

## Taylor Polynomials

**Question** A broker offers you bonds at 90% of their face value. When you cash them in later at their full face value, what percentage profit will you make?

**Answer** The answer is *not* 10%. In fact, since you will get \$1 for each \$0.90 you invest, you get back

$$\frac{1}{0.90} \approx \$1.11$$

for each dollar that you invest, giving you an 11% profit.

Now think about this problem in another way. You are being offered a discount of  $x$  (in this case,  $x = 0.10$ ). To find your profit you need to compute

$$f(x) = \frac{1}{1-x} - 1.$$

We would like to find an easier-to-compute approximation to  $f(x)$ , to see why  $f(0.90) \approx 0.11$ , and to see quickly what would happen for other discount rates. In fact, we shall see very soon that a good approximation is

$$f(x) \approx x + x^2,$$

if  $x$  is close to 0. When  $x = 0.10$ , this gives

$$f(x) \approx 0.10 + 0.01 = 0.11.$$

What we are claiming then is that  $f(x)$  can be approximated by a *polynomial*. This is nice because polynomials are the easiest functions to compute and manipulate. The more general question behind all of this is:

**Question** How can any function  $f(x)$  be approximated, for values of  $x$  close to some point  $a$ , by a polynomial?

**Answer** One very useful answer is given by the following theorem. Its proof is really quite easy from the column method of integration by parts, and we shall give this proof at the end of the section.

**Taylor's Theorem with Remainder**

If  $f(x)$  is  $(n+1)$ -times differentiable, then

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!} + R_n$$

where

$$R_n = \int_a^x \frac{f^{(n+1)}(t)(x-t)^n}{n!} dt$$

Here, as usual,  $n! = n(n-1)(n-2) \cdots 2 \cdot 1$  is called  **$n$  factorial**.

The polynomial

$$T(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!}$$

is called  **$n$ th Taylor polynomial** of  $f$  around  $a$ , or the  **$n$ th Taylor expansion** of  $f$  around  $a$ .<sup>†</sup> The last term  $R_n$  is called the **remainder**, or **error term**, and we shall see how it allows us to estimate how far the polynomial is from equaling the function  $f$ .

For now, let us ignore the remainder, and concentrate on the Taylor polynomials. In fact, it often happens that the remainders  $R_n$  become smaller and smaller, approaching zero, as  $n$  gets large. When this occurs,  $f(x)$  is approximately equal to its  $n$ th Taylor sum;

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!}$$

For this reason, we often call the Taylor sum the **Taylor approximation** of degree  $n$ . The larger  $n$  is, the better the approximation.

**Example 1 Taylor Polynomial**

Expand  $f(x) = \frac{1}{1-x} - 1$  around  $a = 0$ , to get linear, quadratic and cubic approximations.

**Solution** We will be using the formula for the  $n$ th Taylor sum with  $a = 0$ . Thus, we need to calculate  $f(0)$ ,  $f'(0)$ ,  $f''(0)$ , ... Thus let us list  $f(x)$  and all its derivatives, evaluating at  $a = 0$  each time.

$$f(x) = \frac{1}{1-x} - 1 \quad \text{so } f(0) = 0$$

<sup>†</sup> When  $a = 0$ , this is also called the **MacLaurin polynomial**, particularly by Scotsmen.

$$f'(x) = \frac{1}{(1-x)^2} \quad \text{so } f'(0) = 1$$

$$f''(x) = \frac{2}{(1-x)^2} \quad \text{so } f''(0) = 2$$

$$f'''(x) = \frac{6}{(1-x)^2} \quad \text{so } f'''(0) = 6$$

We can now write down the approximations one-at-a-time.

**Degree 1 approximation:**  $f(x) \approx f(a) + f'(a)(x-a)$ .

Substituting, we get:

$$\frac{1}{1-x} - 1 \approx 0 + (1)x = x.$$

So, the linear approximation to  $f(x)$  is the polynomial  $x$ .

**Degree 2 approximation:**  $f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!}$

Substituting, we get:

$$\frac{1}{1-x} - 1 \approx 0 + (1)x + \frac{2x^2}{2!} = x + x^2.$$

Thus the quadratic approximation to  $f(x)$  is the polynomial  $x + x^2$ .

**Degree 3 approximation:**  $f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!}$

Substituting, we get:

$$\frac{1}{1-x} - 1 \approx 0 + (1)x + \frac{2x^2}{2!} + \frac{6x^3}{3!} = x + x^2 + x^3.$$

Thus the cubic approximation to  $f(x)$  is the polynomial  $x + x^2 + x^3$ .

