frac{11.88-12}{0.5/10} = \frac{-0.12}{0.05} = -2.4.$$

So,

$$p = 2P(|Z| \geq 2.4) = 2(0.5 - 0.4918) = 2(0.0082) = 0.0164.$$

Also, α (for 99%) is 0.01.

Since these values are "approximately equal" you can still reject H_o at the 99% level, so the cereal boxes are still not up to par...

Small Sample Hypothesis Testing

This is essentially the same as the testing for large samples, except for the following adjustments:

1. If the sample size is small and the population distribution is approximately normal, we can still use the sample standard deviation in our calculations, provided we use t_α instead of z_α when forming the rejection region. For consistency, we refer the test statistic as t rather than z.
2. When calculating p, we need to use the t-table "backwards" and we can only get an approximate answer without statistical software packages.

Example 3
The emission (in parts of carbon per million) of 10 engines is found to be:

15.6 16.2 22.5 20.5 16.4 19.4 16.6 17.9 12.7 13.9

The mean emission must, according to regulations, be $\mu < 20$ parts per million. Test this at a significance level of $\alpha = 0.01$.
Answer
We have H_0: $\mu \geq 20$, and H_a: $\mu < 20$.
Computations reveal that $\bar{x} = 17.17,\ s = 2.98$. Thus, the t-statistic is

$$t = \frac{\bar{x}-20}{s/\sqrt{n}} = \frac{17.17-20}{2.98/\sqrt{10}} = -3.00$$

For the t-table, the number of degrees of freedom is $\nu = n-1 = 9$, so for the one-tailed test, we use

$t_{0.01} = 2.821.$ `=TINV(0.02,9)`

Since t falls in the rejection region, we can reject H_0 at this level, so the auto manufacturer can claim that the engines meet the standard of less than 20 parts per million at the 99% significance level.

Q What about p for this test?
A Since $t = -3.0$ we look at the $\nu = 9$ row of the t-table to find the value closest to 3.0, and we find $p \approx 0.0075$. In other words, we can also reject H_0 at the 99.25% level if we wanted to.

Homework
Finish up Exercises on Previous Section, and
p. 350 # 34, 38

Topic 15
Confidence Intervals and Hypothesis Testing for the Proportion
(Sections 8.4 and 9.6 in the text)

Suppose you are interested in the percentage of the population that uses Wishy-Washy detergent. Your market research people conduct a telephone survey of 200 domestic workers and find that 32 of them, or 16% of them use Wishy-Washy.

Q1 What is a 95% confidence interval for the proportion of the whole population that uses Wishy-Washy?

To answer the question, let us assume that the proportion p of the population actually uses the product. (We express p as a decimal; $0 \le p \le 1$). This is the population parameter. We can phrase the scenario in terms of a binomial random variable:

Experiment: Select a domestic worker at random; $X = 1$ if the worker uses Wishy-Washy, and 0 if not. The probability of "success" (using the detergent) is p. With X defined like this, if we choose a sample of size n and calculate $\bar{x}$, then

$$\bar{x} = \frac{\text{Number of people using Wishy-Washy}}{n} = \frac{32}{200} = 0.16 \text{ in our example.}$$

This is an *estimate* of the population parameter p, and we call it $\hat{p}$. Thus, $\hat{p} = 0.16$. Similarly, the population mean μ of X is just p, the actual proportion of the population that uses Wishy-Washy. In this way, finding a confidence interval for p amounts to nothing more than finding a confidence interval for a population mean. All we need are:

1. An estimate of the population standard deviation
2. A way of knowing that the sample size is large enough.

1. Estimating Standard Deviation

Since we are repeatedly performing a single Bernoulli trial (selecting a domestic worker and asking about Wishy-Washy), the standard deviation is given by

$$\sigma = \sqrt{p(1-p)}\ .$$

Thus, by the Central Limit Theorem, the standard deviation of $\bar{x} = \hat{p}$ for large samples is approximately

$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}},$$

where n is the same size (200 in our example). However, we don't know what p actually is, so we use the approximation

$$\sigma_{\hat{p}} \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

for the standard deviation.

2. Deciding Whether the Normal Approximation Applies

The usual test for whether a normal approximation is valid involves knowing the actual value of p. Instead, we use the following alternative test, which is similar to the one we used earlier:

The normal approximation is good if the interval $\hat{p} \pm 3\sigma_{\hat{p}}$ *does not include* 0 *or* 1.

Putting all this together gives us the following:

Confidence Interval for Population Proportion *p* (Large Sample)

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

where $\hat{p} = \frac{x}{n}$.