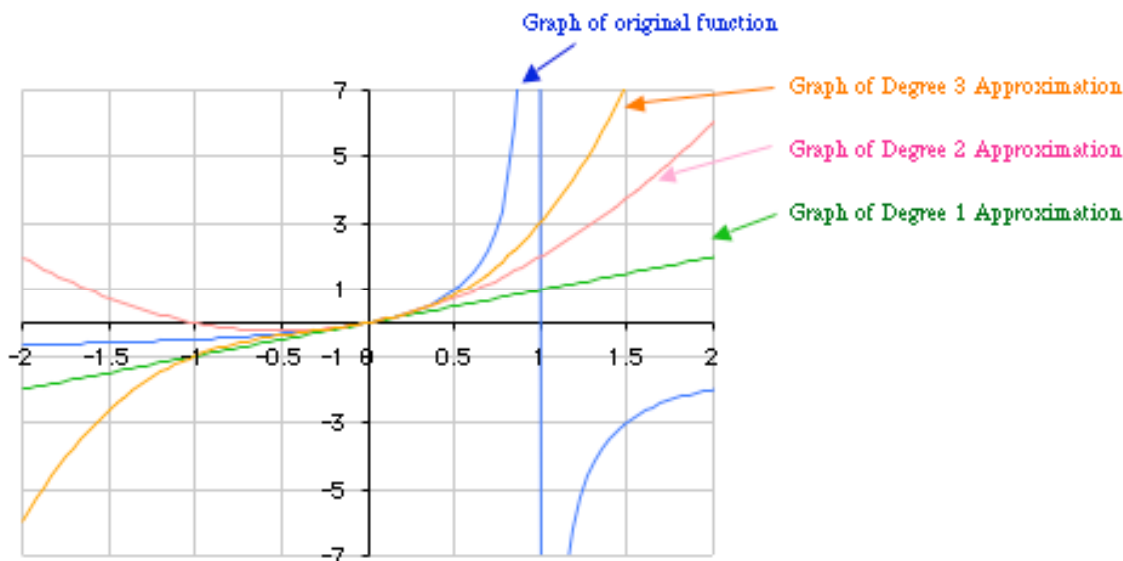
**Before we go on...** Looking at the answers, you probably suspect—correctly—that the fourth degree approximation to  $f(x)$  is  $x + x^2 + x^3 + x^4$ , and so on for the higher degree approximations. If  $x$  happens to have magnitude smaller than 1, these approximations get more and more accurate, and

$$\frac{1}{1-x} - 1 \text{ is exactly equal to } x + x^2 + x^3 + x^4 + \dots + x^n + \dots,$$

a never-ending sum. This infinite sum is called the **Taylor series** of the function  $f$  we are talking about, and tells us something quite interesting: whereas we can

approximate  $f(x)$  by polynomials of larger and larger degree,  $f(x)$  itself is not exactly a polynomial, but rather an “infinite polynomial,” (called a *power series*).

Figure 1 shows the graphs of these approximations, together with the graph of  $f(x) = \frac{1}{1-x} - 1$ .



**Figure 1**

Notice how the graphs of the successive approximations get closer and closer to the curve near  $x = 0$ , but nowhere near the curve when  $x > 1$ .

### Example 2 Taylor Polynomial for $e^x$

Find a 5th degree polynomial approximation for  $e^x$  by expanding the function about zero.

#### Solution

Once again, we have  $a = 0$ , and we need to list all the derivatives up to the fifth, evaluating at 0 as we go.

$$\begin{array}{ll}
 f(x) = e^x & \text{so } f(0) = 1 \\
 f'(x) = e^x & \text{so } f'(0) = 1 \\
 f''(x) = e^x & \text{so } f''(0) = 1 \\
 f'''(x) = e^x & \text{so } f'''(0) = 1 \\
 f^{(4)}(x) = e^x & \text{so } f^{(4)}(0) = 1 \\
 f^{(5)}(x) = e^x & \text{so } f^{(5)}(0) = 1
 \end{array}$$

Since the formula for the fifth degree approximation to  $f(x)$  is

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!} + \frac{f^{(5)}(0)x^5}{5!},$$

we get

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

as our desired degree 5 approximation.

**Before we go on...** This approximation turns out to be quite accurate for small values of  $x$ —even as large as 1. For instance, if we take  $x = 1$ , the left-hand side is 2.7182..., while the right-hand side is 2.71666..., an accuracy to within about 0.002. It turns out (and we shall see why shortly) that the error terms approach zero, no matter what the value of  $x$ , so that

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$$

In particular,

$$e = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} + \cdots$$

This is in fact how many calculators compute  $e^x$  and  $e$ .

If you have a graphing calculator or graphing software, you are urged to graph several approximations, and compare their graphs with that of  $e^x$ .

### Example 3 Taylor Polynomial for $\ln x$

Find the 5th Taylor polynomial for  $f(x) = \ln x$  around 1.

**Solution** This time,  $a = 1$ .\* We need five derivatives of  $f$ :

$$\begin{array}{ll} f(x) = \ln x & \text{so } f(1) = 0 \\ f'(x) = \frac{1}{x} & \text{so } f'(1) = 1 \\ f''(x) = -\frac{1}{x^2} & \text{so } f''(1) = -1 \\ f'''(x) = \frac{2}{x^3} & \text{so } f'''(1) = 2 \\ f^{(4)}(x) = -\frac{6}{x^4} & \text{so } f^{(4)}(1) = -6 \end{array}$$

\* We can't very well take  $a = 0$ . (Why?)

$$f^{(5)}(x) = \frac{24}{x^5} \quad \text{so } f^{(5)}(x) = 24$$

The 5th Taylor polynomial must be

$$\begin{aligned} \ln x &\approx 0 + (1)(x-1) - \frac{(1)(x-1)^2}{2} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!} + \frac{24(x-1)^5}{5!} \\ &\approx (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} \end{aligned}$$

**Before we go on...** If  $|x| < 1$ , then the remainders go to zero as  $n$  gets large, so that we can also represent  $\ln x$  as an infinite polynomial:

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} + \dots$$

We've been making lots of claims about the remainders becoming smaller and smaller as  $n$  gets large, so it is about time that we took a careful look at just what exactly is going on with these remainders. Recall that the remainder is given by the formula

$$R_n = \int_a^x \frac{f^{(n+1)}(t)(x-t)^n}{n!} dt .$$

This formula gives the exact error when  $f(x)$  is approximated by the  $n$ th Taylor sum. The problem is that it is often too cumbersome to evaluate as it stands; what we're going to do is get rid of this formula completely, and replace it by something more manageable. Our philosophy is the following: rather than try to get the exact error, we shall judiciously *overestimate* the magnitude of the error using a far simpler formula.

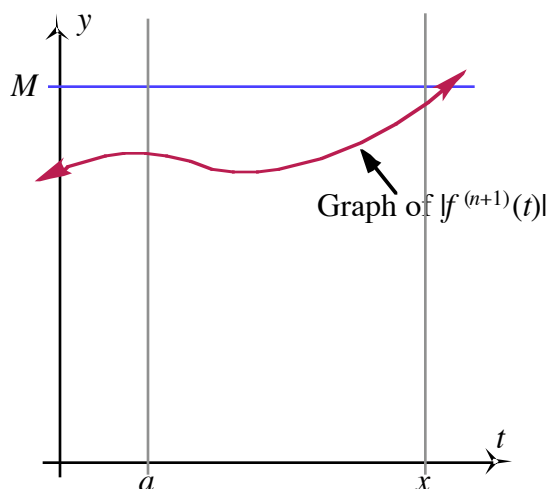
**Question** What's the point of overestimating the error?

**Answer** Look at it this way. The value of  $e$  is 2.718281828... If you say that  $e = 2.718$  to within  $\pm 0.0003$ , you are estimating the error to be at most  $\pm 0.0003$  when  $e$  is replaced by 2.7178. In other words, 0.0003 is an *overestimation* of the actual error; the actual error is 0.000281828... By overestimating the error, you are making a correct claim about the value of  $e$ . If, instead, you were to *underestimate* the error, and claim, for instance, that  $e = 2.718$  to within  $\pm 0.0002$ , you would be dead wrong!<sup>†</sup>

**Question** Fine. Now how do we overestimate the magnitude of  $R_n$ ?

**Answer** First, we look at the magnitude of the  $(n+1)$ st derivative of  $f(t)$  as  $t$  varies between  $a$  and  $x$ , and overestimate that by a single number  $M$ . (See Figure 2.)

<sup>†</sup> and the bridges you design may collapse as a result!



$M$  is an overestimation of the value of  $|f^{(n+1)}(t)|$  for  $t$  between  $a$  and  $x$ .