Acid Test: The formula is valid if the interval $\hat{p} \pm 3\sigma_{\hat{p}}$ does not include 0 or 1, where

$$\sigma_{\hat{p}} \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Example 1 Let us find a 95% CI for the actual percentage of people who use Wishy-Washy (done in class)

Q OK Fine, but even when n is large, the Acid Test may fail if p is very close to 0 or 1 (e.g. as in the chance of being killed in an auto accident). When that happens, we use the "Wilson" estimator of p instead of $\hat{p}$. This is given by

Adjusted CI for Population Proportion *p* (Small Samples or Extreme *p*)

$$\tilde{p} = \frac{x+2}{n+4}$$

with the following CI:

$$\tilde{p} \pm z_{\alpha/2}\sqrt{\frac{\tilde{p}(1-\tilde{p})}{n+4}}$$

Example 2
In a sample of 200 Americans, 3 were victims of violent crime . Estimate the true proportion of Americans who were victims of violent crime using a 95% CI.

Q2 OK Now I know how to find CIs for population proportions. What about doing some hypothesis testing?

A Since we already have everything we need, we can give the following procedure:

Testing a Hypothesis about *p* for a Large Sample
Assumption: The experiment is binomial

H_0: either $p = p_0,\ p \geq p_0,\ or\ p \leq p_0$
H_a: either: $p \neq p_0,\ p < p_0,\ or\ p > p_0$ as usual

Large Sample Test: The interval $p_0 \pm 3\sigma_{p_0}$ does not include 0 or 1.

Test Statistic: $$z = \frac{\hat{p} - p_0}{\sigma_{p_0}}$$

where $$\sigma_{p_0} \approx \sqrt{\frac{p_0(1-p_0)}{n}}$$

Example 3
That battery manufacturer must show that fewer than 5% of its batteries are defective. It tests 300 and finds 10 defective ones. Can the manufacturer rest assured that the number of defectives is less than 5%. (Test at the 95% significance level).

Exercises 13
p. 309 #31, 38
p. 357 #44, 46

Table 1: **Normal Distribution Probabilities:** $P(Z \le z)$

Negative z

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-0.0	0.50000	0.49601	0.49202	0.48803	0.48405	0.48006	0.47608	0.47210	0.46812	0.46414
-0.1	0.46017	0.45620	0.45224	0.44828	0.44433	0.44038	0.43644	0.43251	0.42858	0.42465
-0.2	0.42074	0.41683	0.41294	0.40905	0.40517	0.40129	0.39743	0.39358	0.38974	0.38591
-0.3	0.38209	0.37828	0.37448	0.37070	0.36693	0.36317	0.35942	0.35569	0.35197	0.34827
-0.4	0.34458	0.34090	0.33724	0.33360	0.32997	0.32636	0.32276	0.31918	0.31561	0.31207
-0.5	0.30854	0.30503	0.30153	0.29806	0.29460	0.29116	0.28774	0.28434	0.28096	0.27760
-0.6	0.27425	0.27093	0.26763	0.26435	0.26109	0.25785	0.25463	0.25143	0.24825	0.24510
-0.7	0.24196	0.23885	0.23576	0.23270	0.22965	0.22663	0.22363	0.22065	0.21770	0.21476
-0.8	0.21186	0.20897	0.20611	0.20327	0.20045	0.19766	0.19489	0.19215	0.18943	0.18673
-0.9	0.18406	0.18141	0.17879	0.17619	0.17361	0.17106	0.16853	0.16602	0.16354	0.16109
-1	0.15866	0.15625	0.15386	0.15151	0.14917	0.14686	0.14457	0.14231	0.14007	0.13786
-1.1	0.13567	0.13350	0.13136	0.12924	0.12714	0.12507	0.12302	0.12100	0.11900	0.11702
-1.2	0.11507	0.11314	0.11123	0.10935	0.10749	0.10565	0.10383	0.10204	0.10027	0.09853
-1.3	0.09680	0.09510	0.09342	0.09176	0.09012	0.08851	0.08692	0.08534	0.08379	0.08226
-1.4	0.08076	0.07927	0.07780	0.07636	0.07493	0.07353	0.07215	0.07078	0.06944	0.06811
-1.5	0.06681	0.06552	0.06426	0.06301	0.06178	0.06057	0.05938	0.05821	0.05705	0.05592
-1.6	0.05480	0.05370	0.05262	0.05155	0.05050	0.04947	0.04846	0.04746	0.04648	0.04551
-1.7	0.04457	0.04363	0.04272	0.04182	0.04093	0.04006	0.03920	0.03836	0.03754	0.03673
-1.8	0.03593	0.03515	0.03438	0.03362	0.03288	0.03216	0.03144	0.03074	0.03005	0.02938
-1.9	0.02872	0.02807	0.02743	0.02680	0.02619	0.02559	0.02500	0.02442	0.02385	0.02330
-2	0.02275	0.02222	0.02169	0.02118	0.02068	0.02018	0.01970	0.01923	0.01876	0.01831
-2.1	0.01786	0.01743	0.01700	0.01659	0.01618	0.01578	0.01539	0.01500	0.01463	0.01426
-2.2	0.01390	0.01355	0.01321	0.01287	0.01255	0.01222	0.01191	0.01160	0.01130	0.01101
-2.3	0.01072	0.01044	0.01017	0.00990	0.00964	0.00939	0.00914	0.00889	0.00866	0.00842
-2.4	0.00820	0.00798	0.00776	0.00755	0.00734	0.00714	0.00695	0.00676	0.00657	0.00639
-2.5	0.00621	0.00604	0.00587	0.00570	0.00554	0.00539	0.00523	0.00508	0.00494	0.00480
-2.6	0.00466	0.00453	0.00440	0.00427	0.00415	0.00402	0.00391	0.00379	0.00368	0.00357
-2.7	0.00347	0.00336	0.00326	0.00317	0.00307	0.00298	0.00289	0.00280	0.00272	0.00264
-2.8	0.00256	0.00248	0.00240	0.00233	0.00226	0.00219	0.00212	0.00205	0.00199	0.00193
-2.9	0.00187	0.00181	0.00175	0.00169	0.00164	0.00159	0.00154	0.00149	0.00144	0.00139
-3	0.00135	0.00131	0.00126	0.00122	0.00118	0.00114	0.00111	0.00107	0.00104	0.00100
-3.1	0.00097	0.00094	0.00090	0.00087	0.00084	0.00082	0.00079	0.00076	0.00074	0.00071
-3.2	0.00069	0.00066	0.00064	0.00062	0.00060	0.00058	0.00056	0.00054	0.00052	0.00050
-3.3	0.00048	0.00047	0.00045	0.00043	0.00042	0.00040	0.00039	0.00038	0.00036	0.00035
-3.4	0.00034	0.00032	0.00031	0.00030	0.00029	0.00028	0.00027	0.00026	0.00025	0.00024
-3.5	0.00023	0.00022	0.00022	0.00021	0.00020	0.00019	0.00019	0.00018	0.00017	0.00017
-3.6	0.00016	0.00015	0.00015	0.00014	0.00014	0.00013	0.00013	0.00012	0.00012	0.00011
-3.7	0.00011	0.00010	0.00010	0.00010	0.00009	0.00009	0.00008	0.00008	0.00008	0.00008
-3.8	0.00007	0.00007	0.00007	0.00006	0.00006	0.00006	0.00006	0.00005	0.00005	0.00005
-3.9	0.00005	0.00005	0.00004	0.00004	0.00004	0.00004	0.00004	0.00004	0.00003	0.00003