**Figure 2**

We can then use the overestimation  $M$  to overestimate the error term  $R_n$ :

### Estimation of Error Term

If  $f$  is approximated by the  $n$ th Taylor expansion about  $x = a$ , then the error term  $R_n$  is

no more than  $\frac{M(x-a)^{n+1}}{(n+1)!}$ , or

$$R_n \leq \frac{M(x-a)^{n+1}}{(n+1)!}$$

where  $M$  is an overestimation of the value of  $|f^{(n+1)}(t)|$  for  $t$  between  $a$  and  $x$ .

### Quick Example

Let  $f(x) = \ln x$ ,  $a = 1$ ,  $x = 2$ , and  $n = 3$ . Then  $f^{(n+1)}(t) = f^{(4)}(t) = -\frac{6}{t^4}$ . For values of  $t \geq$

1, its magnitude is 6 or smaller, so we can take  $M = 6$  for  $x = 2$ . If we expand  $f(x)$  about  $a = 1$ , the third degree Taylor polynomial

$$T(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}.$$

See Example 3

$$T(2) = (2-1) - \frac{(2-1)^2}{2} + \frac{(2-1)^3}{3} = \frac{5}{6}$$

is accurate to within

$$\frac{M(x-a)^{n+1}}{(n+1)!} = \frac{6(2-1)^4}{4!} = \frac{1}{4}$$

Thus,  $\ln 2 \approx \frac{5}{6}$  to within  $\pm \frac{1}{4}$ .

**Example 4 Estimating the Error**

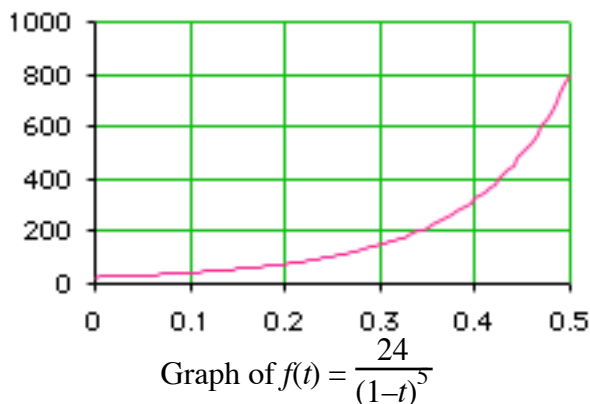
Expand  $f(x) = \frac{1}{(1-x)} - 1$  around  $x = 0$  to get a cubic approximation, and estimate the error if we approximate  $f(x)$  by this cubic for  $x$  between 0 and  $\frac{1}{2}$ .

**Solution** We already saw in Example 1 that the cubic approximation to  $f(x)$  is

$$T(x) = x + x^2 + x^3.$$

To estimate the error, we need to calculate  $M$ . For this, we need to look at the magnitude of the  $(n+1) = 4$ th derivative of  $f(t)$ , which is  $\frac{24}{(1-t)^5}$ . Now  $t$  varies between 0 and  $x$ , which is anywhere between 0 and  $\frac{1}{2}$ . In other words,  $t$  varies between 0 and  $\frac{1}{2}$ .

We now ask ourselves: What is the biggest  $\frac{24}{(1-t)^5}$  can be if  $t$  is between 0 and  $\frac{1}{2}$ ? If you look at the graph of this function of  $t$  (plot a few points or use a graphing calculator) you will see that it increases with  $t$  from 0 to  $\frac{1}{2}$ . (Figure 3).



**Figure 3**

In other words, it is largest when  $t = \frac{1}{2}$ . Its value there is  $\frac{24}{(1-\frac{1}{2})^5} = 24 \times 32 = 768$ . In

other words, we can take  $M = 768$  (or even 800, if we prefer—see the Figure—as long as we don't underestimate it!). Thus the error is

$$R_n \leq \frac{M(x-a)^{n+1}}{(n+1)!}$$



$$R_I \leq \frac{768x^{(3+1)}}{(3+1)!} = \frac{768x^4}{4!} = 32x^4.$$

Thus, for example, if we take  $x = \frac{1}{3}$ , then  $f(\frac{1}{3}) \approx \frac{1}{3} + (\frac{1}{3})^2 + (\frac{1}{3})^3$  to within  $32(\frac{1}{3})^4 = 0.395$

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### Example 5 Approximating e

Calculate the error when e is approximated by  $1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}$ .