Positive *z*

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91308	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997

t-Statistic **Excel:** `=TINV(2*□,df)`

df	0.1	0.05	0.01	0.025	0.005
1	3.078	6.314	31.821	12.706	63.656
2	1.886	2.920	6.965	4.303	9.925
3	1.638	2.353	4.541	3.182	5.841
4	1.533	2.132	3.747	2.776	4.604
5	1.476	2.015	3.365	2.571	4.032
6	1.440	1.943	3.143	2.447	3.707
7	1.415	1.895	2.998	2.365	3.499
8	1.397	1.860	2.896	2.306	3.355
9	1.383	1.833	2.821	2.262	3.250
10	1.372	1.812	2.764	2.228	3.169
11	1.363	1.796	2.718	2.201	3.106
12	1.356	1.782	2.681	2.179	3.055
13	1.350	1.771	2.650	2.160	3.012
14	1.345	1.761	2.624	2.145	2.977
15	1.341	1.753	2.602	2.131	2.947
16	1.337	1.746	2.583	2.120	2.921
17	1.333	1.740	2.567	2.110	2.898
18	1.330	1.734	2.552	2.101	2.878
19	1.328	1.729	2.539	2.093	2.861
20	1.325	1.725	2.528	2.086	2.845
21	1.323	1.721	2.518	2.080	2.831
22	1.321	1.717	2.508	2.074	2.819
23	1.319	1.714	2.500	2.069	2.807
24	1.318	1.711	2.492	2.064	2.797
25	1.316	1.708	2.485	2.060	2.787
26	1.315	1.706	2.479	2.056	2.779
27	1.314	1.703	2.473	2.052	2.771
28	1.313	1.701	2.467	2.048	2.763
29	1.311	1.699	2.462	2.045	2.756
30	1.310	1.697	2.457	2.042	2.750
31	1.309	1.696	2.453	2.040	2.744
32	1.309	1.694	2.449	2.037	2.738
33	1.308	1.692	2.445	2.035	2.733
34	1.307	1.691	2.441	2.032	2.728
35	1.306	1.690	2.438	2.030	2.724
40	1.303	1.684	2.423	2.021	2.704
45	1.301	1.679	2.412	2.014	2.690
50	1.299	1.676	2.403	2.009	2.678
75	1.293	1.665	2.377	1.992	2.643
100	1.290	1.660	2.364	1.984	2.626
200	1.286	1.653	2.345	1.972	2.601
1000	1.282	1.646	2.330	1.962	2.581