**Solution** The given polynomial is the 4th Taylor approximation of  $e^x$  with  $a = 0$  and  $x = 1$  (see Example 2). To get  $M$ , we look at the fifth derivative of  $e^t$ , which is again  $e^t$ , and overestimate that for  $t$  between 0 and 1. Since the largest value it can have in this range is  $e^t = e$ , we can take  $M = e$ , or simpler yet,  $M = 3$ , since we can be sure that 3 is a convenient overestimation of  $e$ . Thus the error is

$$R_4 \leq \frac{3|(1-0)^{4+1}|}{(4+1)!} = \frac{3}{120} = 0.025.$$

Thus, we can be sure that  $e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}$  to within 0.025.

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### Example 6 Approximating the Natural Logarithm

Find the 5th Taylor polynomial for  $f(x) = \ln x$  around  $a = 1$ , and estimate the error term if we use this polynomial to approximate  $\ln 1.1$ .

**Solution** We already calculated this polynomial in Example 3, where we obtained:

$$\ln x \approx (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5}.$$

For the error term, we want to approximate  $\ln 1.1$ , so  $x = 1.1$ . We look at the 6th derivative,  $f^{(6)}(t) = -120/t^6$  and estimate its largest magnitude as  $t$  ranges from 1 to 1.1. It is largest when  $t = 1$ , so that we can take  $M = 120$ . (Note that we throw away the sign in estimating a value for  $M$ .) Thus, the error is

$$R_5 \leq \frac{120(1.1 - 1)^{5+1}}{(5+1)!} = \frac{0.00012}{720} \approx 0.000\ 000\ 167.$$

**Before we go on...** The point of this example is that there is a way to calculate natural logarithms using ordinary arithmetic. If we wish to know  $\ln 1.1$ , this tells us:

$$\ln 1.1 \approx (1.1-1) - \frac{(1.1-1)^2}{2} + \frac{(1.1-1)^3}{3} - \frac{(1.1-1)^4}{4} + \frac{(1.1-1)^5}{5} \approx 0.0953103.$$

How accurate is this? The error is no larger than 0.000000167, so we can say with absolute certainty that  $\ln 1.1$  is somewhere between 0.0953101 and 0.0953105.

Sometimes, we would like to approximate a certain quantity to a specified number of decimal places.

**Question** What exactly does it mean to approximate a quantity to, say, 3 decimal places.

**Answer** If we already know the exact value of the quantity, then we simply round it to 3 decimal places. Notice that doing so may change the quantity by as much as  $0.0005 = 5 \times 10^{-4}$ . (For example, rounding 0.1005 to 3 decimal places produces 0.101, and so we have changed it by 0.0005.) This motivates the following definition:

#### **Accuracy to $n$ Decimal Places**

If  $x$  is a quantity we want to approximate, and  $y$  is an approximation of  $x$ , then we say that  **$y$  approximates  $x$  to  $n$  decimal places** if the magnitude of the difference is no bigger than  $5 \times 10^{-(n+1)}$ ; that is,

$$|y - x| \leq 5 \times 10^{-(n+1)}$$

#### **Quick Examples**

1. 0.12345 approximates 0.123454 to 5 decimal places because  $|0.12345 - 0.123454| = 0.000004 \leq 0.000005 = 0.5 \times 10^{-6}$ .

2. 0.999 approximates 1.000 to 2 decimal places because  $|0.999 - 1.000| = 0.001 \leq 0.005 = 0.5 \times 10^{-3}$ .

#### **Example 7 Approximating to 10 Decimal Places**

How many terms would we need in order to compute  $\ln 1.1$  to 10 decimal places?

**Solution** We want to know what  $n$  will guarantee that the error will be no larger than  $0.5 \times 10^{-11}$ . For this, we will need to estimate the error term  $R_n$  for a general  $n$ , and so we need the  $(n+1)$ st derivative of  $f(x) = \ln x$ . If we go back to the calculation of the derivatives of  $\ln x$ , we can notice that

$$|f^{(n+1)}(t)| = \frac{n!}{(n+1)!}.$$

As in the last example, we have  $a = 1$  and  $x = 1.1$ , so the  $(n+1)$ st derivative is largest when  $t = 1$ , giving  $M = n!$ . Thus,

$$R_n \leq \frac{n!(0.1)^{n+1}}{(n+1)!} = \frac{(0.1)^{n+1}}{n+1}.$$

We want this error to be no larger than  $0.5 \times 10^{-11}$ , so we try a few numbers for  $n$  until the above expression is  $10^{-11}$  or smaller. After some trial and error we notice that

$$\frac{(0.1)^{9+1}}{(9+1)} = 10^{-11},$$

So, we need  $n = 9$  in order to approximate  $\ln 1.1$  to 10 decimal places. That is,

$$\begin{aligned} \ln 1.1 &\approx 0.1 - \frac{0.1^2}{2} + \frac{0.1^3}{3} - \frac{0.1^4}{4} + \frac{0.1^5}{5} - \frac{0.1^6}{6} + \frac{0.1^7}{7} - \frac{0.1^8}{8} + \frac{0.1^9}{9} \\ &\approx 0.095\ 310\ 179\ 8 \end{aligned}$$

which is guaranteed to be accurate to the 10 decimal places shown.

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Let us finish by giving the simple proof of Taylor's Theorem. (You should turn back several pages and look at what Taylor's Theorem says.)

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### ***Proof of Taylor's Theorem***

We start by writing the familiar equation

$$\int_a^x f'(t) dt = f(x) - f(a) \quad \text{Fundamental Theorem of Calculus}$$

so that

$$f(x) = f(a) + \int_a^x f'(t) dt$$

Now comes the clever part. We now use integration by parts in a strange way to evaluate the integral.

$D$	$I$
$+ f'(t)$	$1$
$- f''(t)$	$(t-x)$
$+ f'''(t)$	$\frac{(t-x)^2}{2!}$
$- \dots$	$\dots$
$(-1)^{n-1} f^{(n)}(t)$	$\frac{(t-x)^{n-1}}{(n-1)!}$
$(-1)^n \int f^{(n+1)}(t)$	$\frac{(t-x)^n}{n!}$

Here we used the antiderivative  $(t-x)$  of 1, which we can do if we like. Why do we do this? Omniscience. (Actually, trial an error, to make the next step come out right.) The last arrow is labeled with a  $-$  if  $n$  is odd, and a  $+$  if  $n$  is even. A useful trick:  $(-1)^n$  in  $+1$  if  $n$  is even, and  $-1$  if  $n$  is odd, and we use this below. If we do the integration by parts according to this table, we get

$$\begin{aligned}
 f(x) &= f(a) + \left[ f'(t)(t-x) \right]_a^x - \left[ f''(t) \frac{(t-x)^2}{2!} \right]_a^x + \left[ f'''(t) \frac{(t-x)^3}{3!} \right]_a^x - \dots \\
 &\quad + (-1)^{n-1} \left[ f^{(n)}(t) \frac{(t-x)^n}{n!} \right]_a^x + (-1)^n \int_a^x \frac{f^{(n+1)}(t)(t-x)^n}{n!} dt \\
 &= f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \dots \\
 &\quad + \frac{f^{(n)}(a)(x-a)^n}{n!} + \int_a^x \frac{f^{(n+1)}(t)(x-t)^n}{n!} dt
 \end{aligned}$$

Here we have used many times the fact that  $(a-x)^n = (x-a)^n$  if  $n$  is even, but  $(a-x)^n = -(x-a)^n$  if  $n$  is odd.

### TAYLOR SERIES EXERCISES

In Exercises 1–14, find the 5th Taylor polynomials of the functions around the given points.

1.  $f(x) = x^3 + 2x^2 - 3x + 1; a = 0$
2.  $f(x) = x^3 - x^2 + 4x + 10; a = 0$
3.  $f(x) = x^3 + 2x^2 - 3x + 1; a = 1$
4.  $f(x) = x^3 - x^2 + 4x + 10; a = 1$

## Taylor Polynomials

5.  $f(x) = \ln(1-x); a = 0$

6.  $f(x) = \ln(2-x); a = 0$

7.  $f(x) = e^x; a = 0$

8.  $f(x) = e^{-x}; a = 0$

9.  $f(x) = e^{-x^2}; a = 0$

10.  $f(x) = e^{(x-1)^2}; a = 1$

11.  $\frac{1}{x-1}; x = 0$

12.  $\frac{1}{(1-x)^2}; x = 0$

13.  $\sqrt{x}; x = 1$

14.  $x^{1/3}; x = 1$

In Exercises 15–20, find the Taylor polynomial around  $x = a$  approximating  $f(x)$  to within  $\pm 0.0001$ . Calculate the approximation given by the Taylor polynomial, and compare to the answer given by a calculator.

15.  $f(x) = e^x; a = 0$ , approximate  $f(1)$

16.  $f(x) = e^x; a = 0$ , approximate  $f(1/2)$

17.  $f(x) = \frac{1}{x}; a = 1$ , approximate  $f(1.1)$

18.  $f(x) = \frac{1}{x}; a = 1$ , approximate  $f(0.9)$

19.  $f(x) = \sqrt{x}; a = 100$ , approximate  $f(101)$

20.  $f(x) = \sqrt{x}; a = 100$ , approximate  $f(99)$

21. How many terms of the Taylor series around  $x = 0$  could you use to approximate  $e$  to 3 decimal places?

22. How many terms of the Taylor series around  $x = 0$  could you use to approximate  $e$  to 4 decimal places?

23. To how many decimal places is the approximation

$$\frac{1}{(1-x)} - 1 \approx x + x^2 + x^3$$

in Example 4 accurate when  $x = 0.1$ ?

24. To how many decimal places is the approximation

$$\ln 1.1 \approx 0.1 - \frac{0.1^2}{2} + \frac{0.1^3}{3} - \frac{0.1^4}{4} + \frac{0.1^5}{5}$$

in Example 7 accurate?

## APPLICATIONS

**25. Investing** The Amex Gold BUGS<sup>†</sup> Index was at 150 points in January 2003, decreasing at a rate of 14.5 points/month, and accelerating at 3.6 points/month<sup>2</sup>.

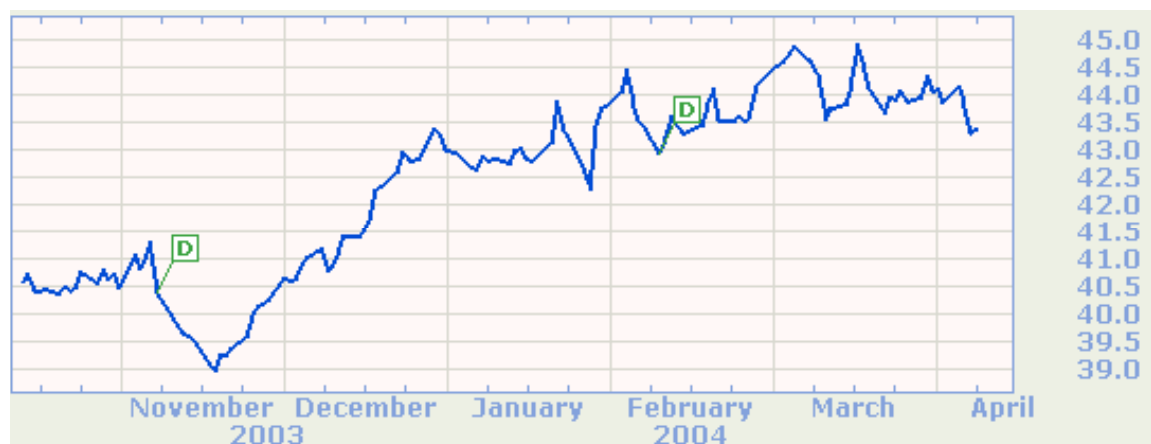


Source: <http://www.amex.com>

(a) Taking  $t$  as time in months since January 2003, obtain the first and second Taylor polynomials of the BUGS index  $b$  as a function of  $t$  around  $t = 0$ .

(b) Use the second order Taylor polynomial to approximate the value of the index in January 2004, and compare the predicted value with the (approximate) actual value.

**26. Investing** The price of Consolidated Edison common stock (ED) was 40.5 at the start of December 1, 2003, increasing at a rate of \$3/month, and decelerating by \$1.10/month<sup>2</sup>.



Source: <http://money.excite.com>

<sup>†</sup> BUGS stands for “basket of unhedged gold stocks.”

- (a) Taking  $t$  as time in months since December 1 2003, obtain the first and second Taylor polynomials of the ED stock price  $p$  as a function of  $t$  around  $t = 0$ .
- (b) Use the second order Taylor polynomial to approximate the ED stock price on April 1 2004, and compare the predicted price with the (approximate) actual price.

### COMMUNICATION & REASONING EXERCISES

27. The linear Taylor polynomial of  $f(x)$  around  $a$  is the equation of the tangent line to the graph of  $f$  at  $a$ . Explain.

28. The quadratic Taylor polynomial of  $f(x)$  around  $a$  is the equation of the parabola tangent to the graph of  $f$  at  $a$  with the same curvature as  $f$  at  $a$ . Explain.

29. An enthusiastic math student, having discovered that

$$\ln x = x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots$$

decides to save money by not buying the scientific feature on his new calculator, figuring that all he has to do is to use this formula to calculate  $\ln x$ . Being somewhat lazy, he decides that the formula

$$\ln x = x - 1 - \frac{(x-1)^2}{2}$$

is “probably accurate to three decimal places.” Comment on this speculation.

30. Your friend tells you that the formula

$$\ln x = x - 1 - \frac{(x-1)^2}{2}$$

is “definitely accurate to three decimal places provided  $x$  is close enough to 1.” How close?

### Answers: Taylor Series Exercises

1.  $x^3 + 2x^2 - 3x + 1$
2.  $x^3 - x^2 + 4x + 10$
3.  $1 + 4(x-1) + 5(x-1)^2 + (x-1)^3 = x^3 + 2x^2 - 3x + 1$
4.  $14 + 5(x-1) + 2(x-1)^2 + (x-1)^3 = x^3 - x^2 + 4x + 10$
5.  $-x - x^2/2 - x^3/3 - x^4/4 - x^5/5$
6.  $\ln 2 - x/2 - x^2/8 - x^3/24 - x^4/64 - x^5/160$
7.  $1 + x + x^2/2 + x^3/6 + x^4/24 + x^5/120$
8.  $1 - x + x^2/2 - x^3/6 + x^4/24 - x^5/120$
9.  $1 - x^2 + x^4/2$
10.  $1 + (x-1)^2 + (x-1)^4/2$
11.  $-1 - x - x^2 - x^3 - x^4 - x^5$

## Taylor Polynomials

12.  $1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5$
13.  $1 + (x-1)/2 - (x-1)^2/8 + (x-1)^3/16 - 5(x-1)^4/128 + 7(x-1)^5/256$
14.  $1 + (x-1)/3 - (x-1)^2/9 + 5(x-1)^3/81 - 10(x-1)^4/243 + 22(x-1)^5/729$
15.  $n = 7: 1 + 1 + 1/2 + 1/3! + 1/4! + 1/5! + 1/6! + 1/7! \approx 2.71825, e^1 = 2.71828. \dots$
16.  $n = 5: 1 + 1/2 + 1/8 + 1/(8 \cdot 3!) + 1/(16 \cdot 4!) + 1/(32 \cdot 5!) \approx 1.6487, e^{1/2} = 1.64872127\dots$
17.  $n = 3: 1 - (0.1) + (0.1)^2 - (0.1)^3 = 0.909, 1/1.1 = 0.90909\dots$
18.  $n = 4: 1 + (0.1) + (0.1)^2 + (0.1)^3 + (0.1)^4 = 1.1111, 1/0.9 = 1.111\dots$
19.  $n = 2: 10 + 1/20 - 1/8000 = 10.0499, \sqrt{101} = 10.049875\dots$
20.  $n = 2: 10 - 1/20 - 1/8000 = 9.94987, \sqrt{99} = 9.9498743\dots$
21. 8 terms altogether ( $n = 7$ )
22. 9 terms altogether ( $n = 8$ )
23.  $R_3 \leq 32x^4 = 32(0.1)^4 = 0.0032$ . Since this is less than 0,005, the approximation is accurate to 2 decimal places.
24.  $R_5 \leq (0.1)^6/6 \approx 0.000\ 000\ 166\ 7$ . Since this is less than 0.000 000 5, the approximation is accurate to 6 decimal places.
25. **(a)** First Taylor Polynomial:  $b(t) \approx 150 - 14.5t$ ; Second Taylor Polynomial:  $b(t) \approx 150 - 14.5t + 1.8t^2$  **(b)**  $b(12) \approx 235.2$  points. The actual value was around 220 points.
26. **(a)** First Taylor Polynomial:  $p(t) \approx 40.5 + 3t$ ; Second Taylor Polynomial:  $p(t) \approx 40.5 + 3t - 0.55t^2$  **(b)**  $p(4) \approx \$43.70$  points. The actual value was around \$44.
27. The linear Taylor polynomial is  $f(a) + f'(a)(x-a)$ , which is the equation of the line through  $(a, f(a))$  with slope  $f'(a)$ .
28. The quadratic polynomial is  $f(a) + f'(a)(x-a) + f''(a)(x-a)^2/2$ , which has the same value, first derivative, and second derivative at  $a$  as does  $f$ .
29. The speculation is unfounded. Take  $x = 1.5$ . Then  $\ln 1.5 \approx 0.4055$ , whereas the second Taylor sum is  $T_2(1.5) = 1.5 - 1 - (1.5 - 1)^2/2 = 0.375$ , so the error is much larger than 0.005.
30. If  $x$  is within 0.114 of 1, then the approximation is guaranteed to be accurate to 3 decimal places.